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Structure of the probability distribution for the GHZ quantum state under local von Neumann measurements

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Abstract: We show that the probability distribution of the Greenberger-Horne-Zeilinger quantum state (GHZ) under local action of independent von Neumann measurements follows a convex distribution of two distributions. The coefficients of the combination are related to the equatorial parts of the measurements, and the distributions associated with those coefficients are associated with the real parts of the measurements. One possible application of the result is that it allows one to split into two pieces the simulation of the GHZ state. Simulating, in worst-case or in average case, a quantum state like the GHZ state with random resources, shared or private, as well as with classical communication resources or even odd resources like nonlocal boxes is a very important in the theory of quantum communication complexity.

Keywords: GHZ, non-locality, simulation of entanglement, equatorial von Neumann measurement, real von Neumann measurement

1. Introduction

The starting point of the result was motivated by the simulation of the Greenberger-Horne-Zeilinger (GHZ, [1], [2], [3], [4], [5]) quantum state. We can think of this simulation problem as a problem in which many people get the description of a von Neumann measurement. Each party does not know the description of any other measurements belonging to the other parties. Each party after having applied his measurement on the subsystem of the state that he shares with the others gets a classical outcome. The joint distribution of the outcomes of every parties follows the distribution studied in this thesis in the case of the GHZ state. The result indicates that in order to simulate the distribution, we can first simulate the equatorial parts of the measurements in order to know which distribution associated to the real parts of the measurements to simulate. Simulating GHZ means simulating the probability distribution of the quantum state, and, more specifically, simulating with shared or private random resources as well as classical communication resources (a classical channel). What does it mean? Intuitively, suppose that an entangled quantum state is shared among n persons. A state is entangled if it cannot be factored partially or completely into tensor products. For instance, we are interested in the state $|\Psi\rangle = \frac{1}{\sqrt{2}}|0^n\rangle + \frac{1}{\sqrt{2}}|1^n\rangle$ called the GHZ state in honor of Greenberger, Horne et Zeilinger, [6]. Each person owns the description of a von Neumann measurement that he can apply to his part of GHZ. Call these n measurements M_j for $j \in \{1, \ldots, n\}$. It is important in what follows that the i^{th} person does not know M_j for $i \neq j$. n persons applying jointly and independently their measurements mean they apply $\otimes_{j=1}^{n} M_j$ to $|\Psi\rangle$. By applying these measurements, they get random outputs $|b_j\rangle$. The 1^{st} gets $|b_1\rangle$, the 2^{nd} gets $|b_2\rangle$, etc. Quantum mechanics asserts these random results are eigenvalues of the M_j. The joint distribution of the these results is what we are trying to simulate classically. Each local von Neumann measurement is represented by a point on the 3-dimensional sphere (called sometimes the Bloch sphere). A point on the sphere is represented by $(\theta, \phi) \in [0, 2\pi) \times [0, \pi)$. The result says that the probability distribution is a convex combination of two distributions. The coefficients of the convex combination are related to the θ_j reprensenting the equatorial parts of the M_j and the two distributions are related to the ϕ_j representing the real parts of M_j.

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For analyzing the structure of the probability distribution that we shall denote from now by $\mathbf{P}$, we will proceed slightly differently, but in an equivalent way, by supposing that the n persons transform $|\Psi\rangle$ by applying a unitary transormation U_j independently from each other. By doing this, the basis reprensenting the eigenvectors is then transformed into the canonical basis used in quantum computing. The eigenvectors are then $|a_j\rangle$ with $a_j \in \{0,1\}$. Hence, the n persons measure in the standard basis of computation once they have applied $U = \otimes_{j=1}^{n} U_j$ to the state $|\Psi\rangle$. The distribution obtained is $\mathbf{P}(a)$ with $a = a_1 \cdots a_n \in \{0,1\}^n$. More precisely, the result is about the structure of $\mathbf{P}(a) = |\langle a|U|\Psi\rangle|^2$.

Being able of simulating classically $\mathbf{P}$ was the subject of many articles. In fact, when saying "simulating GHZ" or "simulating entanglement", it means simulating classically $\mathbf{P}$. How ? With shared random variables and classical communication. For the concepts related to communication complexity like the definitions of worst-case or average case complexity and shared random variables, [7]. The result itself is about the structure of $\mathbf{P}$ and not about communication complexity. Originally, there were A. Einstein, B. Podolsky and N. Rosen (EPR trio, [8]) in 1935 who believed they showed that quantum mechanics (of Copenhagen) is not complete because it is impossible to reproduce quantum correlations with hidden variables (shared random variables) obtained from $|\Psi\rangle$ when $n = 2$. These correlations are the moments of the distribution of $\mathbf{P}$. In [9], it was shown that it is impossible to reproduce these correlations with only shared random variables. Maudlin [10] proposed to add classical communication resource (classical channel) to reproduce those correlations. Maudlin made that suggestion without having the knowledge of the theory of complexity used by computer scientists. From that moment, the theory of communication complexity became important in order to try reproducing $\mathbf{P}$. A sequence of papers were then published on the simulation of $\mathbf{P}$ with $n = 2$ and $n = 3$ with average and worst case complexity with different types of measurements, [11–15].

From the point of view of simulating the distribution $\mathbf{P}$, the result gives a road to follow. In fact, it says that the distribution $\mathbf{P}$ is a convex combination of two distributions denoted by $\mathbf{P}_1$ and $\mathbf{P}_2$. In order to simulate $\mathbf{P}$, firstly one has to to simulate a random bit allowing to decide which of $\mathbf{P}_1$ or $\mathbf{P}_2$ to simulate. This bit is related to the coefficients of the convex combination multiplying $\mathbf{P}_1$ and $\mathbf{P}_2$. The combination being convex, the sum equals 1 and it defines a Bernoulli distribution, biased in general. Secondly, one has to simulate $\mathbf{P}_1$ or $\mathbf{P}_2$ which are discrete distributions over the set $\{0,1\}^n$. The simulation of the Bernoulli distribution is related to the phases of the complex numbers defining the local unitary transformations U_j of the n persons. In an equivalent way, simulating the Bernoulli distritution is only related to the equatorial parts of the local von Neumann M_j if we adopt that way of working. For the simulation of $\mathbf{P}_1$ or of $\mathbf{P}_2$, it is related to the amplitudes of the complex numbers defining the local unitary transformations U_j of the n persons. In an equivalent way, simulating $\mathbf{P}_1$ or $\mathbf{P}_2$ is related to the real parts of the local von Neumann measurements M_j. The cost of simulation (worst-case or average) is then reduced to the sum of the costs for simulating the Bernoulli distribution reprensented by the coefficients of the convex combination and the two distributions $\mathbf{P}_1$ or $\mathbf{P}_2$. Section (2) of this article contains the main result about the structure of $\mathbf{P}$. Section (3) is about the equivalent correspondence between the two ways of working and a small discussion about the simulation of $\mathbf{P}$.

2. Structure of the probability distribution

In what follows, $n \in \mathbb{N}$, $j = 1, \ldots, n$, $a_j \in \{0,1\}$ and $U_j \in U(2)$, where $U(2)$ is the group of unitary 2×2 matrice. The main result concerns the structure of the probability distribution $\mathbf{P}$ over the set $\{0,1\}^n$ where

$$\mathbf{P}(a) = |\langle a|U|\Psi\rangle|^2$$
$$\langle a| = \bigotimes_{j=1}^{n} \langle a_j|$$
$$U = \bigotimes_{j=1}^{n} U_j$$
$$|\Psi\rangle = \tfrac{1}{\sqrt{2}}|0^n\rangle + \tfrac{1}{\sqrt{2}}|1^n\rangle.$$

We remind the reader of the following lemma before proving our theorem.

Lemma 1. *For all $U \in U(2)$, there exist $\varphi \in [0, 2\pi)$, $\psi \in [0, 2\pi)$, $\omega \in [0, 2\pi)$ and $\tau \in \{0,1\}$ such that*

$$U = \begin{pmatrix} e^{\imath\varphi}\cos\omega & -\imath^{\tau} e^{-\imath\psi}\sin\omega \\ \imath^{\tau} e^{\imath\psi}\sin\omega & (-1)^{\tau} e^{-\imath\varphi}\cos\omega \end{pmatrix}.$$

The next remark gives some interesting cases of the preceding lemma. Only the first three cases will be useful in proving the main result.

Remark. 1.If $\tau = 0$ in the preceding lemma (1), then $U \in SU(2)$.

2.If $\tau = 0$, $\varphi = 0$ and $\psi = 0$ in the preceding lemma (1), then U is a rotation.
3.If $\tau = 1$ and $\varphi = 0$ in the preceding lemma (1), then U is also hermitian that is $U = U^\dagger$. In that case, U is an element of order 2.
4.The Pauli matrix σ_1 corresponds to $\omega = \frac{\pi}{2}$, $\varphi = 0$, $\psi = -\frac{\pi}{2}$ and $\tau = 1$.
5.The Pauli matrix σ_2 corresponds to $\omega = \frac{\pi}{2}$, $\varphi = 0$, $\psi = 0$ and $\tau = 1$.
6.The Pauli matrix σ_3 corresponds to $\omega = 0$, $\varphi = 0$, $\psi = 0$ and $\tau = 1$.
7.The Hadamard matrix corresponds to $\omega = \frac{\pi}{4}$, $\varphi = 0$, $\psi = -\frac{\pi}{2}$ and $\tau = 1$.

One more thing before stating the main result, for all $j = 1, \ldots, n$, define α_j and β_j as

$$\alpha_j = e^{\imath\varphi_j} \cos\omega_j$$
$$\beta_j = e^{\imath\psi_j} \sin\omega_j.$$

Therefore every $U_j \in U(2)$ can be represented as

$$U_j = \begin{pmatrix} \alpha_j & -\imath^{\tau_j}\bar{\beta}_j \\ \imath^{\tau_j}\beta_j & (-1)^{\tau_j}\bar{\alpha}_j \end{pmatrix}.$$

The next theorem is the main result of this paper.

Theorem 1.*Let $n \in \mathbb{N}$ players (persons) sharing the state $|\Psi\rangle = \frac{1}{\sqrt{2}}|0^n\rangle + \frac{1}{\sqrt{2}}|1^n\rangle$. Every player receives a fix unitary transformation U_j for $j = 1, \ldots, n$. Every player applies locally on his part his unitary transformation and then measures in the standard basis. The results (outputs) are random bits $a_j \in \{0, 1\}$ for $j = 1, \ldots, n$. Let U_j be given as in the lemma (1), $a = a_1 \cdots a_n \in \{0, 1\}^n$, $\gamma = \sum_{j=1}^n (\varphi_j + \psi_j) \in \mathbb{R}/2\pi\mathbb{Z}$, and $\kappa = \frac{\pi}{2}\sum_{j=1}^n \tau_j \in \mathbb{R}/2\pi\mathbb{Z}$. The joint distribution $\mathbf{P}$, defined by $\mathbf{P}(a) = |\langle a|U|\Psi\rangle|^2$ with $U = \otimes_{j=1}^n |U_j\rangle$ is a convex combination of two distributions as follow*

$$\mathbf{P}(a) = \cos^2\big(\tfrac{\gamma-\kappa}{2}\big)\mathbf{P}_1(a) + \sin^2\big(\tfrac{\gamma-\kappa}{2}\big)\mathbf{P}_2(a),$$

with

$$\mathbf{P}_1(a) = \frac{1}{2}\big(\mathrm{f}_1(a) + \mathrm{f}_2(a)\big)^2,$$
$$\mathbf{P}_2(a) = \frac{1}{2}\big(\mathrm{f}_1(a) - \mathrm{f}_2(a)\big)^2,$$
$$\mathrm{f}_1(a) = \prod_{j=1}^n \cos\big(\omega_j - \tfrac{\pi}{2}a_j\big) \text{ and}$$
$$\mathrm{f}_2(a) = \prod_{j=1}^n -\sin\big(\omega_j - \tfrac{\pi}{2}a_j\big).$$

*Proof.*Let the functions $x_j : \{0, 1\} \to \{\varphi_j, \psi_j\}$ for all $j = 1, \ldots, n$ be defined as

$$x_j(a_j) = \begin{cases} \varphi_j \text{ if } a_j = 0 \\ \psi_j \text{ if } a_j = 1 \end{cases}$$

Moreover, defined the functions $s : \{0, 1\}^n \to \mathbb{R}/2\pi\mathbb{Z}$, $t : \{0, 1\}^n \to \mathbb{R}/2\pi\mathbb{Z}$ and the constants $\gamma \in \mathbb{R}/2\pi\mathbb{Z}$, $\kappa \in \mathbb{R}/2\pi\mathbb{Z}$ respectively by

$$s(a) = \sum_{j=1}^n x_j(a_j),$$
$$t(a) = \frac{\pi}{2}\sum_{j=1}^n \tau_j a_j,$$
$$\gamma = \sum_{j=1}^n \big(\varphi_j + \psi_j\big),$$
$$\kappa = \frac{\pi}{2}\sum_{j=1}^n \tau_j.$$

Since

$$\mathbf{P}(a) = |\langle a|U|\Psi\rangle|^2 \tag{1}$$

$$= \left| \tfrac{1}{\sqrt{2}} \prod_{j=1}^{n} \langle a_j|U_j|0\rangle + \tfrac{1}{\sqrt{2}} \prod_{j=1}^{n} \langle a_j|U_j|1\rangle \right|^2, \tag{2}$$

we then have by the definitions of the functions x_j that the complex arguments (phases) of the terms of the products of the line (2) can be written as

$$\begin{aligned} \arg \langle a_j|U_j|0\rangle &= x_j(a_j) + \tau_j a_j \frac{\pi}{2} \\ \arg \langle a_j|U_j|1\rangle &= -x_j(a_j \oplus 1) + \tau_j a_j \frac{\pi}{2} + \tau_j \frac{\pi}{2}. \end{aligned} \tag{3}$$

Now, the products of the line (2) are written as

$$\prod_{j=1}^{n} \langle a_j|U_j|0\rangle = \mathrm{f}_1(a) \exp\left(\imath \sum_{j=1}^{n} \arg \langle a_j|U_j|0\rangle \right) \tag{4}$$

$$\prod_{j=1}^{n} \langle a_j|U_j|1\rangle = \mathrm{f}_2(a) \exp\left(\imath \sum_{j=1}^{n} \arg \langle a_j|U_j|1\rangle \right). \tag{5}$$

Moreover, by definitions of the function s and the constant γ, we have that

$$\sum_{j=1}^{n} x_j(a_j) = \left(s(a) - \frac{\gamma}{2} \right) + \frac{\gamma}{2} \tag{6}$$

$$\begin{aligned} -\sum_{j=1}^{n} x_j(a_j \oplus 1) &= s(a) - \gamma \\ &= \left(s(a) - \frac{\gamma}{2} \right) - \frac{\gamma}{2}. \end{aligned} \tag{7}$$

Thanks to (6) and to (7), we have that

$$\langle a|U|\Psi\rangle = e^{\imath \left(t(a) + s(a) - \frac{\gamma - \kappa}{2} \right)} \left(\left(\frac{\mathrm{f}_1(a) + \mathrm{f}_2(a)}{\sqrt{2}} \right) \cos\left(\frac{\gamma - \kappa}{2} \right) + \imath \left(\frac{\mathrm{f}_1(a) - \mathrm{f}_2(a)}{\sqrt{2}} \right) \sin\left(\frac{\gamma - \kappa}{2} \right) \right). \tag{8}$$

Therefore, the distribution $\mathbf{P}$ over the set $\{0,1\}^n$ is written as follow

$$\mathbf{P}(a) = \cos^2\left(\frac{\gamma - \kappa}{2} \right) \mathbf{P}_1(a) + \sin^2\left(\frac{\gamma - \kappa}{2} \right) \mathbf{P}_2(a). \tag{9}$$

The only thing that remains to be shown is that $\mathbf{P}_1$ and $\mathbf{P}_2$ are probability distributions. It is clear that $\mathbf{P}_1$ and $\mathbf{P}_2$ are positive for all $a \in \{0,1\}^n$. Hence, it remains to show that $\sum \mathbf{P}_1(a) = 1$. (The proof for $\mathbf{P}_2$ is similar.) To show it, we use the facts that f_1^2 and f_2^2 are distributions so that $(1/2)(\mathrm{f}_1^2 + \mathrm{f}_2^2)$ is a distribution as well so that

$$\frac{1}{2} \sum_{a \in \{0,1\}^n} \left(\mathrm{f}_1^2(a) + \mathrm{f}_2^2(a) \right) = 1.$$

Moreover, we use the fact that

$$\sum_{a \in \{0,1\}^n} (-1)^{a_1 + \ldots + a_n} = 0.$$

Hence,

$$\sum_{a \in \{0,1\}^n} \mathbf{P}_1(a) = 1.$$

■

*Remark.*In addition to the lemma for representing a unitary transformation of size 2×2 by a quadruple, it is worth mentioning that we split the phase of the expression $\langle a|U|\Psi\rangle$ into two parts, the non-interesting part i.e. $e^{\imath\left(t(a)+s(a)-\frac{\gamma-\kappa}{2}\right)}$ and the rest. Shor uses the same technic in analyzing his famous algorithm.

*Remark.*There is a computational advantage in space of using the formula we found for $\mathbf{P}$ instead of using directly the one given by the axioms of quantum mechanics. In fact, in the formula we found, there is no tensor product and therefore we can compute $\mathbf{P}(a_1,\ldots,a_n)$ for large n given the n unitary transformations (or von Neumann measurements). When there are for instance 10 persons each having a 2×2 unitary matrix, then the tensor product of those unitary matrice is a unitary matrix of size 1024×1024.

Theorem (1) shows that the distribution $\mathbf{P}$ over $\{0,1\}^n$ is a linear convex combination of the distributions $\mathbf{P}_1$ and $\mathbf{P}_2$ that are caracterized by the products of the modules (amplitudes) f_1 and f_2. The next corollary establishes a link between the parameters of the unitary transformations and $\mathbf{P}$.

We remind of the definition of a *local* probability distribution (in the physical sense). We will use that definition only to assert that certain distributions have that property.

Definition 1(local distribution). *Let Λ be a random variable with k possibles realizations λ_i with non-zero probability for $i = 1,\ldots,k$. Given fixed unitary transformations U_j with $j = 1,\ldots,n$, a distribution $\mathbf{Q} : \{0,1\}^n \to [0,1]$ is local with respect to Λ and U_j if, for all $a_1a_2\cdots a_n \in \{0,1\}^n$, the following factorisation holds*

$$\mathbf{Q}_{\{U_j\}_{j=1}^n}(a_1,\ldots,a_n) = \sum_{i=1}^{k}\prod_{j=1}^{n}\mathbf{Q}_{U_j,\lambda_i}(a_j)\mathrm{Prob}(\Lambda=\lambda_i).$$

*Remark.*Without entering into the details, simulating classically a local distribution does not cost anything from the point of view of communication complexity.

Corollary 1(Interesting statistical events, choice of interesting parameters, and probabilistic interpretations). *For all $j = 1,\ldots,n$, let the quadruples $(\omega_j,\psi_j,\varphi_j,\tau_j)$ defining the specific following unitary transformations V_j.*

1. *f_1^2 is the local probability distribution obtained when $\otimes_{j=1}^n V_j$ is applied to the state $|0^n\rangle$ and $(\omega_j,\psi_j,\varphi_j,\tau_j) = (\omega_j,0,0,0)$ for all $j = 1,\ldots,n$.*
2. *f_2^2 is the local probability distribution obtained when $\otimes_{j=1}^n V_j$ is applied to the state $|1^n\rangle$ and $(\omega_j,\psi_j,\varphi_j,\tau_j) = (\omega_j,0,0,0)$ for all $j = 1,\ldots,n$.*
3. *$\mathbf{P}_1$ is the local probability distribution obtained when $\otimes_{j=1}^n V_j$ is applied to the state $|\Psi\rangle$ and $(\omega_j,\psi_j,\varphi_j,\tau_j) = (\omega_j,0,0,0)$ for all $j = 1,\ldots,n$.*
4. *$\mathbf{P}_2$ is the local probability distribution obtained when $\otimes_{j=1}^n V_j$ is applied to the state $|\Psi\rangle$ and $(\omega_j,\psi_j,\varphi_j,\tau_j) = (\omega_j,0,0,0)$ for all $j = 1,\ldots,n-2$, $(\omega_j,\psi_j,\varphi_j,\tau_j) = (\omega_j,0,0,1)$ for $j = n-1,n$. (Only two of the parameters τ_j must be equal to 1. We could have taken any other indices instead of $n-1$ and n.)*
5. *$\mathbf{P}_2$ can also be obtained when $\otimes_{j=1}^n V_j$ is applied to $|\Psi\rangle$ and $(\omega_j,\psi_j,\varphi_j,\tau_j)$ $= (\omega_j,0,0,0)$ for all $j = 1,\ldots,n-1$, $(\omega_j,\psi_j,\varphi_j,\tau_j) = (\omega_j,0,-\frac{\pi}{2},1)$ for $j = n$. It is equivalent to apply σ_3 followed by a rotation over the n^{th} subsystem.*
6. *$\frac{1}{2}(\mathrm{f}_1^2+\mathrm{f}_2^2) = \frac{1}{2}(\mathbf{P}_1+\mathbf{P}_2)$ is the probability distribution obtained by applying $\otimes_{j=1}^n V_j$ to $|\Psi\rangle$ and $(\omega_j,\psi_j,\varphi_j,\tau_j) = (\omega_j,0,0,0)$ for all $j = 1,\ldots,n-1$ and $(\omega_n,\psi_n,\varphi_n,\tau_n) = (\omega_n,0,0,1)$. It is interesting to notice that this distribution is local.*
7. *The Bernoulli distribution with parameters $\cos^2\left((\gamma-\kappa)/2\right)$ or $\sin^2\left((\gamma-\kappa)/2\right)$ corresponds to the distribution of the sum $(a_1+\ldots+a_n) \bmod 2$ (parity) obtained by applying $\otimes_{j=1}^n V_j$ to $|\Psi\rangle$* when *the angles ω_j are restricted to $\{\pi/4,3\pi/4,5\pi/4,7\pi/4\}$ which means that*

$$V_j = \begin{pmatrix} e^{\imath\varphi_j}\cos\omega_j & -\imath^{\tau_j}e^{-\imath\psi_j}\sin\omega_j \\ \imath^{\tau_j}e^{\imath\psi_j}\sin\omega_j & \imath^{2\tau_j}e^{-\imath\varphi_j}\cos\omega_j \end{pmatrix} \text{ and } \omega_j \in \{\pi/4,3\pi/4,5\pi/4,7\pi/4\}.$$

*Proof.*The proofs of (1), (2), (3), (4), (5), (6) are immediate. To show (7), notice that if $\omega_j \in \{\pi/4,3\pi/4,5\pi/4,7\pi/4\}$ then

$$\mathbf{P}(a_1,\ldots,a_n) = \frac{1}{2^n} + \frac{(-1)^{n+\sum_{j=1}^n a_j}}{2^n}\left(\prod_{j=1}^{n}\sin(2\omega_j)\right)\cos(\gamma-\kappa).$$

Moreover, for all $j = 1, \ldots, n$

$$\sin(2\omega_j) \in \{-1, +1\} \Longrightarrow \prod_{j=1}^{n} \sin(2\omega_j) \in \{-1, +1\}.$$

Since exactly half the elements of $\{0,1\}^n$ have an even parity and the other half have an odd parity, then we have

$$\begin{aligned}\mathrm{Prob}\Big(\sum_{j=1}^{n} a_j = b\Big) &= 2^{n-1}\left(\frac{1}{2^n} + \frac{(-1)^{n+b}}{2^n}\left(\prod_{j=1}^{n}\sin(2\omega_j)\right)\cos(\gamma-\kappa)\right)\\ &= \frac{1}{2} + \frac{(-1)^{n+b}}{2}\left(\prod_{j=1}^{n}\sin(2\omega_j)\right)\cos(\gamma-\kappa).\end{aligned}$$

Therefore,

$$\mathrm{Prob}\Big(\sum_{j=1}^{n} a_j = b\Big) = \begin{cases} \cos^2\left(\frac{\gamma-\kappa}{2}\right) & \text{if } \begin{cases} n+b \text{ is even and } \prod_{j=1}^{n}\sin(2\omega_j) = 1 \\ \text{or} \\ n+b \text{ is odd and } \prod_{j=1}^{n}\sin(2\omega_j) = -1 \end{cases} \\ \sin^2\left(\frac{\gamma-\kappa}{2}\right) & \text{if } \begin{cases} n+b \text{ is odd and } \prod_{j=1}^{n}\sin(2\omega_j) = 1 \\ \text{or} \\ n+b \text{ is even and } \prod_{j=1}^{n}\sin(2\omega_j) = -1 \end{cases} \end{cases}$$

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*Remark.*The terms $\cos^2\left(\frac{\gamma-\kappa}{2}\right)$ and $\sin^2\left(\frac{\gamma-\kappa}{2}\right)$ are the coefficients of the convex combination of the distribution $\mathbf{P}$ and it explains why (7) is important. In the next section, we will briefly talk about the simulation of $\mathbf{P}$, and to know how simulating $\mathrm{Prob}\Big(\sum_{j=1}^{n} a_j = b\Big)$ will be important for branching i.e. in order to know which of $\mathbf{P}_1$ or $\mathbf{P}_2$ to simulate.

2.1. Marginal and conditional distribution for the n^{th} bit

In this section, $n \geq 2$. Firstly, we will find the marginal distribution $\mathbf{P}(a_{i_1}, \ldots, a_{i_m})$ when $i_j \in I$, $I \in \mathcal{P}(\{1, \ldots, n\})$, $\mathrm{card}(I) = m$, $j \in \{1, \ldots, m\}$, $I \neq \emptyset$, and $I \neq \{1, \ldots, n\}$. Secondly, we will find the conditional distribution n^{th} bit given $a_1, \ldots, a_{n-1}$.

Theorem 2.*Let $n \geq 2$, $I \in \mathcal{P}(\{1, \ldots, n\})$, $\mathrm{card}(I) = m$, $I = \{i_1, \ldots, i_m\}$, $I \neq \emptyset$, and $I \neq \{1, \ldots, n\}$, we have that*

$$\mathbf{P}(a_{i_1}, \ldots, a_{i_m}) = \frac{1}{2}\left(\prod_{j=1}^{m}\cos^2\left(\omega_{i_j} - \frac{\pi}{2}a_{i_j}\right) + \prod_{j=1}^{m}\sin^2\left(\omega_{i_j} - \frac{\pi}{2}a_{i_j}\right)\right).$$

*Proof.*It is only necessary to look at the case $I = \{1, 2, \ldots, n-1\}$ i.e. when $\mathrm{card}(I) = n-1$ and $n \notin I$ because, by symmetry, we easily deduce other cases. We notice first that

$$\mathbf{P}(a_1, \ldots, a_n) = \cos^2\left(\frac{\gamma-\kappa}{2}\right)\mathbf{P}_1(a) + \sin^2\left(\frac{\gamma-\kappa}{2}\right)\mathbf{P}_2(a)$$

Hence using the definition the marginal distribution,

$$\begin{aligned}\mathbf{P}(a_1, \ldots, a_{n-1}) &= \mathbf{P}(a_1, \ldots, a_{n-1}, 0) + \mathbf{P}(a_1, \ldots, a_{n-1}, 1)\\ &= \frac{1}{2}\left(\prod_{j=1}^{n-1}\cos^2\left(\omega_j - \frac{\pi}{2}a_j\right) + \prod_{j=1}^{n-1}\sin^2\left(\omega_j - \frac{\pi}{2}a_j\right)\right).\end{aligned}$$

By symmetry, we have the desired result.

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*Remark.*For all $I \in \mathcal{P}(\{1,\ldots,n\})$ such that $I \neq \emptyset$ and $I \neq \{1,\ldots,n\}$, the marginal distribution associated to the set of indices I is *local.*

Now, we look more closely to the conditional distribution $\mathbf{P}(a_n|a_1,\ldots,a_{n-1})$, which is possible since we know the marginal distribution $\mathbf{P}(a_1,\ldots,a_{n-1})$.

Theorem 3.*Let*

$$u = \prod_{j=1}^{n-1} \cos(\omega_j - \frac{\pi}{2}a_j)$$

$$v = \prod_{j=1}^{n-1} -\sin(\omega_j - \frac{\pi}{2}a_j)$$

$$t = \arctan\left(\frac{v}{u}\right).$$

Being given the first $(n-1)$ *bits* $a_1,\ldots,a_{n-1}$*, the conditional distribution of the* n^{th} *bit is given by*

$$\mathbf{P}(a_n|a_1,\ldots,a_{n-1}) = \cos^2\left(\omega_n - \frac{\pi}{2}a_n - t\right)\cos^2\left(\frac{\gamma-\kappa}{2}\right) + cos^2\left(\omega_n - \frac{\pi}{2}a_n + t\right)\sin^2\left(\frac{\gamma-\kappa}{2}\right).$$

*Proof.*By definition,

$$\mathbf{P}(a_n|a_1,\ldots,a_{n-1}) = \frac{\mathbf{P}(a_1,\ldots,a_n)}{\mathbf{P}(a_1,\ldots,a_{n-1})}.$$

Hence, fixing the first $(n-1)$ bits $a_1,\ldots,a_{n-1}$, letting

$$u = \prod_{j=1}^{n-1} \cos(\omega_j - \frac{\pi}{2}a_j)$$

$$v = \prod_{j=1}^{n-1} -\sin(\omega_j - \frac{\pi}{2}a_j)$$

$$t = \arctan\left(\frac{v}{u}\right)$$

and using the fact that for all $p, q \in \mathbb{R}$ and $x \in [0, 2\pi)$,

$$p\cos(x) + q\sin(x) = \sqrt{p^2+q^2}\cos(x+h)$$

$$h = \arctan\left(\frac{q}{p}\right) + \begin{cases} 0 \text{ if } p \leq 0 \\ \pi \text{ if } p > 0 \end{cases}$$

we therefore have that

$$\mathbf{P}(a_n|a_1,\ldots,a_{n-1}) = \cos^2\left(\frac{\gamma-\kappa}{2}\right)\cos^2\left(\omega_n - \frac{\pi}{2}a_n - t\right) + \sin^2\left(\frac{\gamma-\kappa}{2}\right)\cos^2\left(\omega_n - \frac{\pi}{2}a_n + t\right).$$

It is not necessary to add π *to* t *when* $u > 0$ *since we square the cosinus.*

■

Remark.$\mathbf{P}(a_n|a_1,\ldots,a_{n-1})$ is a convex combination of two distributions.

3. From general von Neumann measurements to equatorial and real measurements

3.1. Correspondences between specific 2×2 *unitary transformations and measurements on the 3-dimensional sphere*

We remind that a measure over a qubit is a hermitian operator represented by a triple $(x, y, z) \in \mathbb{R}^3$ such that

$$M = x\sigma_1 + y\sigma_2 + z\sigma_3 = \begin{pmatrix} z & x - \imath y \\ x + \imath y & -z \end{pmatrix} \quad \text{and} \quad x^2 + y^2 + z^2 = 1.$$

and therefore using using the spherical coordinates, the pair (θ, ϕ) can be used for representing a measure over a qubit. We have that $x = \cos\theta\cos\phi$, $y = \sin\theta\cos\phi$ and $z = \sin\phi$ o $(\theta, \phi) \in [0, 2\pi) \times [0, \pi)$. A measure over a qubit $M = x\sigma_1 + y\sigma_2 + z\sigma_3$ is *real* if $y = 0$. A measure over a qubit $M = x\sigma_1 + y\sigma_2 + z\sigma_3$ is *equatorial* if $z = 0$.

The next corollary explains the meaning, in terms of equatorial and real measurements, of the convex parts of $\mathbf{P}$ i.e. the coefficients (Bernoulli distribution) and the two distributions $\mathbf{P}_1$ and $\mathbf{P}_2$. In fact, we give an equivalent picture in terms of von Neumann measurements of the unitary transformations involved in the corollary (1). In fact, a player may instead of applying a unitary transformation on his part followed by measurement in the standard basis apply a von Neumann measurement directely getting eigenvalues ± 1. In fact, here -1 replaces 1 in the standard basis of computation and $+1$ replaces 0 since the eigenvectors $|0\rangle = |+1\rangle$ and $-|1\rangle = |-1\rangle$. In general, the spectrum (set of eigenvalues) of a measure $M = x\sigma_1 + y\sigma_2 + z\sigma_3$ such that $x^2 + y^2 + z^2 = 1$ is $\{-1, +1\}$. A unitary transformation U is given by

$$U = \begin{pmatrix} \alpha & -\imath^\tau\bar{\beta} \\ \imath^\tau\beta & (-1)^\tau\bar{\alpha} \end{pmatrix} \text{ and } \alpha = e^{i\varphi}\cos\omega,\ \beta = e^{i\psi}\sin\omega$$

Doing a change of basis, we have that $M = U\mathrm{diag}(1, -1)U^\dagger$. Hence, denoting $w = x + \imath y$,

$$M = \begin{pmatrix} z & \bar{w} \\ w & -z \end{pmatrix} = \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{pmatrix}$$

gives

$$\begin{aligned} z &= |\alpha|^2 - |\beta|^2 = \cos 2\omega \quad \text{and} \\ w &= \quad 2\imath^\tau\bar{\alpha}\beta \quad = \imath^\tau e^{i(\psi-\varphi)}\sin 2\omega. \end{aligned}$$

To represent a measure M, the spherical coordinates $(\theta, \phi) \in [0, 2\pi) \times [0, \pi)$ may be used to obtain a geometric interpretation. Since $z = \sin\phi = \cos 2\omega$, we have $\phi = \frac{\pi}{2} + 2\omega$, and since $w = e^{\imath(\psi-\varphi+\tau\frac{\pi}{2})}\sin(2\omega)$, we have $\theta = \psi - \varphi + \tau\frac{\pi}{2}$.

Corollary 2. *1.A unitary matrix U represented by $(\omega, 0, 0, 0)$ corresponds to a real measurement.*
2.A unitary matrix U represented by $(\omega, \varphi, \psi, \tau)$ and $\omega \in \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$ corresponds to an equatorial measurement.
3.By denoting $b_j \in \{-1, +1\}$ for $j \in \{1, \ldots, n\}$ we have the following correspondence between the outputs

$$\begin{aligned} a_1 + \ldots + a_n &\equiv 0 \mod 2 \Leftrightarrow b_1 \cdots b_n = +1 \\ a_1 + \ldots + a_n &\equiv 1 \mod 2 \Leftrightarrow b_1 \cdots b_n = -1. \end{aligned}$$

3.2. A small section on the classical simulation of $\mathbf{P}(a)$

In this section, we will establish a road map for the simulation in terms of von Neumann measurements instead of unitary transformations. Thanks to the corollary (1), if there were protocols for simulating classically with communication, in worst-case or average case complexity, and with shared random variables both of $\mathbf{P}_1$ (or $\mathbf{P}_2$) and the Bernoulli distribution involved in the convex coefficients of $\mathbf{P}$, then it would be possible to simulate $\mathbf{P}$. The next corollary (3) is equivalent to the corollary (1) with the difference that it is expressed in terms of von Neumann measurements. The corollary (3) asserts that the problem of simulating $\mathbf{P}$ under applying locally von Neumann measurements can be split into two problems, the one of simulating equatorial measurements (coefficients of the convex combination) and the one of simulating real measurements ($\mathbf{P}_1$ or $\mathbf{P}_2$).

Corollary 3.*Let $\mathrm{P}_{\mathrm{GHZ-R}}$ be a protocol simulating $\mathbf{P}_1$ or $\mathbf{P}_2$. Moreover, let $\mathrm{P}_{\mathrm{GHZ-E}}$ be a protocol simulating the Bernoulli distribution with parameter $\cos^2\big((\gamma - \kappa)/2\big)$. If $\mathrm{P}_{\mathrm{GHZ-E}}$ and $\mathrm{P}_{\mathrm{GHZ-R}}$ exist with worst-case or average case complexity, then we can create a protocol, denoted by $\mathrm{P}_{\mathrm{GHZ}}$, for simulating $\mathbf{P}$ such that Cost$(P_{GHZ}) \leq$ Cost$(P_{GHZ-E})+$ Cost$(P_{GHZ-R}) + n$.*

*Proof.*First, the inputs and outputs of $\mathrm{P}_{\mathrm{GHZ}}$ are respectively the parameters describing the unitary transformations i.e. $\{(\varphi_j, \psi_j, \omega_j, \tau_j)\}_{j=1}^n$ and an element $a \in \{0, 1\}^n$. The inputs and outputs of $\mathrm{P}_{\mathrm{GHZ-E}}$ are respectively $\{(\varphi_j, \psi_j, \tau_j)\}_{j=1}^n$ and a bit $c \in \{0, 1\}$ such that $\mathrm{Prob}(c = 0) = \cos^2\big((\gamma - \kappa)/2\big)$. The inputs and outputs of $\mathrm{P}_{\mathrm{GHZ-R}}$ are respectively $\{(\omega_j, \tau_j)\}_{j=1}^n$ and an element $a \in \{0, 1\}^n$ such that $\mathrm{Prob}(a|c = 0) = \mathbf{P}_1(a)$.

We create $\mathrm{P}_{\mathrm{GHZ}}$ as follow :

1.$\mathrm{P}_{\mathrm{GHZ}}$ calls $\mathrm{P}_{\mathrm{GHZ-E}}$ as a subroutine with inputs $\{(\varphi_j, \psi_j, \tau_j)\}$ for $j = 1, \ldots, n$. $\mathrm{P}_{\mathrm{GHZ-E}}$ returns c and, if $c = 0$, then $\mathrm{P}_{\mathrm{GHZ}}$ goes to (2) otherwise $\mathrm{P}_{\mathrm{GHZ}}$ goes to (3).

2.$\mathrm{P}_{\mathrm{GHZ}}$ calls $\mathrm{P}_{\mathrm{GHZ-R}}$ as a subroutine with inputs $\{(\omega_j, 0)\}$ for $j = 1, \ldots, n$. $\mathrm{P}_{\mathrm{GHZ}}$ returns $a = a_1 \cdots a_n$.
3.$\mathrm{P}_{\mathrm{GHZ}}$ calls $\mathrm{P}_{\mathrm{GHZ-R}}$ as a subroutine with inputs $\{(\omega_j, 0)\}$ for $j = 1, \ldots, n-2$ and $\{(\omega_j, 1)\}$ for $j = n-1, n$. $\mathrm{P}_{\mathrm{GHZ}}$ returns $a = a_1 \cdots a_{n-1} a_n$.

Clearly $\mathbf{P}(a)$ is the probability of the value returned by $\mathrm{P}_{\mathrm{GHZ}}$ and Cost$(P_{GHZ}) \leq$ Cost$(P_{GHZ-E})+$Cost$(P_{GHZ-R})+n$. The n extra bits are necessary for simulating the equatorial part consisting of the distribution of the parity i.e. $\mathrm{Prob}\Big(\sum_{j=1}^{n} a_j = c\Big)$. Without loss of generality, the players can restrict $\omega_j = \pi/4$ for simulating the equatorial part. The $(n-1)$ first players send their bit to the n^{th} player, and the n^{th} player simulates on his part the bit c that he returns to the other players for a total of n bits.

■

3.3. Locality of the marginal and conditional distribution of n^{th} bit

In this section, using the fact that $\mathbf{P}(a_1, \ldots, a_n) = \mathbf{P}(a_1, \ldots, a_{n-1})\mathbf{P}(a_n|a_1, \ldots, a_{n-1})$, we look at the simulation of $\mathbf{P}(a_1, \ldots, a_n)$. In fact, as it was shown before, all the marginal distributions are local, and, hence it does not cost anything for simulating $\mathbf{P}(a_1, \ldots, a_{n-1})$. It seems therefore that the big difficulty for simulating $\mathbf{P}(a_1, \ldots, a_n)$ "belongs" the n^{th} person when $n \geq 2$. The conditional distribution $\mathbf{P}(a_n|a_1, \ldots, a_{n-1})$ is given by

$$\mathbf{P}(a_n|a_1, \ldots, a_{n-1}) = \cos^2\left(\frac{\gamma-\kappa}{2}\right)\cos^2\left(\omega_n - \frac{\pi}{2}a_n - t\right) + \sin^2\left(\frac{\gamma-\kappa}{2}\right)\cos^2\left(\omega_n - \frac{\pi}{2}a_n + t\right).$$

with

$$t = \arctan\left(\frac{v}{u}\right),$$

$$u = \prod_{j=1}^{n-1} \cos(\omega_j - \frac{\pi}{2}a_j),$$

$$v = \prod_{j=1}^{n-1} -\sin(\omega_j - \frac{\pi}{2}a_j).$$

Having an idea of the discrete distribution of t over $\{0,1\}^{n-1}$ and how simulating it would be useful.

4. Conclusion

In this paper, we showed that the discrete probability distribution $\mathbf{P}$ is a convex combination of two distributions, $\mathbf{P}_1$ and $\mathbf{P}_2$. The coefficients of the combination are associated with the phases of the complex numbers defining the unitary transformations. The distributions, $\mathbf{P}_1$ and $\mathbf{P}_2$, are related to the modules of the complex numbers defining the unitary transformations. From the point of view commmunication complexity, the usefulness of the result is that we can split the simulation of the distribution $\mathbf{P}$ of the GHZ quantum state into two simulations. Among open questions, we wonder whether or not $\mathbf{P}_1$ and $\mathbf{P}_2$ are local? If $\mathbf{P}_1$ and $\mathbf{P}_2$ are local, then it would follow that the entanglement of GHZ, which is a maximally entangled state, would be contained in the coefficients multiplying $\mathbf{P}_1$ and $\mathbf{P}_2$?

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