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Analytical Solution to the Dirac equation in 3+1 Space-Time Dimensions

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Abstract: We consider the general Dirac equation in 3+1 space-time dimensions. We derive an analytical expression for the general solution of the Dirac equation. This solution has been generated by transforming the Dirac equation for one spinor component to a generalized 3D-Ricatti type of equation.

Keywords: Dirac equation, analytical solution.

1. Introduction

The Dirac equation is a relativistic extension of the Schrödinger equation, which describes the time-evolution of a relativistic quantum mechanical particle [1]. It is a relativistically covariant first order linear differential equation in space and time. It describes a spinor particle at relativistic energies below the threshold of pair production. It also embodies the features of quantum mechanics as well as special relativity. However, despite all the work that has been done over the years on this equation, its exact solution has been limited to a very small set of potentials. Exactly solvable potentials are of great interest both from the pure mathematical point of view and for testing the validity of perturbative, numerical and semi-classical approximations of physical systems. Furthermore, in some limiting cases or for under some special circumstances they may constitute analytic solutions to realistic problems. Besides, it is a fact that exact solutions are important because of the conceptual understanding of physics that can only be brought about by the analysis of such solutions. Exact solvability of a given Dirac equation with its boundary conditions usually entails the exact knowledge of all its eigenfunctions and the corresponding energy spectrum. However, in recent years there have been efforts in classifying all types of solvable problems based on symmetry considerations. Most of the known exactly solvable problems fall within distinct classes of what is referred to as ”shape invariant potentials”[2], supersymmetric quantum mechanics[2], potential algebras[3], and ”point canonical transformations”[4] are three methods among many which are used in the search for exact solutions of the wave equation. Finally, we would like to mention that interest in the solutions of the Dirac equation for the case of spin and pseudo-spin symmetry has surged due to the great simplification it entails to the associated spinor equations, a critical investigation of these cases was recently accomplished[5].

Since the original work of Dirac in the early part of last century up until 1989 only the relativistic Coulomb problem was solved exactly. In 1989, the relativistic extension of the oscillator problem (Dirac-Oscillator) was formulated and solved by Moshinsky and Szczepaniak [6]. However, recently an effective approach for solving the Dirac equation with spherical symmetry was introduced by Alhaidari[7]. His method was initiated by the observation that different potentials can be grouped into symmetry classes; for example, the non-relativistic Coulomb, oscillator and S-wave Morse problems constitute one such class. Therefore, the solution of two problems in one class implies solution for the remaining one. This ap-

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proach was then applied successfully in obtaining solutions for the relativistic extension of a class of shape invariant potentials such as the Dirac-Scarf, Dirac-Rosen-Morse I & II, Dirac-Pschl-Teller, Dirac-Eckart, Dirac-Hulthn, and Dirac-Woods-Saxon potentials[8].

In this article we would like to present an efficient method to solve analytically the three dimensional Dirac equation. This solution will be generated by transforming the Dirac equation for one spinor component to a Ricatti type of equation which will then solved using a recently derived fast converging method[9–11].

2. Solution of the Dirac equation

The free Dirac equation is given by

$$(i\gamma^{\mu}\partial_{\mu} - M)\Psi(t, \overrightarrow{r}) = 0, \tag{1}$$

where we have used the Einstein summation convention for repeated indices. Taking into account the vector $V(\to r)$ and scalar $S(\to r)$ interaction potentials the Dirac equation can be rewritten in its spinor component form as follows

$$\begin{aligned}&[M + S(\to r) + V(\to r)]\,\phi^{+} + (-i\to\sigma.\to\nabla)\,\phi^{-}\\ &= \varepsilon\phi^{+},\end{aligned} \tag{2}$$

$$\begin{aligned}&(-i\to\sigma.\to\nabla)\,\phi^{+} + [-M - S(\to r) + V(\to r)]\,\phi^{-}\\ &= \varepsilon\phi^{-},\end{aligned} \tag{3}$$

where $\phi^{+} = \begin{pmatrix}\phi_{+}^{+}\\ \phi_{-}^{+}\end{pmatrix}$ and $\phi^{-} = \begin{pmatrix}\phi_{+}^{-}\\ \phi_{-}^{-}\end{pmatrix}$ are the upper and lower two-component spinors of Ψ, respectively. From the above equations we obtain

$$\phi^{\mp} = \frac{(-i\to\sigma.\to\nabla)\,\phi^{\pm}}{[\varepsilon \pm M \pm S(\to r) - V(\to r)]} \tag{4}$$

The Dirac equation after algebraic manipulations for ϕ^{+} then becomes

$$\begin{aligned}&(-i\to\sigma.\to\nabla)^{2}\,\phi^{+}\\ &+h(\to r)\,(-i\to\sigma.\to\nabla)\,\phi^{+} + g(\to r)\phi^{+} = 0,\end{aligned} \tag{5}$$

where $h(\to r)$and $g(\to r)$ are functions in 2×2 dimensions.

$$h(\to r) = (-i\to\sigma.\to\nabla)\,\frac{1}{[\varepsilon + M + S(\to r) - V(\to r)]}, \tag{6}$$

$$g(\to r) = [M + S(\to r) + V(\to r) - \varepsilon]\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{7}$$

Let us first define the operator

$$\pounds = -i\to\sigma.\to\nabla \tag{8}$$

The above Dirac equation (5) can be then written in terms of this operator as follows

$$\pounds^{2}\left[\phi^{+}\right] + h(\to r)\pounds\left[\phi^{+}\right] + g(\to r)\phi^{+} = 0. \tag{9}$$

It is worth to mention that

$$\pounds^{2} = -\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\Delta \tag{10}$$

Let us try to find the inverse of this operator $\pounds^{-1}$. We consider $H(\to r)$, so that

$$H(\to r) = (-i\to\sigma.\to\nabla)\,G, \tag{11}$$

where G is 2×2 dimensional function verifying

$$\begin{aligned}&\int\frac{\partial}{\partial x_i}G(\to r)dx_j = \int\frac{\partial}{\partial x_j}G(\to r)dx_i,\\ &\qquad i, j = 1, 2, 3\end{aligned} \tag{12}$$

here x_i are defined as following: $(x_1, x_2, x_3) = (x, y, z)$.

Let us calculate

$$\int(i\to\sigma.d\to r)H(\to r), \tag{13}$$

Since $\sigma_i^2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and $\sigma_i\sigma_j + \sigma_j\sigma_i = 0$ for $i \neq j$ and by using the restriction for the 2×2 dimensional function G we get

$$\begin{aligned}&\int(i\to\sigma.d\to r)H(\to r)\\ &= \int(i\to\sigma.d\to r)\pounds(G(\to r)) = G(\to r).\end{aligned} \tag{14}$$

Let us calculate

$$-i\to\sigma.\to\nabla\left[\int(i\to\sigma.d\to r)G(\to r)\right] \tag{15}$$

$$\begin{aligned}&-i\to\sigma.\to\nabla\left[\int(i\to\sigma.d\to r)G(\to r)\right]\\ &= \sigma_x^2\frac{\partial}{\partial x}\int Gdx + \sigma_x^2\frac{\partial}{\partial y}\int Gdy + \sigma_x^2\frac{\partial}{\partial z}\int Gdx\\ &+\sigma_x\sigma_y\frac{\partial}{\partial x}\int Gdy + \sigma_y\sigma_x\frac{\partial}{\partial y}\int Gdx\\ &+\sigma_x\sigma_y\frac{\partial}{\partial x}\int Gdy + \sigma_y\sigma_x\frac{\partial}{\partial y}\int Gdx\\ &+\sigma_y\sigma_z\int\frac{\partial}{\partial y}Gdz + \sigma_y\sigma_z\int\frac{\partial}{\partial z}Gdy\end{aligned} \tag{16}$$

which gives

$$-i\to\sigma.\to\nabla\left[\int(i\to\sigma.d\to r)G(\to r)\right] = G(\to r), \tag{17}$$

in other words

$$-i\to\sigma.\to\nabla\left[\int(i\to\sigma.d\to r)G(\to r)\right] \tag{18}$$

$$= \pounds \left[\int (i \to \sigma . d \to r) G(\to r) \right] = G(\to r).$$

so we have

$$\pounds \left[\int (i \to \sigma . d \to r) G(\to r) \right] = G(\to r), \tag{19}$$

and

$$\int (i \to \sigma . d \to r) \pounds \left[G(\to r) \right] = G(\to r), \tag{20}$$

we can write then that

$$\pounds^{-1}(G(\to r)) = \int (i \to \sigma . d \to r) G(\to r), \tag{21}$$

for any 2×2 dimensional function G verifying

$$\int \frac{\partial}{\partial x_i} G(\to r) dx_j = \int \frac{\partial}{\partial x_j} G(\to r) dx_i$$
$$i, j = 1, 2, 3 \tag{22}$$

Let us analyze the last condition, we can deduce more restricted condition by deriving this condition by $\frac{\partial}{\partial x_i}$ and then by $\frac{\partial}{\partial x_i}$ and we get a more restricted condition

$$\frac{\partial^2}{\partial x_i^2} G = \frac{\partial^2}{\partial x_j^2} G, \tag{23}$$

where, as an exception to the general rule, repeated indices are not summed over in (27). If we define the following coordinate $(\xi_1, \zeta_2, \zeta_3) = (x, iy, z)$ then our previous equation reduces to

$$\begin{aligned} &\Delta_{1,2} G = 0, \\ &\qquad and \\ &\Delta_{2,3} G = 0, \end{aligned} \tag{24}$$

where $\Delta_{i,j} = \frac{\partial^2}{\partial \zeta_i^2} + \frac{\partial^2}{\partial \zeta_j^2}$.Let us return to our main problem the Dirac equation and let us define a 2×2 dimensional function F verifying the condition and

$$\phi^+ = \exp\left[\pounds^{-1} F\right] A \tag{25}$$

$$= \exp\left[\int (i \to \sigma . d \to r) F(\to r) \right] A,$$

in other words

$$\begin{aligned} F &= \pounds \left[\ln \left[\phi^+ . A^T \right] \right] \\ &= (-i \to \sigma . \to \nabla) \ln \left[\phi^+ . A^T \right] \end{aligned} \tag{26}$$

A is a two dimensional constant vector. The Dirac equation becomes then

$$F^2(\to r) + (-i \to \sigma . \to \nabla) F(\to r) \tag{27}$$
$$+ h(\to r) F(\to r) + g(\to r) = 0.$$

Using the same technique developed recently for Schrödinger equation [9–11] we can write

$$F(\to r) = F_0(\to r) + \eta(\to r), \tag{28}$$

where $F_0(\to r)$ and $\eta(\to r)$ are 2×2 dimensional functions. $F_0(\to r)$ represents the adiabatic part of the solution verifying

$$F_0^2(\to r) + h(\to r) F_0(\to r) + g(\to r) = 0, \tag{29}$$

with formal solution

$$F_{0\pm}(\to r) = \frac{-h(\to r) \pm \left(\left[h(\to r) \right]^2 - 4 g(\to r) \right)^{\frac{1}{2}}}{2}. \tag{30}$$

neglecting the $\eta(\to r)^2$ we get

$$F_0^2(\to r) + (-i \to \sigma . \to \nabla) \eta(\to r) \tag{31}$$
$$+ \left[h(\to r) + 2 F_0(\to r) \right] \eta(\to r) = 0.$$

The 2-dimensional constant vectors $A_\pm$ in the general solution (42) are to be fixed by the boundary conditions.

3. Conclusion

We have succeeded in solving the three dimensional Dirac equation by reducing the one component spinor equation to a generalized Ricatti equation which was then solved using a recently developed fast converging technique, the final form of the solution is given by equation (42). The present method can be applied to a broad class of relativistic problems ranging from atomic to laser applications. This approach promises that it is more accurate than the usual perturbative approach and goes beyond the adiabatic limit.

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