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Analytical Solutions to the Dirac equation in 1+1 Space-Time Dimension

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Abstract: We consider the general Dirac equation in 1+1 space-time dimension with vector, scalar and pseudo-scalar interactions. We give general procedure on how to obtain its exact solution which can be put in closed form only for special types of potential. In addition we have given an analytical approximate solution for general potentials.

Keywords: Dirac equation, analytical solution.

1. Introduction

The Dirac equation is a relativistic quantum mechanical wave equation formulated by British physicist Paul Dirac in 1928 and provides a description of elementary spin-$\frac{1}{2}$ particles, such as electrons, consistent with both the principles of quantum mechanics and the theory of special relativity [1] . The Dirac equation is the most frequently used wave equation for the description of particle dynamics in relativistic quantum mechanics. It is a relativistically covariant first order linear differential equation in space and time. It describes a spinor particle at relativistic energies below the threshold of pair production. It also embodies the features of quantum mechanics as well as special relativity. However, despite all the work that has been done over the years on this equation, its exact solution has been limited to a very small set of potentials.

Exact solutions of the Dirac equation allow for a better understanding of relativistic energy spectrum and their non relativistic limit should result in the corresponding Schrodinger spectrum. Hence a lot of efforts have been deployed to obtain exact solution of the Dirac equation. The use of supersymmetric quantum mechanic along with shape invariance, in particular, provided a very elegant way to obtain exact solutions [2]. Adiabatic approximations, mainly the JWKB approximation, the variational and perturbative approaches have also been used extensively in the past to obtain approximate solutions [3] . In this work we extend the recent approach developed by one of the authors [5,6] for obtaining analytical solutions of the Schrodinger equation that goes beyond the adiabatic approximation. In addition we also consider several exactly solvable potentials obtained by imposing simplifying conditions to our Schrodinger-like equation for the spinor components.

The Dirac equation for a free structureless fermion of mass m is given by:

$$\left(i\hbar\gamma^{\mu}\partial_{\mu} - mc\right)\psi = 0 \tag{1}$$

where c represents the speed of light and $\{\gamma^{\mu}\}_{\mu=0}^{1}$ are square matrices satisfying $\{\gamma^{\mu},\gamma^{\nu}\} = 2g^{\mu\nu}$ with the Mikowski space-time metric $g = diag(+-)$ and the 2-gradiant is defined by $\partial_{\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)$.

After algebraic manipulations [7] we obtain the Dirac equation in 1+1 dimension with most general potential coupling in $u(1)$ gauge invariant

$$\begin{pmatrix} mc^2 + S + qV - \varepsilon & \hbar c\frac{d}{dx} + iqcU + W \\ -\hbar c\frac{d}{dx} - iqcU + W & -m - S + qV - \varepsilon \end{pmatrix} \times \begin{pmatrix} \phi^{+} \\ \phi^{-} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2}$$

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where $S(x)$ is a scalar potential, $W(x)$ is a pseudo-scalar potential and q is the coupling parameter. V and U are components of the 2-vector potential $A = (V, cU)$.

Therefore, we obtain the following first order differential relations between the two components in the conventional relativistic units, $\hbar = c = 1$,

$$\left(\frac{d}{dx} + iqU + W\right)\phi^- = \Omega_+\phi^+,$$

$$\left(\frac{d}{dx} + iqU - W\right)\phi^+ = \Omega_-\phi^-, \tag{3}$$

where $\Omega_\pm = \pm(\varepsilon - V_\pm(x)) - m$ and $V_\pm = qV \pm S$. Let us assume that the potential function $U(x)$ is integrable over the desired interval in configuration space and use the following gauge transformation to eliminate the potential $U(x)$, $\phi^\pm(x) = \zeta^\pm(x)\exp\left[-iq\int_x U(y)dy\right]$, which is a local phase transformation that does not change the physics due to the $u(1)$ symmetry of the problem (local gauge invariance). Hence, we can rewrite Eq. (3) in terms $\zeta^\pm(x)$ as follows

$$(\frac{d}{dx} + W)\zeta^- = \Omega_+\zeta^+,$$

$$(\frac{d}{dx} - W)\zeta^+ = \Omega_-\zeta^-, \tag{4}$$

Eliminating one component in terms of the other results in the following second order differential equation

$$\begin{bmatrix} -\frac{d^2}{dx^2} + \frac{\Omega'_\mp}{\Omega_\mp}\left(\frac{d}{dx} \mp W\right) \\ +W^2 \pm \frac{dW}{dx} + \Omega_+\Omega_- \end{bmatrix}\zeta^\pm = 0 \tag{5}$$

$\Omega_+\Omega_- = (m-S)^2 - (\varepsilon - qV)^2$. It is important to mention that the solution space of the Dirac equation consists of two disconnected subspaces; one is for "positive" energy and the other for "negative" energy. One of these two subspaces is constructed from the solution of Eq. (5) for ζ^+, which will be substituted into Eq. (4b) to give ζ^- that belongs to the same subspace. However, the other subspace is obtained from Eq. (5) with the bottom sign and then Eq. (4a).

We are now interested in solving the differential equation (5) which, in addition to (4), are equivalent to the original Dirac equation. Exploring the analogy between (5) and the Schrodinger equation we can rewrite (5) as follows

$$\left[-\frac{d^2}{dx^2} + \frac{\frac{d\Omega_\mp}{dx}}{\Omega_\mp}\frac{d}{dx} + a(x)\right]\zeta^\pm = 0 \tag{6}$$

where

$$\begin{aligned} a(x) &= \Omega_+\Omega_- + W^2 \pm \frac{dW}{dx} \mp \frac{\frac{d\Omega_\mp}{dx}}{\Omega_\mp}W \\ &= (m-S)^2 - (\varepsilon - qV)^2 \\ &\quad + W^2 \pm \frac{dW}{dx} \mp \frac{\frac{d\Omega_\mp}{dx}}{\Omega_\mp}W. \end{aligned} \tag{7}$$

At this point, we define a point canonical transformation which eliminates the first derivative term in equation (6). This is accomplished through the following transformation

$$\mu_\pm = \frac{\zeta^\pm}{\sqrt{\Omega_\mp}}, \tag{8}$$

then equation (7) becomes

$$\left[\frac{d^2}{dx^2} + V_\pm^{eff}(x)\right]\mu_\pm(x) = 0$$

$$V_\pm^{eff}(x) = \frac{2\frac{d^2\Omega_\mp}{dx^2}\Omega_\mp - 3\left(\frac{d\Omega_\mp}{dx}\right)^2}{4\Omega_\mp^2} \tag{9}$$

$$-W^2 \mp \frac{dW}{dx} \pm \frac{\frac{d\Omega_\mp}{dx}}{\Omega_\mp}W - \Omega_+\Omega_-$$

Thus the original Dirac equation has been transformed into two Schrodinger-like equations for one component coupled with their corresponding kinetic balance equations for the other component of the spinor. Using the method recently developed by one of the authors [5,6,8] we can write the general solution of (6) as follows

$$\begin{aligned} \zeta^\pm(x) &= \sqrt{\Omega_\mp}A_{1\pm}\exp i\left[\int_{x_0}^x \left(f_{1\pm}(y) + \eta_{1\pm}(y)\right)dy\right] \\ &+ \sqrt{\Omega_\mp}A_{2\pm}\exp i\left[\int_{x_0}^x \left(f_{2\pm}(y) + \eta_{2\pm}(y)\right)dy\right], \end{aligned} \tag{10}$$

With

$$f_{1\pm}(x) = \sqrt{V_\pm^{eff}(x)};\, f_{2\pm}(x) = -\sqrt{V_\pm^{eff}(x)},$$

and

$$\begin{aligned} \eta_{1\pm}(x) &= \exp\left[-2\int_{x_0}^x \left(f_{1\pm}(y)\right)dy\right] \\ &\times \int_{x_0}^x \left\{-\frac{df_{1\pm}}{dy}\exp\left[2\int_{y_0}^y \left(f_{1\pm}(z)\right)dz\right]\right\}dy, \\ \eta_{2\pm}(x) &= \exp\left[-2\int_{x_0}^x \left(f_{2\pm}(y)\right)dy\right] \\ &\times \int_{x_0}^x \left\{-\frac{df_{2\pm}}{dy}\exp\left[2\int_{y_0}^y \left(f_{2\pm}(z)\right)dz\right]\right\}dy, \end{aligned} \tag{11}$$

where $A_{1\pm}$ and $A_{1\pm}$ are complex amplitudes that can be determined by the boundary conditions $\zeta^\pm(x_0)$ and $\zeta'^\pm(x_0)$. The numerical implementation of this method its fast convergence was discussed in detail in a previous publications [5,6,9,10]. In addition to the above general solution, we can also generate additional special potential solutions by looking for potential forms $V(x)$, $S(x)$ and $W(x)$ for which the Schrödinger-like equation (9) has exact solutions. For this purpose we suggest to simplify the above equation using the following strategy, first we eliminate the first term in the effective potential this will result in requiring

$$2\frac{d^2\Omega_\mp}{dx^2}\Omega_\mp - 3\left(\frac{d\Omega_\mp}{dx}\right)^2 = 0, \quad (12)$$

with solution given by

$$\Omega_\pm = \frac{1}{(a_\pm x + b_\pm)^2}, \quad (13)$$

where $a_\pm$ and $b_\pm$ are free parameters. This form should be used along with the definition of $\Omega_\pm$ in equation (3)to deduce the form of the solvable potentials $V_\pm(x)$. Next we use this form of $\Omega_\pm$ and choose $W_\pm(x)$ so that it satisfies the following Riccati type equation

$$-W_\pm^2 \mp \frac{dW_\pm}{dx} \pm \frac{\Omega'_\mp}{\Omega_\mp}W_\pm = 0, \quad (14)$$

having the following solution

$$W_\pm(x) = \frac{F_\pm}{(a_\pm x + b_\pm)}, \quad (15)$$

where $F_\pm$ is an additional free parameter. This special choose of $\Omega_\mp$ and $W_\pm$ reduces our previous Schrödinger-like equation to the following simple form

$$\left[\frac{d^2}{dx^2} - \frac{1}{(a_+x+b_+)^2\,(a_-x+b_-)^2}\right]\mu_\pm(x) = 0 \quad (16)$$

with the general solution

$$\begin{aligned}\mu_\pm(x) = C_{1\pm}\Bigg\{&(a_+x+b_+)^{\frac{1}{2}\left[1+\sqrt{1-\frac{4}{(a_-b_+-a_+b_-)^2}}\right]}\\ &\times(a_-x+b_-)^{\frac{1}{2}\left[1-\sqrt{1-\frac{4}{(a_-b_+-a_+b_-)^2}}\right]}\Bigg\}\\ +C_{2\pm}\Bigg\{&(a_+x+b_+)^{\frac{1}{2}\left[1-\sqrt{1-\frac{4}{(a_-b_+-a_+b_-)^2}}\right]}\\ &\times(a_-x+b_-)^{\frac{1}{2}\left[1-\sqrt{1-\frac{4}{(a_-b_+-a_+b_-)^2}}\right]}\Bigg\}. \quad (17)\end{aligned}$$

The solution of the Dirac equation for the spin symmetric case $S = qV$ has been considered in recent literature [4] in this case $\Omega_- = -(\varepsilon + m)$ and hence the corresponding Schrödinger-like equation simplifies to the following form

$$\begin{bmatrix} -\frac{d^2}{dx^2} + \frac{dW}{dx} + W^2 \\ +2S(x)(m+\varepsilon) - (\varepsilon^2 - m^2) \end{bmatrix}\mu_+(x) = 0. \quad (18)$$

This is a generalized type of Klein-Gordon equation whose analytical solution can be derived for certain specific forms of $W(x)$ and $S(x)$. Furthermore similar approach used for generating analytical solution for Schrödinger-like equation (see eq(6) and eq.(9)) can be used to solve the general Klein-Gordon equation given by eq (18) .

In summary we have solved the general Dirac equation in 1+1 dimension and obtained simple solutions for some special potentials. This approach can be extended to deal with the Dirac equation in 2+1 and 3+1 dimensions with cylindrical and spherical symmetry, respectively. The radial Dirac equation will be analogous to the one in 1+1 dimension with $x \to r$ and W→ $W + \kappa/r$, where κ is the spin orbit quantum number. However, in these cases the affine term in the differential operator does not manage because κ cannot be zero (in fact in 2+1 dimensions κ is equal $\pm 1/2$, $\pm 3/2$, $\pm 5/2$...while in 3+1 dimensions κ is equal ± 1, ± 2, ± 3...). Furthermore this procedure will be extended to real 3+1 Dirac equation following a similar procedure used to solve the 3D Schrödinger equation.

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