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About a King-type operator

P. I. Braica, O. T. Pop and A. D. Indrea

Grigore Moisil School, 1 Mileniului Street, 440037 Satu Mare, Romania

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Abstract: In this paper we discuss some properties of a King-type operator. We give an approximation theorem and a Voronovskaja type theorem for this operator.

Keywords: Bernstein's polynomials, King-type operator, Voronovskaja theorem.

1. Introduction

In [6], J.P. King defined linear positive operators which generalize the classical Bernstein operators. These operators reproduce the test functions e_0 and e_2. In the papers [1], [2], [4], [5] and [7], operators of King's type are studied. In this paper, we define new King-type operators which reproduce the test functions e_1 and e_2. After studying its approximation properties, we give a Voronovkaja-type theorem.

2. Preliminaries

Let N be the set of positive integers and $N_0 = N \cup \{0\}$. In this section, we recall some notions and results which we will use in this article. Following [8], we consider $I \subset R$, I an interval and we shall use the function sets: $E(I)$, $F(I)$ which are subsets of the set of real functions defined on I, $B(I) = \{f | f : I \to R,\ f$ bounded on $I\}$, $C(I) = \{f | f : I \to R,\ f$ continuous on $I\}$ and $C_B(I) = B(I) \cap C(I)$. For $x \in I$, consider the function $\psi_x : I \to R$, $\psi_x(t) = t - x$, for any $t \in I$. Let a, b, a', b' be real numbers, $I \subset R$ interval, $a < b, a' < b'$, $[a, b] \subset I, [a', b'] \subset I$, and $[a, b] \cap [a', b'] \neq \phi$. For any $m \in N$, consider the functions $\varphi_{m,k} : I \to R$ with the property that $\varphi_{m,k}(x) \geq 0$ for any $x \in [a', b']$, for any $k \in \{0, 1, 2, ..., m\}$ and the linear positive functionals $A_{m,k} : E([a, b]) \to R$, for any $k \in \{0, 1, 2, ..., m\}$. For $m \in N$, define the operator: $L^*_m : E([a, b]) \to F(I)$ by

$$(L^*_m f)(x) = \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}(f), \tag{1}$$

for any $f \in E([a, b])$, for any $x \in I$ and for $i \in N_0$, define $T^*_{m,i}$ by

$$(T^*_{m,i} L^*_m)(x) = m^i (L^*_m \psi^i_x)(x) = m^i \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}(\psi^i_x), \tag{2}$$

for any $x \in [a, b] \cap [a', b']$. In the following, let s be a fixed natural number, s even and we suppose that the operators $(L^*_m)_{m\geq 1}$ verify the condition: there exists the smallest $\alpha_s, \alpha_{s+1} \in [0; \infty)$ so that

$$\lim_{m\to\infty} \frac{(T^*_{m,j} L^*_m)(x)}{m^{\alpha_j}} = B_j(x) \in R, \tag{3}$$

for any $x \in [a, b] \cap [a', b'], j \in \{s, s + 2\}$ and

$$\alpha_{s+2} < \alpha_s + 2. \tag{4}$$

If $I \subset R$ is a given interval and $f \in C_B(I)$, then, the first order modulus of smoothness of f is the function $\omega(f; .) : [0; \infty) \to R$ defined for any $\delta \geq 0$ by

$$\omega(f; \delta) = sup\{|f(x') - f(x'')| : x', x'' \in I; |x' - x''| \leq \delta\}.$$

*Remark.*For $m \in N$, the L^*_m operators are linear and positive.

Theorem 1.[8] *Let $f : [a, b] \to R$ be a function. If $x \in [a, b] \cap [a', b']$ and f is a s times derivable function in x, the function $f^{(s)}$ is continuous in x, then*

$$\lim_{m\to\infty} m^{s-\alpha_s} \left[(L^*_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!} (T^*_{m,i} L^*_m)(x) \right] = 0. \tag{5}$$

* Corresponding author: e-mail: petrubr@yahoo.com, ovidiutiberiu@yahoo.com, adrian.indrea@yahoo.com

If f is a s times derivable function on $[a,b]$, the function $f^{(s)}$ is continuous on $[a,b]$ and there exists $m(s)\in N$ and $k_j\in R$ so that for any natural number $m, m\geq m(s)$ and for any $x\in[a,b]\cap[a',b']$ we have

$$\frac{(T^*_{m,j}L^*_m)(x)}{m^{\alpha_j}}\leq k_j, \tag{6}$$

where $j\in\{s,s+2\}$, then the convergence given in (2.5) is uniform on $[a,b]\cap[a',b']$ and

$$m^{s-\alpha_s}\left|(L^*_m f)(x)-\sum_{i=0}^{s}\frac{f^{(i)}(x)}{m^i i!}(T^*_{m,i}L^*_m)(x)\right|\leq \tag{7}$$

$\leq \frac{1}{s!}(k_s+k_{s+2})\omega\left(f^{(s)};\frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\right)$, *for any $x\in[a,b]\cap[a',b']$, for any natural number $m, m\geq m(s)$.*

3. The study the convergence and a Voronovskaja-type theorem

In the following, we consider a fixed number $m_0\in N$, $m_0>2$. For the function $f:[0;1]\to R$, we define the sequence of operators $(B^*_m f)_{m\geq m_0}$ by

$$(B^*_m f)(x)=\frac{(m-1)x}{mx-1}\left(1-\frac{1}{m}\right)^{-m}$$
$$\sum_{k=0}^{m} mk(1-x)^{m-k}\left(x-\frac{1}{m}\right)^k f\left(\frac{k}{m}\right), \tag{8}$$

for any $m\geq m_0$ and any $x\in\left[\frac{1}{m_0-1},1\right]$. For fixed m_0, from $m\geq m_0$ it results that $m>m_0-1$. Then, from $x\geq\frac{1}{m_0-1}$ we have $mx-1\geq\frac{m-m_0+1}{m_0-1}>0$, so $mx-1\neq 0$ for any $x\in\left[\frac{1}{m_0-1},1\right]$.

In the following, we use the construction and the results from the first section. For the operator above, we have $\varphi_{m,k}(x)=\frac{(m-1)x}{mx-1}\left(1-\frac{1}{m}\right)^{-m} mk(1-x)^{m-k}\left(x-\frac{1}{m}\right)^k$ and $A_{m,k}=f\left(\frac{k}{m}\right)$, for any $m\geq m_0$, any $k\in\{0,1,2,...,m\}$ and any $x\in\left[\frac{1}{m_0-1},1\right]$.

If $m\in N$, $m\geq m_0$, then the operator B^*_m is linear and positive.

The verify is immediate.

Lemma 1.*The identities*

$$(B^*_m e_0)(x)=\frac{(m-1)x}{mx-1}, \tag{9}$$

$$(B^*_m e_1)(x)=x, \tag{10}$$

$$(B^*_m e_2)(x)=x^2, \tag{11}$$

$$(T^*_{m,0}B^*_m)(x)=\frac{(m-1)x}{mx-1}, \tag{12}$$

$$(T^*_{m,1}B^*_m)(x)=\frac{mx(x-1)}{mx-1}, \tag{13}$$

$$(T^*_{m,2}B^*_m)(x)=\frac{m^2x^2(1-x)}{mx-1} \tag{14}$$

hold for any $m\in N, m\geq m_0$ and any $x\in\left[\frac{1}{m_0-1},1\right]$.

*Proof.*We have that

$(\mathrm{B}^*_m e_0)(x)=\frac{(m-1)x}{mx-1}\left(1-\frac{1}{m}\right)^{-m}$
$\sum_{k=0}^{m} mk(1-x)^{m-k}\left(x-\frac{1}{m}\right)^k$
$=\frac{(m-1)x}{mx-1}\left(1-\frac{1}{m}\right)^{-m}\left(1-x+x-\frac{1}{m}\right)^m$
$=\frac{(m-1)x}{mx-1}\left(1-\frac{1}{m}\right)^{-m}\left(1-\frac{1}{m}\right)^m$
$=\frac{(m-1)x}{mx-1}$,

$(\mathrm{B}^*_m e_1)(x)=\frac{(m-1)x}{mx-1}\left(1-\frac{1}{m}\right)^{-m}$
$\sum_{k=0}^{m} mk(1-x)^{m-k}\left(x-\frac{1}{m}\right)^k\frac{k}{m}$
$=\frac{(m-1)x}{mx-1}\left(1-\frac{1}{m}\right)^{-m}\left(x-\frac{1}{m}\right)$
$\sum_{k=1}^{m} m-1k-1(1-x)^{(m-1)-(k-1)}\left(x-\frac{1}{m}\right)^{k-1}$
$=\frac{(m-1)x}{mx-1}\left(1-\frac{1}{m}\right)^{-m}$
$\frac{mx-1}{m}\left(1-x+x-\frac{1}{m}\right)^{m-1}=x$,

$(B^*_m e_2)(x)=\frac{(m-1)x}{mx-1}\left(1-\frac{1}{m}\right)^{-m}$
$\sum_{k=0}^{m}\frac{k^2}{m^2}mk(1-x)^{m-k}\left(x-\frac{1}{m}\right)^k$
$=\frac{(m-1)x}{mx-1}\left(1-\frac{1}{m}\right)^{-m}\cdot$
$\left(\frac{m-1}{m}\left(x-\frac{1}{m}\right)^2\sum_{k=2}^{m} m-2k-2(1-x)^{m-k}\right.$
$\left(x-\frac{1}{m}\right)^{k-2}+\frac{1}{m}\left(x-\frac{1}{m}\right)\cdot$
$\left.\sum_{k=1}^{m} C^{m-1}_{k-1}(1-x)^{m-k}\left(x-\frac{1}{m}\right)^{k-1}\right)=x^2,$

which means that (3.2)-(3.4) hold.

By using the relations (3.2)-(3.4) we have that $(\mathrm{T}^*_{m,0}B^*_m)(x)=(B^*_m e_0)(x)=\frac{(m-1)x}{mx-1}$, $(T^*_{m,1}B^*_m)(x)=m(B^*_m\psi_x)(x)=m((B^*_m e_1)(x)-x(B^*_m e_0)(x))$
$=\frac{mx(x-1)}{mx-1}$,
$(T^*_{m,2}B^*_m)(x)=m^2(B^*_m\psi^2_x)(x)$
$=m^2((B^*_m e_2)(x)-2x(B^*_m e_1)(x)+x^2(B^*_m e_0)(x))$, from where (3.5)-(3.7) follows.

Lemma 2. *We have that*

$$B_0(x) = \lim_{m\to\infty} (T^*_{m,0}B^*_m)(x) = 1 \tag{15}$$

$$B_2(x) = \lim_{m\to\infty} \frac{(T^*_{m,2}B^*_m)(x)}{m} = x(1-x) \tag{16}$$

for any $x \in \left[\frac{1}{m_0-1}, 1\right]$ *and*

$$(T^*_{m,0}B^*_m)(x) \le m_0 - 1 = k_0 \tag{17}$$

$$\frac{(T^*_{m,2}B^*_m)(x)}{m} \le \frac{m_0}{4} = k_2 \tag{18}$$

for any $x \in \left[\frac{1}{m_0-1}, 1\right]$.

Proof. The relations (3.8)-(3.9) results immediately from Lemma 1. The function $\frac{x}{mx-1}$ is decreasing on $\left[\frac{1}{m_0-1}, 1\right]$, so the maximum is obtained for $x = \frac{1}{m_0-1}$. On the other hand, the inequality $x(1-x) \le \frac{1}{4}$ holds, for any $x \in \left[\frac{1}{m_0-1}, 1\right]$. Then $\frac{(T^*_{m,2}B^*_m)(x)}{m} = m\frac{x}{mx-1}\, x(1-x) \le \frac{m}{4(m-m_0+1)}$, and because $\frac{m}{m-m_0+1} \le m_0$, for any $m \in N, m \ge m_0$, inequality (3.11) is obtained. Similarly, we have that $(T_{m,0}B^*_m)(x) = (m-1)\frac{x}{mx-1} \le \frac{m-1}{m-m_0+1} \le m_0 - 1$, so (3.10) results.

Theorem 2. *Let* $f : [0,1] \to R$ *be a continuous function on* $[0,1]$. *Then*

$|(B^*_m f)(x) - f(x)| \le |f(x)|\frac{1-x}{mx-1} + \frac{x}{mx-1}$
$\left(m - 1 + \frac{1}{\delta}\sqrt{(m-1)x(1-x)}\right)\omega(f;\delta)$,
$|(B^*_m f)(x) - f(x)| \le \frac{(m_0-2)M}{m-m_0+1} + \frac{2(m-1)}{m-m_0+1}\omega$
$\left(f; \frac{1}{2\sqrt{m-1}}\right)$ (18) for any $\delta > 0, m \in N, m \ge m_0$ and $x \in \left[\frac{1}{m_0-1}, 1\right]$, where M= $\sup\left\{|f(x)| : x \in \left[\frac{1}{m_0-1}, 1\right]\right\}$.

Proof. We have $(B^*_m \psi_x^2)(x) = \frac{(T^*_{m,2}B^*_m)(x)}{m^2}$ and by taking (3.7) into account we obtain $(B^*_m \psi_x^2)(x) = \frac{x^2(1-x)}{mx-1}$. Now, by using Shisha-Mond Theorem (see [3]), we obtain inequality (3.12). We take that $\frac{1-x}{mx-1} \le \frac{m_0-2}{m-m_0+1}$, $\frac{x}{mx-1} \le \frac{1}{m-m_0+1}$ and $x(1-x) \le \frac{1}{4}$ for any $x \in \left[\frac{1}{m_0-1}, 1\right]$, any $m \in N, m \ge m_0$ and we consider $\delta = \frac{1}{2\sqrt{m-1}}$. Then, from (3.12), (3.13) follows.

Lemma 3. *Let* $f : [0,1] \to R$ *be a continuous function on* $[0,1]$.

There exists $m(0)$ *so that*

$$\left|(B^*_m f)(x) - \frac{(m-1)x}{mx-1}f(x)\right| \tag{19}$$

$$\le \frac{5m_0-1}{4}\omega\left(f; \frac{1}{\sqrt{m}}\right),$$

$$|(B^*_m f)(x) - f(x)| \le |f(x)|\,\frac{1-x}{mx-1} \tag{20}$$

$$+\frac{5m_0-1}{4}\omega\left(f; \frac{1}{\sqrt{m}}\right)$$

for any $x \in \left[\frac{1}{m_0-1}, 1\right]$ *and any* $m \in N$, $m \ge m(0)$, $m(0)$ *introduced in Theorem 1.*

Proof. The relations (3.14) results from Theorem 1 for $s = 0$, Lemma 1 and Lemma 2. By using the inequality

$|a-c| - |b-c| \le |a-b|$, where $a, b, c \in R$, we have that

$$|(B^*_m f)(x) - f(x)| - \left|\frac{(m-1)x}{mx-1}f(x) - f(x)\right|$$

$$\le \left|(B^*_m f)(x) - \frac{(m-1)x}{mx-1}f(x)\right|,$$

and taking (3.14) into account, the inequality (3.15) is obtained.

Theorem 3. *Let* $f : [0,1] \to R$ *be a continuous function on* $[0,1]$. *Then*

$$\lim_{m\to\infty} (B^*_m f)(x) = f(x) \tag{21}$$

uniformly on $\left[\frac{1}{m_0-1}, 1\right]$. *There exists* $m(0)$ *so that*

$$|(B^*_m f)(x) - f(x)| \le \frac{(m_0-2)M}{m-m_0+1} \tag{22}$$

$$+\frac{5m_0-1}{4}\omega\left(f; \frac{1}{\sqrt{m}}\right)$$

for any $x \in \left[\frac{1}{m_0-1}, 1\right]$ *and any* $m \in N$, $m \ge m(0)$.

Proof. By using the inequality $\frac{1-x}{mx-1} \le \frac{m_0-2}{m-m_0+1}$ demonstrated in Theorem 2 and taking (3.15) into account, the inequality (3.17) is obtained. The relation (3.16) results from (3.17).

Theorem 4. *Let* $f : [0,1] \to R$ *be a continuous function on* $[0,1]$. *If* $x \in \left[\frac{1}{m_0-1}, 1\right]$, f *is two times differentiable in* x *and* $f^{(2)}$ *is continuous in* x, *then*

$$\lim_{m\to\infty} m\left((B^*_m f)(x) - \frac{(m-1)x}{mx-1}f(x)\right)$$
$$= (x-1)f^{(1)}(x) + \frac{x(1-x)}{2}f^{(2)}(x),$$
$$\lim_{m\to\infty} m\left((B^*_m f)(x) - f(x)\right)$$
$$= \frac{1-x}{x}f(x) + (x-1)f^{(1)}(x) + \frac{x(1-x)}{2}f^{(2)}(x).$$

Proof. Relation (3.18) results from Theorem 1 for $s = 2$, Lemma 1 and Lemma 2. From (3.18), it results (3.19).

Remark. Theorem 4 is a Voronovskaja's type theorem.

References

[1] Agratini, O., *An asymptotic formula for a class of approximation processes of King's type*, Studia Sci. Math. Hungar., **47** (2010), No. 4, 435–444

[2] Agratini, O., *On a class of linear positive bivariate operators of King type*, Studia Univ. "Babeş-Bolyai", Matematica, **LI** (2006), No. 4, 13–22

[3] Bărbosu, D., *Introduction in numerical analysis and approximation theory*, Ed. Univ. de Nord Baia Mare, 2009 (in Romanian)

[4] Cárdenas-Morales, D., Garrancho, P., Munos-Delgado, F. J. *Shape preserving approximation by Bernstein-type operators which fix polinomials*, Appl. Math. Comput. **182** (2006), 1615–1622

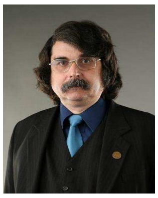
[5] Gonska, H., Piţul, P., *Remarks on a article of J.P. King*, Comment. Math. Univ. Carolin., **46** (2005), No. 4, 645–652

[6] King, J. P., *Positive linear operators which preserve x^2*, Acta Math. Hungar., **99** (2003), No. 3, 203–208

[7] Özarslan, M. A. and Duman, O., *A new approach in obtaining a better estimation in approximation by positive linear operators* Commun Fac. Sci. Univ. Ank. Sér. A1 Math. Stat., **58** (2009), No. 1, 17–22

[8] Pop, O. T., *The generalization of Voronovskaja's theorem for a class of linear and positive operators*, Rev. Anal. Numer. Théor. Approx., **34** (2005), No. 1, 79–91

Petru Braica is a professor of mathematics at the "Grigore Moisil" Secondary School in Satu Mare, Romania and postgraduate at North University of Baia Mare. His research field consists of the uniform approximation by linear positive operators. Recently, he researched uniform generalized approximation operators in terms of fundamental polynomials with fixed test functions.



Ovidiu Pop is a professor of mathematics in the Department of Mathematics of the National College "Mihai Eminescu" in Satu Mare, Romania. In 2006, he received the Ph.D. in Mathematics at Babeş-Bolyai University of Cluj-Napoca. His research filds include the theory of linear positive operators. He published over 60 scientific papers, 8 having been published in ISI journals.

Adrian Indrea is a professor of mathematics in the Department of Mathematics of the National College "Doamna Stanca" in Satu Mare, Romania. He is interested in the theory of linear positive operators.