Mathematical Sciences Letters
An International Journal

@2013 NSP
Natural Sciences Publishing Cor.

Positive integer powers of certain complex tridiagonal matrices

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Received: 19 Jan. 2012; Revised: 8 May 2012; Accepted: 2 Jul. 2012

Published online: 1 Jan. 2013

Abstract: In this paper, we firstly present a general expression for the entries of the rth $(r \in \mathbb{N})$ power of certain n-square are complex tridiagonal matrix, in terms of the Chebyshev polynomials of the first kind. Secondly, we obtain two complex factorizations for Fibonacci and Pell numbers. We also give some Maple 13 procedures in order to verify our calculations.

Keywords: Tridiagonal matrices; Eigenvalues; Eigenvectors; Jordan's form; Chebyshev polynomials.

1 Introduction

In recent years, computing the integer powers of tridiagonal matrices has been a very popular problem. Rimas investigated positive integer powers of certain tridiagonal matrices of odd and even order depending on the Chebyshev polynomials [1,2]. The authors generalized some papers of Rimas [3-8]. Eigenvalues of certain tridiagonal matrices are investigated in many papers [9-10].

In this paper, we obtain the entries of positive integer powers of an n-square complex tridiagonal matrix of the form

$$B=\begin{bmatrix} a+b & b & & & & \\ b & a & b & & 0 & \\ & b & a & \ddots & & \\ & & \ddots & \ddots & b & \\ & 0 & & b & a & b \\ & & & & b & a+b \end{bmatrix}, \tag{1}$$

where $b \neq 0$ and $a,b \in \mathbb{C}$. We also give complex factorization formulas for the Fibonacci and Pell numbers.

Now, we are beginning with following lemma.

Lemma 1 *[11] Let* $\{H(n),\ n=1,2,\ldots\}$ *be sequence of tridiagonal matrices of the form*

$$H(n)=\begin{bmatrix} h_{1,1} & h_{1,2} & & & \\ h_{2,1} & h_{2,2} & h_{2,3} & 0 & \\ & h_{3,2} & h_{3,3} & \ddots & \\ & 0 & \ddots & \ddots & h_{n-1,n} \\ & & & h_{n,n-1} & h_{n,n} \end{bmatrix}.$$

Then the successive determinants of $H(n)$ are given by the recursive formula:

$$|H(1)|=h_{1,1},$$
$$|H(2)|=h_{1,1}h_{2,2}-h_{1,2}h_{2,1},$$
$$|H(n)|=h_{n,n}|H(n-1)|-h_{n-1,n}h_{n,n-1}|H(n-2)|.$$

Let $\{H^{\dagger}(n),\ n=1,2,\ldots\}$ be sequence of tridiagonal matrices of the form

$$H^{\dagger}(n)=\begin{bmatrix} h_{1,1} & -h_{1,2} & & & \\ -h_{2,1} & h_{2,2} & -h_{2,3} & & \\ & -h_{3,2} & h_{3,3} & \ddots & \\ & & \ddots & \ddots & -h_{n-1,n} \\ & & & -h_{n,n-1} & h_{n,n} \end{bmatrix}.$$

Since the matrices $H(n)$ and $H^{\dagger}(n)$ have the same recursive formula, it can be written that

$$|H(n)|=|H^{\dagger}(n)|. \tag{2}$$

2 Main Results

In this section, we give the eigenvalues and eigenvectors of the matrix B given by (1).

Let U be the following n-square tridiagonal matrix

$$U=\begin{bmatrix} 1 & 1 & & & & \\ 1 & 0 & 1 & & 0 & \\ & 1 & 0 & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & 0 & & 1 & 0 & 1 \\ & & & & 1 & 1 \end{bmatrix}.$$

By using (2), we write the characteristic polynomial of U as the following:

$$|tI-U|=\begin{vmatrix} t-1 & 1 & & & & \\ 1 & t & 1 & & 0 & \\ & 1 & t & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & 1 & t & 1 \\ & & & & 1 & t-1 \end{vmatrix}, \tag{3}$$

and from [2], the eigenvalues of U are

$$t_k = -2\cos\frac{k\pi}{n},\ \text{for } k = 1, 2, \ldots, n \tag{4}$$

where t_k denotes *k*th eigenvalue of the matrix U.

Lemma 2 *Let Q be the following n-square tridiagonal matrix*

$$Q = \begin{bmatrix} a+1 & 1 & & & & \\ 1 & a & 1 & & 0 & \\ & 1 & a & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & 0 & & 1 & a & 1 \\ & & & & 1 & a+1 \end{bmatrix} \tag{5}$$

where $a \in \mathbb{C}$. Then the eigenvalues of Q are

$$\mu_k = a - 2\cos\frac{k\pi}{n},\ \text{for } k = 1, 2, \ldots, n \tag{6}$$

where μ_k denotes kth eigenvalue of the matrix Q.

Proof. Since eigenvalues of Q are the roots of its characteristic polynomial, we can write the characteristic polynomial of Q from (2) to be

$$|\mu I - Q| = \begin{vmatrix} \mu-a-1 & 1 & & & & \\ 1 & \mu-a & 1 & & 0 & \\ & 1 & \mu-a & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & 0 & & 1 & \mu-a & 1 \\ & & & & 1 & \mu-a-1 \end{vmatrix}.$$

Substituting $t = \mu - a$ and taking (3) and (4) into account, we find the eigenvalues of the matrix Q as

$$\mu_k = a - 2\cos\frac{k\pi}{n},\ \text{for } k = 1, 2, \ldots, n.$$

Theorem 3 *Let the matrix B be as in (1). Then the eigenvalues of B are*

$$\lambda_k = a - 2b\cos\frac{k\pi}{n},\ \text{for } k = 1, 2, \ldots, n \tag{7}$$

where λ_k denotes kth eigenvalue of the matrix B and $b \neq 0$.

Proof. We need a relation between the matrices B and Q in order to prove the theorem. Dividing all entries of the matrix B by nonzero b, we get a new *n*-square matrix M normalized the upper and lower sub-diagonals as the following

$$M=\begin{bmatrix} \frac{a}{b}+1 & 1 & & & & \\ 1 & \frac{a}{b} & 1 & & 0 & \\ & 1 & \frac{a}{b} & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & 0 & & 1 & \frac{a}{b} & 1 \\ & & & & 1 & \frac{a}{b}+1 \end{bmatrix}.$$

Taking (5) and (6) into account, we find the eigenvalues of the matrix M to be

$$\frac{a}{b}-2\cos\frac{k\pi}{n},\ \text{for } k=1,2,\dots,n.$$

Since the eigenvalues of the matrix B are just b times the eigenvalues of the matrix M, we get

$$\lambda_k=a-2b\cos\frac{k\pi}{n},\ \text{for } k=1,2,\dots,n,$$

and the proof is completed.

Each eigenvector of the matrix B is the solution of the following homogeneous linear equation system

$$\left(\lambda_j I-B\right)x=0, \tag{8}$$

where λ_j is the *j*th eigenvalue of the matrix B $\left(1\le j\le n\right)$. Solving the set of system (8), we find the eigenvectors of the matrix B as

$$x_{jk}=T_{\frac{2k-1}{2}}\left(\frac{\delta_j}{2}\right)\ \text{for } j,k=1,2,\dots,n, \tag{9}$$

where $T_k(x)$ is the *k*th degree Chebyshev polynomial of the first kind [12]:

$$T_k\left(x\right)=\cos k\left(\arccos x\right),\ -1\le x\le 1,$$

and

$$\delta_j=\frac{\lambda_j-a}{b}.$$

General expression for the entries of B^r

Consider the relation $B^r=NJ^rN^{-1}$, where J is the Jordan's form of B and N is the transforming matrix. In order to get the general expression for the entries of B^r, we firstly find the matrices J and N.

Since all the eigenvalues λ_k $(k=1,2,\dots,n)$ are simple, each eigenvalue λ_k corresponds single Jordan cell $J_i\left(\lambda_k\right)$ in the matrix J. Taking this into account we write down the Jordan's form of the matrix B

$$J=diag\left(\lambda_1,\lambda_2,\dots,\lambda_n\right). \tag{10}$$

Let us find the transforming matrix N and its inverse N^{-1}. Denoting *j*th column of N by N_j, we have $N=\left(N_1,N_2,\dots,N_n\right)$. From (9) we get

$$N_j=\begin{bmatrix} T_{\frac{1}{2}}\left(\frac{\delta_j}{2}\right) \\ T_{\frac{3}{2}}\left(\frac{\delta_j}{2}\right) \\ \vdots \\ T_{\frac{2n-1}{2}}\left(\frac{\delta_j}{2}\right) \end{bmatrix}, \quad \text{for } j=1,2,\ldots,n. \tag{11}$$

By (11), we obtain the transforming matrix N as:

$$N=\begin{bmatrix} T_{\frac{1}{2}}\left(\frac{\delta_1}{2}\right) & T_{\frac{1}{2}}\left(\frac{\delta_2}{2}\right) & \cdots & T_{\frac{1}{2}}\left(\frac{\delta_n}{2}\right) \\ T_{\frac{3}{2}}\left(\frac{\delta_1}{2}\right) & T_{\frac{3}{2}}\left(\frac{\delta_2}{2}\right) & \cdots & T_{\frac{3}{2}}\left(\frac{\delta_n}{2}\right) \\ \vdots & \vdots & \ddots & \vdots \\ T_{\frac{2n-1}{2}}\left(\frac{\delta_1}{2}\right) & T_{\frac{2n-1}{2}}\left(\frac{\delta_2}{2}\right) & \cdots & T_{\frac{2n-1}{2}}\left(\frac{\delta_n}{2}\right) \end{bmatrix}. \tag{12}$$

Denoting the jth column of the inverse matrix N^{-1} by τ_j $\left(N^{-1}=\left(\tau_1,\tau_2,\ldots,\tau_n\right)\right)$, from [1], we get

$$\tau_j=\begin{bmatrix} f_1 T_{\frac{2j-1}{2}}\left(\frac{\delta_1}{2}\right) \\ f_2 T_{\frac{2j-1}{2}}\left(\frac{\delta_2}{2}\right) \\ \vdots \\ f_n T_{\frac{2j-1}{2}}\left(\frac{\delta_n}{2}\right) \end{bmatrix}, \quad \text{for } j=1,2,\ldots,n \tag{13}$$

where

$$f_k=\begin{cases} \frac{2}{n}, & \text{if } k=1,2,\ldots,n-1, \\ \frac{1}{n}, & \text{if } k=n. \end{cases}$$

Taking into account (13), we write down the matrix N^{-1} as

$$N^{-1}=\frac{1}{n}\begin{bmatrix} 2T_{\frac{1}{2}}\left(\frac{\delta_1}{2}\right) & 2T_{\frac{3}{2}}\left(\frac{\delta_1}{2}\right) & \cdots & 2T_{\frac{2n-3}{2}}\left(\frac{\delta_1}{2}\right) & 2T_{\frac{2n-1}{2}}\left(\frac{\delta_1}{2}\right) \\ 2T_{\frac{1}{2}}\left(\frac{\delta_2}{2}\right) & 2T_{\frac{3}{2}}\left(\frac{\delta_2}{2}\right) & \cdots & 2T_{\frac{2n-3}{2}}\left(\frac{\delta_2}{2}\right) & 2T_{\frac{2n-1}{2}}\left(\frac{\delta_2}{2}\right) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2T_{\frac{1}{2}}\left(\frac{\delta_{n-1}}{2}\right) & 2T_{\frac{3}{2}}\left(\frac{\delta_{n-1}}{2}\right) & \cdots & 2T_{\frac{2n-3}{2}}\left(\frac{\delta_{n-1}}{2}\right) & 2T_{\frac{2n-1}{2}}\left(\frac{\delta_{n-1}}{2}\right) \\ T_{\frac{1}{2}}\left(\frac{\delta_n}{2}\right) & T_{\frac{3}{2}}\left(\frac{\delta_n}{2}\right) & \cdots & T_{\frac{2n-3}{2}}\left(\frac{\delta_n}{2}\right) & T_{\frac{2n-1}{2}}\left(\frac{\delta_n}{2}\right) \end{bmatrix}. \tag{14}$$

By combining (10), (12) and (14) and using the equality $B^r=NJ^rN^{-1}$, we compute the rth powers of the matrix B of order n. *(i,j)*th entry of the matrix $B^r=\left[s_{ij}\right]$ can be given as:

$$s_{ij}=\sum_{k=1}^{n} f_k\left(\lambda_k\right)^r T_{\frac{2i-1}{2}}\left(\frac{\delta_k}{2}\right) T_{\frac{2j-1}{2}}\left(\frac{\delta_k}{2}\right) \text{ for } i,j=1,2,\ldots,n,$$

or, by substituting $\delta_k=\frac{\lambda_k-a}{b}$,

$$s_{ij}=\sum_{k=1}^{n} f_k\left(\lambda_k\right)^r T_{\frac{2i-1}{2}}\left(\frac{\lambda_k-a}{2b}\right) T_{\frac{2j-1}{2}}\left(\frac{\lambda_k-a}{2b}\right) \text{ for } i,j=1,2,\ldots,n. \tag{15}$$

3 Numerical Considerations

In this section, we give two examples. One of them is *4*-square real matrix and the other is *3*-square complex matrix. We calculated *3*th and *2*th power of these matrices, respectively. These examples can be verified by using Maple procedure given in Appendix A.

Example 4 *Let* B_1 *be a* 4×4 *real tridiagonal matrix (for* $a=2$ *and* $b=3$*, given in (1)) as in the following:*

$$B_1=\begin{bmatrix}5&3&0&0\\3&2&3&0\\0&3&2&3\\0&0&3&5\end{bmatrix}.$$

3th power of B_1 *is computed as in the following.*

From (7), eigenvalues of the matrix B_1 *can be written fo* $k=1,2,3,4$ *as:*

$$\lambda_k=2-6\cos\frac{k\pi}{4},$$

namely, $\lambda_1=2-3\sqrt{2}$, $\lambda_2=2$, $\lambda_3=2+3\sqrt{2}$ *and* $\lambda_4=8$*. We also write the transforming matrix* N_1*, whose columns consist of eigenvectors of* B_1*, and its inverse as:*

$$N_1=\begin{bmatrix}T_{\frac{1}{2}}\left(\frac{\lambda_1-2}{6}\right)&T_{\frac{1}{2}}\left(\frac{\lambda_2-2}{6}\right)&T_{\frac{1}{2}}\left(\frac{\lambda_3-2}{6}\right)&T_{\frac{1}{2}}\left(\frac{\lambda_4-2}{6}\right)\\T_{\frac{3}{2}}\left(\frac{\lambda_1-2}{6}\right)&T_{\frac{3}{2}}\left(\frac{\lambda_2-2}{6}\right)&T_{\frac{3}{2}}\left(\frac{\lambda_3-2}{6}\right)&T_{\frac{3}{2}}\left(\frac{\lambda_4-2}{6}\right)\\T_{\frac{5}{2}}\left(\frac{\lambda_1-2}{6}\right)&T_{\frac{5}{2}}\left(\frac{\lambda_2-2}{6}\right)&T_{\frac{5}{2}}\left(\frac{\lambda_3-2}{6}\right)&T_{\frac{5}{2}}\left(\frac{\lambda_4-2}{6}\right)\\T_{\frac{7}{2}}\left(\frac{\lambda_1-2}{6}\right)&T_{\frac{7}{2}}\left(\frac{\lambda_2-2}{6}\right)&T_{\frac{7}{2}}\left(\frac{\lambda_3-2}{6}\right)&T_{\frac{7}{2}}\left(\frac{\lambda_4-2}{6}\right)\end{bmatrix}=\frac{1}{2}\begin{bmatrix}z&x&y&1\\-y&-x&z&-1\\y&-x&-z&1\\-z&x&y&-1\end{bmatrix}$$

and

$$N_1^{-1}=\frac{1}{4}\begin{bmatrix}2T_{\frac{1}{2}}\left(\frac{\lambda_1-2}{6}\right)&2T_{\frac{3}{2}}\left(\frac{\lambda_1-2}{6}\right)&2T_{\frac{5}{2}}\left(\frac{\lambda_1-2}{6}\right)&2T_{\frac{7}{2}}\left(\frac{\lambda_1-2}{6}\right)\\2T_{\frac{1}{2}}\left(\frac{\lambda_2-2}{6}\right)&2T_{\frac{3}{2}}\left(\frac{\lambda_2-2}{6}\right)&2T_{\frac{5}{2}}\left(\frac{\lambda_2-2}{6}\right)&2T_{\frac{7}{2}}\left(\frac{\lambda_2-2}{6}\right)\\2T_{\frac{1}{2}}\left(\frac{\lambda_3-2}{6}\right)&2T_{\frac{3}{2}}\left(\frac{\lambda_3-2}{6}\right)&2T_{\frac{5}{2}}\left(\frac{\lambda_3-2}{6}\right)&2T_{\frac{7}{2}}\left(\frac{\lambda_3-2}{6}\right)\\T_{\frac{1}{2}}\left(\frac{\lambda_4-2}{6}\right)&T_{\frac{3}{2}}\left(\frac{\lambda_4-2}{6}\right)&T_{\frac{5}{2}}\left(\frac{\lambda_4-2}{6}\right)&T_{\frac{7}{2}}\left(\frac{\lambda_4-2}{6}\right)\end{bmatrix}=\frac{1}{4}\begin{bmatrix}z&-y&y&-z\\x&-x&-x&x\\y&z&-z&-y\\1&1&1&1\end{bmatrix},$$

Where $x=\sqrt{2}$, $y=2\cos\frac{\pi}{8}$ *and* $z=2\cos\frac{3\pi}{8}$*. Then we get*

$$B_1^3=N_1J_1^3N_1^{-1}=\begin{bmatrix}233&171&81&27\\171&143&117&81\\81&117&143&171\\27&81&171&233\end{bmatrix}.$$

(i,j)th entry of the B_1^3*, can be verified by formula given in (15).*

Example 5 *Let* B_2 *be a* 3×3 *complex tridiagonal matrix (for* $a=1-\mathbf{i}$ *and* $b=5+2\mathbf{i}$*, given in (1)) as in the following:*

$$B_2=\begin{bmatrix} 6+\mathbf{i} & 5+2\mathbf{i} & 0 \\ 5+2\mathbf{i} & 1-\mathbf{i} & 5+2\mathbf{i} \\ 0 & 5+2\mathbf{i} & 6+\mathbf{i} \end{bmatrix}.$$

2th power of B_2 *is computed as in the following.*

From (7), eigenvalues of the matrix B_2 *can be written for* $k=1,2,3$ *as:*

$$\lambda_k=(1-\mathbf{i})-(10+4\mathbf{i})\cos\frac{k\pi}{3},$$

namely, $\lambda_1=-4-3\mathbf{i}$, $\lambda_2=6+\mathbf{i}$ *and* $\lambda_3=11+3\mathbf{i}$. *We also write the transforming matrix* N_2, *whose columns consist of eigenvectors of the* B_2, *and its inverse as:*

$$N_2=\begin{bmatrix} T_{\frac{1}{2}}\left(\frac{\lambda_1-2}{6}\right) & T_{\frac{1}{2}}\left(\frac{\lambda_2-2}{6}\right) & T_{\frac{1}{2}}\left(\frac{\lambda_3-2}{6}\right) \\ T_{\frac{3}{2}}\left(\frac{\lambda_1-2}{6}\right) & T_{\frac{3}{2}}\left(\frac{\lambda_2-2}{6}\right) & T_{\frac{3}{2}}\left(\frac{\lambda_3-2}{6}\right) \\ T_{\frac{5}{2}}\left(\frac{\lambda_1-2}{6}\right) & T_{\frac{5}{2}}\left(\frac{\lambda_2-2}{6}\right) & T_{\frac{5}{2}}\left(\frac{\lambda_3-2}{6}\right) \end{bmatrix}=\frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} & 2 \\ -2 & 0 & 2 \\ 1 & -\sqrt{3} & 2 \end{bmatrix}$$

and

$$N_2^{-1}=\frac{1}{3}\begin{bmatrix} 2T_{\frac{1}{2}}\left(\frac{\lambda_1-2}{6}\right) & 2T_{\frac{3}{2}}\left(\frac{\lambda_1-2}{6}\right) & 2T_{\frac{5}{2}}\left(\frac{\lambda_1-2}{6}\right) \\ 2T_{\frac{1}{2}}\left(\frac{\lambda_2-2}{6}\right) & 2T_{\frac{3}{2}}\left(\frac{\lambda_2-2}{6}\right) & 2T_{\frac{5}{2}}\left(\frac{\lambda_2-2}{6}\right) \\ T_{\frac{1}{2}}\left(\frac{\lambda_3-2}{6}\right) & T_{\frac{3}{2}}\left(\frac{\lambda_3-2}{6}\right) & T_{\frac{5}{2}}\left(\frac{\lambda_3-2}{6}\right) \end{bmatrix}=\frac{1}{3}\begin{bmatrix} 1 & -2 & 1 \\ \sqrt{3} & 0 & -\sqrt{3} \\ 1 & 1 & 1 \end{bmatrix}.$$

Then we get

$$B_2^2=N_2J_2^2N_2^{-1}=\begin{bmatrix} 56+32\mathbf{i} & 35+14\mathbf{i} & 21+20\mathbf{i} \\ 35+14\mathbf{i} & 42+38\mathbf{i} & 35+14\mathbf{i} \\ 21+20\mathbf{i} & 35+14\mathbf{i} & 56+32\mathbf{i} \end{bmatrix}.$$

(i,j)th entry of the B_2, *can be verified by formula given in (15).*

4. Complex Factorization of Fibonacci and Pell Numbers

In this section we find two complex factorization formulas for the Fibonacci and Pell numbers in terms of determinant of the matrix B given in (1). Calculations given in this section can be verified by using Maple 13 procedures given in Appendix B.

In [11], authors obtained that

$$\left|tridiag_n(\mathbf{i},1,\mathbf{i})\right|=F_{n+1}. \tag{16}$$

In [13], authors obtained that

$$m\left|tridiag_n(1,2\mathbf{i},1)\right|=P_{n+1} \tag{17}$$

where

$$m=\begin{cases} 1 & n\equiv 0\,(\mathrm{mod}\,4), \\ -\mathbf{i} & n\equiv 1\,(\mathrm{mod}\,4), \\ -1 & n\equiv 2\,(\mathrm{mod}\,4), \\ \mathbf{i} & n\equiv 3\,(\mathrm{mod}\,4). \end{cases}$$

Equality (17) can be written as

$$\left|tridiag_n\left(\mathbf{i},2,\mathbf{i}\right)\right| = P_{n+1}. \tag{18}$$

Theorem 6 *Let* B *be n-square matrix as in (1). Then*

$$\det(B) = \begin{cases} (1+2\mathbf{i})F_n & \text{if } a=1 \text{ and } b=\mathbf{i}, \\ (2+2\mathbf{i})P_n & \text{if } a=2 \text{ and } b=\mathbf{i}, \end{cases} \tag{19}$$

where F_n *and* P_n *denote nth Fibonacci and Pell numbers.*

Proof. By applying the Laplace expansion according to the first and last rows of B, we get

$$\begin{aligned}|B| &= (a+b)^2\left|tridiag_{n-2}(b,a,b)\right|. \\ &\quad -2b^2(a+b)\left|tridiag_{n-3}(b,a,b)\right| + b^4\left|tridiag_{n-4}(b,a,b)\right|.\end{aligned} \tag{20}$$

If we choose $a=1$ and $b=\mathbf{i}$ in (20), and take (16) into account, we write

$$\begin{aligned}\det(B) &= (1+\mathbf{i})^2\left|tridiag_{n-2}(\mathbf{i},1,\mathbf{i})\right| + 2(1+\mathbf{i})\left|tridiag_{n-3}(\mathbf{i},1,\mathbf{i})\right| + \left|tridiag_{n-4}(\mathbf{i},1,\mathbf{i})\right| \\ &= (1+\mathbf{i})^2F_{n-1} + 2(1+\mathbf{i})F_{n-2} + F_{n-3} \\ &= (1+2\mathbf{i})F_n.\end{aligned}$$

If we also choose $a=2$ and $b=\mathbf{i}$ in (20), and take (18) into account, we write

$$\begin{aligned}\det(B) &= (2+\mathbf{i})^2\left|tridiag_{n-2}(\mathbf{i},2,\mathbf{i})\right| + 2(2+\mathbf{i})\left|tridiag_{n-3}(\mathbf{i},2,\mathbf{i})\right| + \left|tridiag_{n-4}(\mathbf{i},2,\mathbf{i})\right| \\ &= (2+\mathbf{i})^2P_{n-1} + 2(2+\mathbf{i})P_{n-2} + P_{n-3} \\ &= (2+2\mathbf{i})P_n,\end{aligned}$$

thus, the proof completes.

Conclusion 7 *Let n-square matrix* B *be as in (1). Then, complex factorizations of the Fibonacci and Pell numbers are*

$$F_n = \prod_{k=1}^{n-1}\left[1-2\mathbf{i}\cos\frac{k\pi}{n}\right]$$

and

$$P_n = \prod_{k=1}^{n-1}\left[2-2\mathbf{i}\cos\frac{k\pi}{n}\right].$$

Proof. Since eigenvalues of B for $k=1,2,\ldots,n$ are

$$\lambda_k = a-2b\cos\frac{k\pi}{n}$$

from (7), determinant of B can be written as

$$\det(B) = \prod_{k=1}^{n}\left[a-2b\cos\frac{k\pi}{n}\right].$$

By using (190), we write

$$(1+2\mathbf{i})F_n = \prod_{k=1}^{n}\left[1-2\mathbf{i}\cos\frac{k\pi}{n}\right]$$

and

$$(2+2\mathbf{i})P_n = \prod_{k=1}^{n}\left[2-2\mathbf{i}\cos\frac{k\pi}{n}\right].$$

So, we obtain

$$F_n = \frac{1}{1+2\mathbf{i}}\prod_{k=1}^{n}\left[1-2\mathbf{i}\cos\frac{k\pi}{n}\right]$$
$$= \prod_{k=1}^{n-1}\left[1-2\mathbf{i}\cos\frac{k\pi}{n}\right]$$

as in [11] and

$$P_n = \frac{1}{2+2\mathbf{i}}\prod_{k=1}^{n}\left[2-2\mathbf{i}\cos\frac{k\pi}{n}\right]$$
$$= \prod_{k=1}^{n-1}\left[2-2\mathbf{i}\cos\frac{k\pi}{n}\right].$$

Thus, proof is completed.

Appendix A. Following Maple 13 procedure calculates the rth power of $n-$ square complex tridiagonal matrix given in (1) and (i,j) *th* entry of B^r for $b \neq 0$.

```
>restart:
with(LinearAlgebra):
power:=proc(n,r,a,b,i,j)
local c,s,B,f, lambda, delta, T, M;
c:=(i,j)->piecewise(i=1 and j=1,a+b,i=n and j=n,a+b,i=j,a,abs(i-j)=1,b,0);
B:=Matrix(n,n,c):
lambda:=(k)->a-2*b*cos(k*Pi/n);
delta:=(j)->((lambda(j)-a)/b);
T:=(k,x)->evalf(cos(k*arccos(x)));
f:=(k)->piecewise(k=n,1/n,2/n);
s:=(i,j)->sum(f(k)*(lambda(k))^r*T((2*i-1)/2,delta(k)/2)*T((2*j-1)/2,delta(k)/2),k=1..n);
M:=Matrix(n,n,s);
print(M);
print(s(i,j));
end proc:
power( , , , , , );
```

Appendix B. (i) Following Maple 13 procedure calculates $n-$ square matrix B given in (1) for $a = 1$ and $b = \mathbf{i}$, determinant of B and complex factorization formula for Fibonacci numbers given in Conclusion 7.

```
>restart:
with(LinearAlgebra):
F:=proc(n)
local c,B,Factorization;
c:=(i,j)->piecewise(i=1 and j=1,1+I,i=n and j=n,1+I,i=j,1,abs(i-j)=1,I,0);
B:=Matrix(n,n,c):
Factorization:=(1/(1+2*I))*product(1-2*I*cos(k*Pi/n),k=1..n);
print(B);
print(Determinant(B));
print(evalf(Factorization));
end proc:
F( );
```

(ii) Following Maple 13 procedure calculates $n-$ square matrix B given in (1) for $a=2$ and $b=\mathbf{i}$, determinant of B and complex factorization formula for Pell numbers given in Conclusion 8.

```
>restart:
with(LinearAlgebra):
P:=proc(n)
local c,B,Factorization;
c:=(i,j)->piecewise(i=1 and j=1,2+I,i=n and j=n,2+I,i=j,2,abs(i-j)=1,I,0);
B:=Matrix(n,n,c):
Factorization:=(1/(2+2*I))*product(2-2*I*cos(k*Pi/n),k=1..n);
print(B);
print(Determinant(B));
print(evalf(Factorization));
end proc:
P( );
```

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