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Laurent Expansion of q-Zeta Functions

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Abstract: The main purpose of this paper is to establish a Laurent expansion of Hurwitz and Riemann q-zeta functions by means of a slightly different version of the Euler-Maclaurin summation formula from what is usually given. Further results are also given.

Keywords: Hurwitz and Riemann q-zeta functions; Laurent expansion; Stieltjes constants.
Mathematics Subject Classification (2010):11M41; 05A30; 41A58.

1 Introduction

The Hurwitz zeta-function $\zeta(s,a)$ is defined by

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \tag{1.1}$$

where $\sigma = \Re(s) > 1$ and a is any complex number [1]. (Normally, it is assumed that $0 < a \leq 1$ in the definition of $\zeta(s,a)$). Observe that $\zeta(s,1) = \zeta(s)$, where $\zeta(s)$ denotes the Riemann zeta-function. The function $\zeta(s,a)$ can be analytically continued into the entire complex s-plane and is holomorphic except for a simple pole at $s = 1$ with residue 1. In a neighbourhood of $s = 1, \zeta(s,a)$ has the Laurent expansion in powers of $s-1$

$$\zeta(s,a) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n(a)}{n!}(s-1)^n \tag{1.2}$$

where the quantities $\gamma_n(a)$ are known as the generalized Stieltjes constants which are given by the limit [2]

$$\gamma_n(a) = \lim_{m\to\infty} \left[\sum_{k=0}^{m} \frac{\log^n(k+a)}{k+a} - \frac{\log^{n+1}(m+a)}{n+1} \right] \tag{1.3}$$

and when $n = 0$, we have $\gamma_0(a) = -\psi(a)$ where $\psi(a)$ is the digamma function. When $a = 1$ the expansion (1.2) reduces to the Laurent expansion of the Riemann zeta function

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{n!}(s-1)^n \tag{1.4}$$

where $\gamma_n = \gamma_n(1)$ is the (ordinary) Stieltjes constants.

Tsumura [3] defined q-analogue of the Hurwitz zeta function for a complex number s and $0 < a \leq 1, 0 < q < 1$ as

$$\zeta_q(s,a) = \sum_{n=0}^{\infty} \frac{q^{-n-1}}{(q^{-n-1}[n]_q + aq^{-1})^s} \tag{1.5}$$

where $[x]_q = (1-q^x)/(1-q)$. When $a = 1$, $\zeta_q(s,a)$ reduces to the Riernann q-zeta function

$$\zeta_q(s) = \sum_{n=1}^{\infty} \frac{q^{-n}}{(q^{-n}[n]_q)^s} \tag{1.6}$$

the right-hand side of this series converges when $\Re(s) > 1$. Tsumura proved that the series $\zeta_q(s)$ converges when $\Re(s) > 1$. $\zeta_q(s)$ may be analytically continued to the whole complex plane, except for a simple pole at $s = 1$ with residue $(q-1)/\log q$ (see also [4,5]). He defined q-zeta functions (1.5) and (1.6) for $0 < q < 1$, but we see that the two series in the right hand sides of (1.5) and (1.6) converge absolutely for $|q| < 1$ and thus we will consider $|q| < 1$ throughout this paper.

In this paper, we are seeking to establish a Laurent expansion of Hurwitz and Riemann q-zeta functions by means of a slightly different version of the Euler-Maclaurin summation formula from what is usually given. Let ℓ is a natural number and $f(t)$ is a smooth function defined for all real numbers t between α and ℓ, then

$$\textstyle\sum_{\alpha<k\leq n} f(k) = \int_\alpha^n f(t)dt + \sum_{k=1}^{\ell}(-1)^k \frac{B_k}{k!} f^{(k-1)}(n) + \sum_{k=1}^{\ell}(-1)^{k+1} P_k(\alpha) f^{(k-1)}(\alpha) + R_\ell \tag{1.7}$$

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where the reminder term R_ℓ is defined as

$$R_\ell = (-1)^{\ell+1} \int_\alpha^n P_\ell(t) f^{(\ell)}(t) dt$$

Here, B_k denotes the kth Bernoulli number, and $P_k(t)$ is the kth periodic Bernoulli function defined by

$$P_k(t) = \frac{1}{k!} B_k(\{t\}) \quad \text{with} \quad P_1(t) = B_1(\{t\}) = \{t\} - \frac{1}{2}$$

where $B_k(t)$ is the kth Bernoulli polynomial and $\{t\}$ is the fractional part of the real number t.

2 Integral representation of q-zeta functions

In [2], Berndt introduced some representations for the Hurwitz zeta function. He gave integral representation of the Hurwitz zeta function $\zeta(s,a)$ as follows

$$\zeta(s,a) = \frac{a^{1-s}}{s-1} + \frac{a^{-s}}{2} - s \int_0^\infty \frac{\{t\} - 1/2}{(t+a)^{s+1}} dt \tag{2.1}$$

He also gave classical formulae related to Hurwitz zeta function and Lerch's expression $\ln \Gamma(a) = \zeta'(0,a) - \zeta'(0)$. In the present section, we are seeking to establish integral representations of q-zeta functions which tend to the Berndt's results.

Theorem 2.1.
For $0 < a \le 1, m \le \alpha < m+1; m = 0,1,2,\cdots$ and $\sigma > 1$, the Hurwitz q-zeta function (1.5) can be represented in the integral form as

$$\begin{aligned}\zeta_q(s,a) &= \sum_{k=0}^{m} \frac{q^{(k+1)(s-1)}}{([k]_q + aq^k)^s} + \frac{q-1}{\log q} \frac{1}{s-1} \frac{q^{(\alpha+1)(s-1)}}{([\alpha]_q + aq^\alpha)^{s-1}} + \frac{q^{(\alpha+1)(s-1)}}{([\alpha]_q + aq^\alpha)^s} \left(\{\alpha\} - \frac{1}{2} \right) \\ &\quad + \int_\alpha^\infty \frac{q^{-2(t+1)}(\{t\} - 1/2)((a-aq-1)q^t + 1 - s)}{(q^{-t-1}[t]_q + aq^{-1})^{s+1}} dt \end{aligned} \tag{2.2}$$

In particular, when $a = 1$, the Riemann q-zeta function has the integral form

$$\begin{aligned}\zeta_q(s) &= \sum_{k=0}^{m} \frac{q^{(k+1)(s-1)}}{[k+1]_q^s} + \frac{q-1}{\log q} \frac{1}{s-1} \frac{q^{(\alpha+1)(s-1)}}{[\alpha+1]_q^{s-1}} + \frac{q^{(\alpha+1)(s-1)}}{[\alpha+1]_q^s} \left(\{\alpha\} - \frac{1}{2} \right) \\ &\quad + \int_{\alpha+1}^\infty \frac{q^{-2t}(\{t\} - 1/2)(1 - s - q^t)}{(q^{-t}[t]_q)^{s+1}} dt \end{aligned} \tag{2.3}$$

Proof. Let the smooth function

$$f(t) = \frac{q^{-t-1}}{(q^{-t-1}[t]_q + aq^{-1})^s} \tag{2.4}$$

and $\ell = 1$ in Euler-Maclaurin summation formula (1.7). Then

$$\sum_{k=m+1}^{n} f(k) = \int_\alpha^n f(t) dt - B_1 f(n) + P_1(\alpha) f(\alpha) + \int_\alpha^n P_1(t) f'(t) dt$$

If $\sigma > 1$, upon letting $n \to \infty$ and after some simple calculations, we obtain the integral representation (2.2). The integral representation (2.3) can be obtained by putting $a = 1$ in (2.2) with changing the variable of the integral from $t+1$ to t. This ends the proof.

The integral representation (2.2) of Hurwitz q-zeta function can be rewritten in different forms by assigning various values of α. Some of these integral representations are listed below

I) For $0 < a \le 1, \alpha = 1 - a$ and $\sigma > 1$, the Hurwitz q-zeta function (1.5) can be represented in the integral form as

$$\begin{aligned}\zeta_q(s,[a]_q) &= \frac{q^{s-1}}{[a]_q^s} + \frac{q-1}{\log q} \frac{q^{-a(s-1)}}{s-1} + q^{-a(s-1)} \left(\{1-a\} - \frac{1}{2} \right) \\ &\quad + \int_{1-a}^\infty \frac{q^{-2(t+1)}(\{t\} - 1/2)(1 - s - q^{t+a})}{(q^{-t-1}[t+a]_q)^{s+1}} dt \end{aligned} \tag{2.5}$$

which appears in Berndt [[2](5.2),] as $q \to 1$ and if we put $a = 1$, we return to (2.3) when $\alpha = 0$.

II) For $0 < a \le 1, \alpha = m = 0$ and $\sigma > 1$, the Hurwitz q-zeta function (1.5) can be represented in the integral form as

$$\zeta_q(s,a) = \frac{q^{s-1}}{2\alpha^s} + \frac{q-1}{\log q} \frac{1}{s-1} \frac{q^{s-1}}{a^{s-1}} + \int_0^\infty \frac{q^{-2(t+1)}(\{t\} - 1/2)((a-aq-1)q^t + 1 - s)}{(q^{-t-1}[t]_q + aq^{-1})^{s+1}} dt \tag{2.6}$$

$$= \frac{q^{s-1}}{a^s} + \frac{q-1}{\log q} \frac{1}{s-1} \frac{q^{s-1}}{a^{s-1}} + \int_0^\infty \frac{q^{-2(t+1)}\{t\}((a-aq-1)q^t + 1 - s)}{(q^{-t-1}[t]_q + aq^{-1})^{s+1}} dt \tag{2.7}$$

which tends to (2.1) as $q \to 1$.

3 Laurent expansion

In the present section, we are seeking to provide Laurent expansion of q-zeta functions. To do this, let the integral

$$I(s;q) = \int_0^\infty \frac{q^{-2(t+1)}\{t\}((a-aq-1)q^t + 1 - s)}{(q^{-t-1}[t]_q + aq^{-1})^{s+1}} dt, \quad \sigma > 1 \tag{3.1}$$

which can be rewritten as

$$I(s;q) = \int_0^\infty \frac{q^{(t+1)(s-1)}\{t\}((a-aq-1)q^t + 1 - s)}{([t]_q + aq^t)^{s+1}} dt, \quad \sigma > 1 \tag{3.2}$$

Unfortunately, this integral just converges for $\sigma > 1$ due to exist the factor $q^{t(s-1)}$ in the integral $I(s;q)$ unlike what happens in the classical case where if $q \to 1$, the integral $I(s,1)$ converges by analytic continuation for $\sigma > -1$ (see, Berndt [2]). In contrast to the situation of Berndt, this does not help much and indeed we want to establish an equivalent form to $I(s;q)$ which can be continued analytically.

Lemma 3.1 The integral $I(s;q)$ defined by (3.1) can be expanded in the series

$$\begin{aligned}I(s;q) &= \frac{q-1}{\log q} \frac{1}{s-1} - \frac{q-1}{\log q} \frac{1}{s-1} \frac{q^{s-1}}{a^{s-1}} - \frac{q^{s-1}}{a^s} \\ &\quad + \sum_{n=0}^\infty \frac{(-1)^n (s-1)^n}{n!} \lim_{m \to \infty} \left[\sum_{k=0}^{m} \frac{\log^n(q^{-k-1}[k]_q + aq^{-1})}{[k]_q + aq^k} - \frac{q-1}{\log q} \frac{\log^{n+1}(q^{-m-1}[m]_q + aq^{-1})}{n+1} dt \right] \end{aligned} \tag{3.3}$$

which is continued meromorphically to the whole complex s-plane except at the point $s=1$.

Proof. Let $f(t)$ be the function defined by (2.4). Then

$$\begin{aligned}
I(s;q) &= \int_0^{\infty}\{t\}f'(t)dt = \sum_{k=1}^{\infty}\int_{k-1}^{k}\{t\}f'(t)dt \\
&= \sum_{k=1}^{\infty}\int_{k-1}^{k}(t-k+1)f'(t)dt = \sum_{k=1}^{\infty}\left[f(k)-\int_{k-1}^{k}f(t)dt\right] \\
&= \sum_{k=1}^{\infty}\left[\frac{e^{-(s-1)\log(q^{-k-1}[k]_q+aq^{-1})}}{[k]_q+aq^k}-\int_{k-1}^{k}\frac{e^{-(s-1)\log(q^{-t-1}[t]_q+aq^{-1})}}{[t]_q+aq^t}dt\right] \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n(s-1)^n}{n!}\lim_{m\to\infty}\left[\sum_{k=1}^{m}\frac{\log^n(q^{-k-1}[k]_q+aq^{-1})}{[k]_q+aq^k}-\sum_{k=1}^{m}\int_{k-1}^{k}\frac{\log^n(q^{-t-1}[t]_q+aq^{-1})}{[t]_q+aq^t}dt\right]
\end{aligned}$$

By virtue of convergence of the exponential function everywhere, the summation and limit were permuted in the previous relation. Now we are interested in calculating the inner integral to obtain

$$\begin{aligned}
I(s;q) &= \sum_{n=0}^{\infty}\frac{(-1)^n(s-1)^n}{n!}\left[\frac{q-1}{\log q}\frac{\log^{n+1}(aq^{-1})}{n+1}-\frac{\log^n(aq^{-1})}{a}\right] \\
&+ \sum_{n=0}^{\infty}\frac{(-1)^n(s-1)^n}{n!}\lim_{m\to\infty}\left[\sum_{k=0}^{m}\frac{\log^n(q^{-k-1}[k]_q+aq^{-1})}{[k]_q+aq^k}-\frac{q-1}{\log q}\frac{\log^{n+1}(q^{-m-1}[m]_q+aq^{-1})}{n+1}\right] \\
&= \frac{q-1}{\log q}\frac{1}{s-1}-\frac{q-1}{\log q}\frac{1}{s-1}\frac{q^{s-1}}{a^{s-1}}-\frac{q^{s-1}}{a^s} \\
&+ \sum_{n=0}^{\infty}\frac{(-1)^n(s-1)^n}{n!}\lim_{m\to\infty}\left[\sum_{k=0}^{m}\frac{\log^n(q^{-k-1}[k]_q+aq^{-1})}{[k]_q+aq^k}-\frac{q-1}{\log q}\frac{\log^{n+1}(q^{-m-1}[m]_q+aq^{-1})}{n+1}\right]
\end{aligned}$$

It is obvious that the previous series is continued meromorphically to the whole complex s-plane except at the point $s=1$ and so the integral $I(s;q)$. This completes the proof.

By inserting the relation (3.3) into the relation (2.7) with noting that the Laurent expansion for any function (if exists) is unique, the main theorem in this paper can be formulated as follows

Theorem 3.2. The Hurwitz q-zeta function $\zeta_q(s,a)$ can be continued meromorphically to the whole complex s-plane and that it has a simple pole at $s=1$ with residue $(q-1)/\log q$. Moreover, it can be expanded in the Leaurent expansion as

$$\zeta_q(s,a) = \frac{q-1}{\log q}\frac{1}{s-1}+\sum_{n=0}^{\infty}\frac{(-1)^n(s-1)^n}{n!}\gamma_n(a;q) \quad (3.4)$$

where

$$\gamma_n(a;q) = \lim_{m\to\infty}\left[\sum_{k=0}^{m}\frac{\log^n(q^{-k-1}[k]_q+aq^{-1})}{[k]_q+aq^k}-\frac{q-1}{\log q}\frac{\log^{n+1}(q^{-m-1}[m]_q+aq^{-1})}{n+1}\right] \quad (3.5)$$

Remark 3.3. It is known that the Hurwitz q-zeta function $\zeta_q(s,a)$ tends to the Hurwitz zeta function $\zeta(s,a)$ as $q\to 1$ and thus the expansion (3.4) tends to the expansion (1.4) as $q\to 1$, consequently $\gamma_n(a;q)$ tend to the generalized Stieltjes constants $\gamma_n(a)$ defined by (1.4) for all $n=0,1,2,\cdots$. This means that $\gamma_n(a;q)$ can be said to have q-analogue of the generalized Stieltjes constants $\gamma_n(a)$ for all $n=0,1,2,\cdots$.

Corollary 3.4. The Reimann q-zeta function $\zeta_q(s)$ can be continued meromorphically to the whole complex s-plane and that it has a simple pole at $s=1$ with residue $(q-1)/\log q$. Moreover, it can be expanded in the Laurent expansion as

$$\zeta_q(s) = \frac{q-1}{\log q}\frac{1}{s-1}+\sum_{n=0}^{\infty}\frac{(-1)^n(s-1)^n}{n!}\gamma_n(q) \quad (3.6)$$

where

$$\gamma_n(q) = \gamma_n(1,q) = \lim_{m\to\infty}\left[\sum_{k=1}^{m}\frac{\log^n(q^{-k}[k]_q)}{[k]_q}-\frac{q-1}{\log q}\frac{\log^{n+1}(q^{-m}[m]_q)}{n+1}\right]. \quad (3.7)$$

The q-digamma function $\psi_q(z)$ is defined as the logarithmic derivative of the q-gamma function [6]

$$\psi_q(z) = \frac{d}{dz}(\ln\Gamma_q(z)) = \frac{\Gamma_q'(z)}{\Gamma_q(z)}$$

where $\Gamma_q(z)$ is the q-gamma function defined as

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, \quad z\neq 0,-1,-2,\ldots$$

Therefore, the q-digamma function $\psi_q(z)$ can be expressed as

$$\psi_q(z) = -\ln(1-q)+\frac{\ln q}{1-q}\sum_{n=0}^{\infty}\frac{q^{n+z}}{[n+z]_q}, \quad z\neq 0,-1,-2,\cdots$$

For more details on the q-digamma function, see [7] which presented some properties and expansions associated with the q-digamma function.

Theorem 3.5. For $0<a\leq 1$, we have

$$\gamma_0([a]_q;q) = \frac{1-q}{\log q}\psi_q(a). \quad (3.8)$$

Proof. From (3.5), we have

$$\begin{aligned}
\gamma_0([a]_q;q) &= \lim_{m\to\infty}\left[\sum_{k=0}^{m}\frac{1}{[k]_q+[a]_qq^k}-\frac{q-1}{\log q}\frac{\log(q^{-m-1}[m]_q+[a]_qq^{-1})}{n+1}\right] \\
&= \lim_{m\to\infty}\left[\sum_{k=0}^{m}\frac{1}{[a+k]_q}-\frac{q-1}{\log q}\log(q^{-m-1}[a+m]_q)\right] \\
&= \lim_{m\to\infty}\left[\sum_{k=0}^{m}\frac{1-q^{a+k}+q^{a+k}}{[a+k]_q}-(1-q)(m+1)-\frac{q-1}{\log q}\log([a+m]_q)\right] \\
&= \lim_{m\to\infty}\left[\sum_{k=0}^{m}\frac{q^{a+k}}{[a+k]_q}-\frac{q-1}{\log q}\log([a+m]_q)\right] \\
&= \sum_{k=0}^{\infty}\frac{q^{a+k}}{[a+k]_q}-\frac{1-q}{\log q}\log(1-q) = \frac{1-q}{\log q}\psi_q(a).
\end{aligned}$$

This ends the proof.

From the Laurent expansion of the Hurwitz q-zeta function (3.4) and the relation (3.8), we can deduce that

$$\lim_{s\to 1}\left[\zeta_q(s,[a]_q)-\frac{q-1}{\log q}\frac{1}{s-1}\right] = \gamma_0([a]_q;q) = \frac{1-q}{\log q}\psi_q(a). \quad (3.9)$$

In particular, when $a=1$, we get

$$\lim_{s\to 1}\left[\zeta_q(s)-\frac{q-1}{\log q}\frac{1}{s-1}\right] = \gamma_0(q) = \frac{1-q}{\log q}\psi_q(1) = \gamma_q \quad (3.10)$$

where γ_q is the q-analogue of the Euler-Mascheroni constant defined by

$$\gamma_q = \sum_{k=1}^{\infty} \frac{q^k}{[k]_q} - \frac{1-q}{\ln q} \ln(1-q) = \lim_{m\to\infty} \left[\sum_{k=1}^{m} \frac{q^k}{[k]_q} - \frac{q-1}{\ln q} \ln[m]_q \right]. \tag{3.11}$$

which tends to the Euler-Mascheroni constant γ as $q \to 1$ [8].

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