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A Reliable Algorithm for Solving Tenth-order Boundary Value Problems

Muhammad Aslam Noor[1,2], *Eisa Al-Said*[1] *and Syed T. Mohyud-Din*[3]

[1] Mathematics Department, COMSATS Institute of Information Technology, Pakistan
[2] IMathematics Department, College of Science, King Saud University, Riyadh, KSA
[3] Basic Sciences, HITEC University, Taxila Cantt, Islamabad, Pakistan

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Abstract: In this paper, we use the variational iterative method for solving the tenth-order boundary value problems. The analytical solutions of the equations have been obtained in terms of convergent series with easily computable components. Several examples are given to check the efficiency of the proposed algorithm. The fact that variational iteration method solves nonlinear problems without using the Adomian's polynomials is a clear advantage of this technique over the decomposition method.

Keywords: Variational iterative method, Boundary value Problems, Iterative schemes, Approximate solution.

1. Introduction

In this paper, we consider the general tenth-order boundary value problem of the type:

$$y^{(iv)} + f(x)y(x) = g(x),$$
$$x \in [a, b]$$

with boundary conditions

$$y(a) = \alpha_0, \quad y''(a) = \epsilon_0,$$
$$y^{(iv)} = (a) = \xi_0,$$
$$Y^{(vi)}(a) = \sigma_0, \quad y^{(viii)} = \eta_0$$
$$y(b) = \alpha_1, \quad y''(b) = \epsilon_1,$$
$$y^{(iv)}(b) = \xi_0,$$
$$y^{(vi)}(b) = \sigma_1, \quad y^{(viii)}(b) = \eta_1.$$

Here the functions $f(x)$ $g(x)$ are continuous functions and $\alpha_0, \alpha_1, \epsilon_0, \epsilon_1, \xi_0, \xi_1, \sigma_0, \sigma_2, \eta_0, \eta_1$ are constants. This type of higher-order boundary value arises in the mathematical modelling of the viscoelastic flows and other branches of mathematical, physical and engineering sciences. It is well known that when a layer of fluid is heated from below and is subject to the action of rotation, instability may set in as ordinary convection which may be modeled by a tenth-order boundary value problem, see [2-5,12-14,23,24] and the references therein. Due to their importance, much attention has been given to develop several numerical method for solving these boundary value problems.

These numerical and analytical techniques including finite-difference, polynomial and non-polynomial spline, homotopy perturbation, variational of parameters, variational iterative methods and decomposition have been employed for solving such problems, see [2-12,18-24] and the references therein. Some of these techniques have their inbuilt deficiencies, like divergence of the results at the points adjacent to the boundary and calculation of the so-called Adomian's polynomials. Moreover, the performance of most of the finite difference methods used so far is well known that they provide the solution at grid points only.

Noor and Mohyud-Din [12,18-21] employed homotopy perturbation method and a new technique variational iteration decomposition method (which is an elegant combination of variational iteration method and the decomposition method) for solving higher-order boundary value problems. Inspired and motivated by the ongoing research in this area, we apply the variational iteration method for solving the tenth-order boundary value problems. It is worth mentioning that our proposed technique can handle

* Corresponding author: e-mail: noormaslam@hotmail.com , syedtauseefs@hotmail.com

any boundary value problem with a set of boundary conditions defined at any order derivatives.

He [6-10] developed the variational iteration method for solving linear, nonlinear, initial and boundary value problems. It is worth mentioning that the origin of variational iteration method is traced back to Inokuti, Sekine and Mura [11]. The basic motivation of this paper is to apply the variational iteration method to solve the linear and nonlinear tenth-order boundary value problems. It is shown that the variational iteration method provides the solution in a rapid convergent series with easily computable components. We write the correct functional for the tenth-order boundary value problem and calculate the Lagrange multiplier optimally via variational theory.

The use of Lagrange multiplier reduces the successive application of the integral operator and minimizes the computational work. Moreover, the selection of the initial value is done by introducing an essential modification which increases the efficiency of the proposed algorithm. The VIM solves effectively, easily and accurately a large class of linear, nonlinear, partial, deterministic or stochastic differential equations with approximate solutions which converge very rapidly to accurate solutions. Several examples are given to illustrate the reliability and performance of the proposed method.

2. Variational Iterative Method

To illustrate the basic concept of the technique, we consider the following general differential equation L(u)+N(u)=g(x), where where L is a linear operator, N a nonlinear operator and $g(x)$ is the inhomogeneous term. According to variational iteration method [6-12, 18-21], we can construct a correct functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda\left(L(u_n(s)) + N(\tilde{u_n}(s)) - g(s)\right) ds,$$

where λ is a Lagrange multiplier [6-12, 18-21], which can be identified optimally via variational iteration method. The subscripts n denotes the nth approximation, $\tilde{u_n}$ is considered as a restricted variation, that is, $\delta\tilde{u_n} = 0$, is called as a correct functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in [6-11]. In this method, it is required first to determine the Lagrange multiplier λ optimally via variational theory. The successive approximation $u_{n+1}, \quad n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0, consequently, the solution is given by u=$\lim_{n\longrightarrow\infty} u_n$.

3. Numerical Applications

In this section, we apply the variational iteration method developed by He for solving the tenth-order boundary value problems. We introduce a slight modification in the selection of the initial value which increases the efficiency of the proposed iterative scheme. For the sake of comparison, we take the same examples as in [23,24].

Example 3.1 [23,24]. Consider the following nonlinear boundary value problems of tenth-order $\mathrm{y}^{(x)}(x) = e^{-x}y^2(x), \quad 0 < x < 1$,with boundary condition

$$\begin{aligned} &y(0) = 1, \quad y''(0) = y^{(iv)}(0) \\ &= y^{vi)}(0) = y^{(viii)}(0) = 1, \\ &y(1) = e, \quad y''(1) = y^{(iv)}(1) \\ &= y^{vi)}(1) = y^{(viii)}(0) = e. \end{aligned}$$

The exact solution of this problem is $y(x) = e^x$. and the correct functional is given as $\mathrm{y}_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)\left[\frac{d^{10}y_n}{dx^{10}} - e^{-x}\tilde{y}_n^2(x)\right] ds$.Making the correct functional stationary and using $\lambda = \frac{1}{9!}(s-x)^9$, as the Lagrange multiplier [12], we have the following iterative scheme

$$\begin{aligned} y_{n+1}(x) &= y_n(x) + \int_0^x \frac{1}{9!}(s-x)^9 \\ &\quad \left[\frac{d^{10}y_n}{dx^{10}} - e^{-x}y_n^2 9x)\right] ds \\ &= 1 + Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 \\ &\quad + \frac{1}{4!}x^4 + \frac{C}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}Dx^7 \\ &\quad + \frac{1}{8!}x^8 + \frac{1}{9!}Ex^9 \\ &\quad + \int_0^x \frac{1}{9!}(s-x)^9\left[\frac{d^{10}y_n}{dx^{10}} - e^{-x}y_n^2\right] ds, \end{aligned}$$

where

$$\begin{aligned} &A = y'(0), \quad B = y'''(0), \\ &C = y^{(v)}(0), \quad D = y^{(vii)}(0), \\ &E = y^{(ix)}(0). \end{aligned}$$

Consequently, we obtain the following approximants:

$$y_0(x) = 1,$$

$$\begin{aligned} y_1(x) &= 1 + Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 \\ &\quad + \frac{1}{4!}x^4 + \frac{C}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}Dx^7 \\ &\quad + \frac{1}{8!}x^8 + \frac{1}{9!}Ex^9 + \frac{1}{10!}x^{10} + \ldots, \end{aligned}$$

$$\begin{aligned} y_2(x) &= 1 + Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 \\ &\quad + \frac{1}{4!}x^4 + \frac{C}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}Dx^7 \\ &\quad + \frac{1}{8!}x^8 + \frac{1}{9!}Ex^9 + \frac{1}{10!}x^{10} \\ &\quad + \frac{1}{11!}x^{11} + \frac{1}{12!}x^{12} + \frac{2}{11!}Ax^{11} \end{aligned}$$

$$+\left(-\frac{4}{12!}A+\frac{1}{239500800}\right)x^{12}+\ldots,$$

$$\vdots$$

The series solution is given as:

$$\begin{aligned}
y(x) &= 1+Ax+\frac{1}{2!}x^2+\frac{1}{3!}Bx^3 \\
&+\frac{1}{4!}x^4+\frac{1}{5!}Cx^5+\frac{1}{6!}x6^{+}\frac{1}{7!}Dx^7 \\
&+\frac{1}{8!}x^8+\frac{1}{9!}Ex^9+\frac{1}{10!}D^{10} \\
&+\left(-\frac{1}{19958400}A+\frac{1}{39916800}\right)x^{11} \\
&+\left(-\frac{1}{119750400}A+\frac{1}{159667200}\right)x^{12} \\
&+O(x^{13}).
\end{aligned}$$

Using the boundary conditions at $x=1$, we obatin

$A=1.00001436, \quad B=0.999858964,$
$C=1.001365775$
$D=0.987457318, \quad E=1.0932797434$

From the above relations, we obtain the series solution of the nonlinear tenth-order boundary value problem as:

$$\begin{aligned}
y(x) &= 1+1.00001436+\frac{1}{2!}x^2+0.1666431607x^3+\frac{1}{4!}x^4 \\
&+0.008344714791x^5+\frac{1}{6!}x^6+0.00019524071x^7+\frac{1}{8!}x^8+\ldots,
\end{aligned}$$

which is in full agreement with [23].

Example 3.2 [23,24]. Consider the following tenth-order linear boundary value problem of the type: $y^{(x)}(x)=-8e^x+y''(x), \quad 0<x<1.$with boundary condition

$$\begin{aligned}
&y(0)=1, \quad y'(0)=0, \quad y''(0)=-1, \\
&y''(0)=-2, \quad y^{(iv)}(0)=-3 \\
&y(1)=0, \quad y'(1)=-e, \\
&y''(1)=-2e, \quad y'''(1)=-3e, \\
&y^{(iv)}(1)=-4e.
\end{aligned}$$

The exact solution of this problem is $y(x)=(1-x)e^x$. and the correct functional is given as

$$\begin{aligned}
y_{n+1}(x) &= y_n(x) \\
&+\int_0^x \lambda(s)\left[\frac{d^{10}y_n}{dx^{10}}-(-8x^x+\tilde{y}_n''(x))\right]ds.
\end{aligned}$$

Making the correct functional stationary and using $\lambda=\frac{1}{9!}(s-x)^9$, as the Lagrange multiplier [12], we have the following iterative scheme

$$\begin{aligned}
y_{n+1}(x) &= y_n(x)+\int_0^x \frac{1}{9!}(s-x)^9 \\
&\left[\frac{d^{10}y_n}{dx^{10}}-(-8e^x+y_n''(x))\right]ds \\
&= 1-\frac{1}{2!}x^2-\frac{1}{3}x^3-\frac{1}{4!}x^4+\frac{1}{5!}Ax^5 \\
&+\frac{1}{6!}Bx^6+\frac{1}{7!}Cx^7+\frac{1}{8!}Dx^8 \\
&+\frac{1}{9!}Ex^9+\int_0^x \frac{1}{9!}(s-x)^9 \\
&\left(\frac{d^{10}y_n}{dx^{10}}-(-8e^x+y_n''(x)\right)ds,
\end{aligned}$$

where

$A=y^{(v)}(0), \quad B=y^{(vi)}(0),$
$C=y^{(vii)}(0), \quad D=y(viii)(0),$
$E=y^{(ix)}(0).$

Consequently, we obtain the following approximants:

$$\begin{aligned}
y_0(x) &= 1, \\
y_1(x) &= 1-8e^x+8+8x+\frac{7}{2!}x^2 \\
&+x^3+\frac{5}{4!}x^4+\left(\frac{1}{15}+\frac{1}{5!}A\right)x^5 \\
&+\left(\frac{1}{90}+\frac{1}{6!}Bx^6\right)+\left(\frac{1}{630}+\frac{1}{7!}Cx^7\right) \\
&+\left(\frac{1}{7!}+\frac{1}{8!}D\right)x^8+\left(\frac{1}{45360}+\frac{1}{9!}E\right)x^9 \\
y_2(x) &= 1-16e^x+16+16x+\frac{15}{2}x^2 \\
&+\frac{7}{3}x^3+\frac{13}{4!}x^4+\left(\frac{2}{15}+\frac{1}{5!}A\right)x^5 \\
&+\left(\frac{1}{45}+\frac{1}{6!}B\right)x^6+\left(\frac{1}{315}+\frac{1}{7!}C\right)x^7 \\
&+\left(\frac{2}{7!}+\frac{1}{8!}D\right)x^8 \\
&+\left(\frac{2}{45360}+\frac{E}{9!}\right)x^9+\frac{1}{518400}x^{10}
\end{aligned}$$

$+1\frac{}{6652800x^{11}+\frac{1}{95800320}x^{12}=\ldots}$The series solution of the problem is given by

$$\begin{aligned}
y(x) &= 17-16e^x+16x=\frac{15}{2!}x^2 \\
&+\frac{7}{3}x^3+\frac{13}{4!}x^4+\left(\frac{2}{15}+\frac{1}{15!}\right)x^5 \\
&+\left(\frac{1}{45}+\frac{B}{6!}\right)x^6+\left(\frac{1}{315}+\frac{C}{7!}\right)x^7 \\
&+\left(\frac{2}{7!}+\frac{D}{8!}\right)x^8+\frac{1}{9!}(8+E)x^9+\ldots,
\end{aligned}$$

Now using the boundary conditions at $x=1$, we have

$A=-4, \quad B=-5, \quad C=-6,$
$D=-7, \quad E=-8.$

Consequently, the series solution is given by

$$y(x) = 1 - \frac{1}{2}x^2 - \frac{1}{3}x^3$$
$$-\frac{1}{8}x^4 - \frac{1}{30}x^5 - \frac{1}{144}x^6 + \ldots,$$

and in the closed form as: y(x)=(1-x)e^x.

Conclusions

In this paper, we have used the variational iteration method for solving linear and nonlinear boundary value problems for tenth-order. The method is applied in a direct way without using linearization, perturbation or restrictive assumptions. It may be concluded that VIM is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. It is worth mentioning that the method is capable of reducing the volume of the computational work as compare to the classical methods while still maintaining the high accuracy of the numerical result, the size reduction amounts to the improvement of performance of approach. The fact that the variational iterative method solves nonlinear problems without using the Adomian's polynomials is a clear advantage of this technique over the decomposition method. It is worth mentioning that the variational iterative method can be used to solve the obstacle, unilateral and contact problems associated with the variational inequalities. For more details and information, see [1, 13-17] and the references therein. This is another direction for future research. The interested readers are advised to explore this interesting field and discover the novel and innovative applications of the variational iterative method and its variaous modifications.

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Prof. Dr. Muhammad Aslam Noor earned his PhD degree from Brunel University, London, UK (1975) in the field of Numerical Analysis and Optimization (Applied Mathematics). He has a vast experience of teaching and research at university level in various countries including Iran, Pakistan, Saudi Arabia, Canada and United Arab Emirates. His field of interest and specialization is versatile in nature. It covers many areas of Mathematical and Engineering Sciences. He has been recognized as Top Mathematician of the Muslim World by OIC and Top Leading Scientist and Engineer in Pakistan. He has been awarded by the President of Pakistan: President's Award for pride of performance on August 14, 2008. He has published extensively and has to his credit more than 720 research papers in leading world class scientific journals.

Prof. Dr. Eisa Al-Said is Professor at King Saud University, Riyadh, Saudi Arabia. His field of interest and specialization is Numerical Analysis of Differential equations, Variational Inequalities and related areas. He has published several papers in reputed international journals of Mathematical and Engineering Sciences

Prof. Dr. Syed Tauseef Mohyud-Din is currently Professor at HITEC University, Cantt, Pakistan. He is also Chairman, Basic, Sciences at HITEC University, Cantt, Pakistan. His field of interest is Numerical Analysis and optimization. He has published more than 250 research papers in Mathematical and Engineering Sciences.