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Nonlinear Pedestrian-flow Model: Uniform Well-posedness and Global Existence

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Abstract: This paper is concerned with the existence, uniqueness and propagation of monotonous properties for a certain class of Cauchy problems for first-order Hamilton-Jacobi equations for which initial data is a gradient function. In particular, we show that the monotonicity properties are propagated under certain assumptions. The main result is gained by the method of characteristics and a priori estimates under a monotonicity criterion. Applications of equations studying here refer to control problems in crowd dynamics.

Keywords: First-order Hamilton-Jacobi equations, monotonicity, a priori estimates.

1. Introduction

This article is meant with analytical investigations of the following hyperbolic system of first-order PDEs:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (h(u)\cdot\nabla)u = f(x) \\ ut = 0, x = u_0(x) \end{cases} \tag{1}$$

where $u(t,x) = (u_1(t,x), u_2(t,x)) : \mathbb{R}_+ \times \mathbb{R}^2 \mapsto \mathbb{R}^2$ is the unknown vector function, the external field $h : \mathbb{R}^2 \to \mathbb{R}^2$ and the initial data $u_0 : \mathbb{R} \to \mathbb{R}^2$ are given functions, $f : \mathbb{R}^2 \to \mathbb{R}^2$ is prescribed and

$$(h(u)\cdot\nabla)u = \left(\sum_{j=1}^{2} h_j \frac{\partial u_1}{\partial x_j}, \sum_{j=1}^{2} h_j \frac{\partial u_2}{\partial x_j} \right).$$

Different choices of the terms in the system (1) allow the derivations of various models of living systems existing in the pertinent literature. In particular, three prototypes problems are: crowd dynamics, see among others, papers [1–4] and the recent review [5]; vehicular traffic [6–9]; movement and growth of animal populations [10]. In these cases, $h(u)$ is a nonlinear function of the density u and $f(x)$ is the reaction term, which models the interactions among the entities.

It is worth stressing that the class of equations considered in this paper also includes the Hamilton-Jacobi-Bellman equation which arises in the framework of optimal control problems and differential game to front propagation, image enhancement [11, 12], mathematical finance [13], and so on.

Specifically the analysis developed in the present paper refers to the well-posedness problem of the system (1), namely existence and uniqueness of solutions, smooth properties and shock. Moreover the solution $u(t,\cdot)$ of (1) loses its initial regularity at a finite-time if and only if an eigenvalue of the initial velocity gradient crosses the negative real axis. To gain these results, we take advantage of the necessary condition of the propagation of monotonicity performed by Lions in [14]. Under the compatibility condition on the data and the nonlinearity (see (H1)-(H2) below), we establish global existence and uniqueness of smooth solution of (1).

The contents of the present paper are organized into three more sections which follow this introduction. In more details, after introducing a prototype model of Eq. (1), we begin by reviewing the main difficulty of our problem in Section 2. This section is also concerned with the definition of the main assumption under which we perform the whole analysis. Section 3 is devoted to the global well-posedness of the model described in Section 2 and then

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we illustrate seven prototype examples taken from various fields. Finally Section 4 deals with the propagation of the monotonicity where we derive a priori estimates using PDEs techniques.

2. Basic Equation: A Prototype Model

In the context of crowd dynamics, the model (1) can be derived from the optimality principle of the following finite-horizon control problem and is dictated by the following reasonable picture. Let S be a system composed of pedestrians evolving in a domain $\Omega \subset \mathbb{R}^2$. Assume that a pedestrian is at some location $x(0)$ at time $t=0$ in the domain Ω and would like to end up at some better location $x(t)$ at a later time $t=T>0$. Each location x in the domain Ω has some cost $J(x,T) \in \mathbb{R}$ at this final time T, which is small when x is a desirable location and large otherwise, so that the pedestrian would like to minimize a cost function of the following form:

$$J(x,T) = \int_0^T L(x'(t))dt + \phi(T,x(T)). \tag{2}$$

The functional $L : \Omega \to \mathbb{R}$, is a cost for changing states $L(v)dt$ measures the marginal cost of moving at a given velocity v for time dt and trajectory $x : [0,T] \to \Omega$. Function ϕ is a final cost (incentive to reach a certain area). One only needs that the functions L and ϕ to be convex. Moreover, we can also consider the case $L = L(t,x,v)$, therefore Eq. (2) reads:

$$J(x,T,v) = \int_0^T L(t,x(t),v(t))dt + \phi(T,x(T)). \tag{3}$$

The goal now is to select a trajectory that minimizes the functional (2). One way to compute the optimal trajectory is to solve the Euler-Lagrange equation associated to Eq. (2), with the boundary condition that the initial position $x(0)$ is fixed. From this, to derive a PDE equation, let us take $0 \leqslant t_0 \leqslant T$ and $x_0 \in \Omega$ and define the optimal cost $u(t_0,x_0)$ at the point (t_0,x_0) in space-time to be

$$u(t_0,x_0) = \inf \left[\int_{t_0}^T L(x'(t))dt + \phi(T,x(T)) \right] \tag{4}$$

over all (smooth) paths $x : [t_0,T] \to \Omega$ starting at $x(t_0) = x_0$ and with an arbitrary endpoint $x(T)$. Informally, this is the cost that the pedestrian would place on being at position x_0 at time t_0. For times t_0 less than T, it turns out that under some regularity hypotheses, the optimal cost function u obeys a partial differential equation, known as the Hamilton-Jacobi-Bellman equation, which we shall heuristically derive as follows. Assume that, the pedestrian finds himself at position x_0 at some time $t_0 < T$ and is deciding where to go next. Presumably there is some optimal velocity v in which the pedestrian should move in (*a priori*, this velocity need not be unique). So, if dt is an infinitesimal time, the pedestrian should move at this velocity for time dt, ending up at a new position $x_0 + dt$ at time $t_0 + dt$, and incurring a travel cost of $L(v)dt$. At this point, the optimal cost for the remainder of the pedestrian's journey is given by $u(t_0+dt, x_0+vdt)$ by definition of u. This leads to the heuristic formula

$$u(t_0+dt,x_0+vdt) = u(t_0,x_0) + L(v)dt$$

which by Taylor expansion (and by omitting higher order terms) gives:

$$\begin{aligned} u(t_0+dt,x_0+vdt) &= u(t_0,x_0) \\ &\qquad [\partial_t u(t_0,x_0) + v\cdot\nabla_x u(t_0,x_0) + L(v)]\, dt. \end{aligned}$$

On the one hand, v is being chosen to minimize the final cost. Thus, we see that v should be chosen to minimize the following expression:

$$v\cdot\nabla_x u(t_0,x_0) + L(v).$$

Note that, from the strict convexity that minimum v will be unique, and will be some function of $\nabla_x u(t_0,x_0)$. We introduce the Legendre transform $H : \Omega \to \mathbb{R}$ of $L : \Omega \to \mathbb{R}$ by the following formula:

$$H(p) := \sup_{v\in\Omega} [v\cdot p - L(v)],$$

which represents the gradient of u ($p = \nabla u(x)$). Then, under the assumption that L is an even function, we get

$$\min_{v\in\Omega}\{v\cdot\nabla_x u(t_0,x_0) + L(v)\} = -H(\nabla_x u(t_0,x_0)).$$

We conclude that

$$u(t_0,x_0) = u(t_0,x_0) + dt\, [\partial_t u(t_0,x_0) - H(\nabla_x u(t_0,x_0))]$$

leading to the following Hamilton-Jacobi-Bellman equation:

$$-\frac{\partial u}{\partial t} - H(\nabla_x u) = 0. \tag{5}$$

Equation (5) is being solved backwards in time, as the optimal cost is prescribed at the final time $t=T$, but we are interested in its value at earlier times, and in particular when $t=0$. Hence, given a smooth Hamiltonian $H : \mathbb{R}^2 \to \mathbb{R}$ and a smooth initial data $g : \mathbb{R}^2 \to \mathbb{R}$, we consider the corresponding initial-value problem, i.e. the solution $\varphi(t,x) = u(T-t,x)$ of the problem

$$\begin{cases} \dfrac{\partial\varphi}{\partial t} + H(\nabla_x\varphi) = 0 & \text{in } \mathbb{R}^2\times(0,T) \\ \varphi(0,x) = g(x) & \text{in } \mathbb{R}^2. \end{cases} \tag{6}$$

The equation (6), which plays a dominant role in our study, shows up in a variety of contexts dictated by different modeling of H's. While it is still reasonable for the crowd to have a target, we will assume that the initial value problem for models of crowd dynamics (6) is stated in unbounded domains for individuals who have the objective to reach a point of the whole space (e.g. a shopping area).

It well known that, in general, the nonlinear partial differential equation (6) can not be solved analytically. The solutions usually develop singularities in their derivatives even with smooth initial conditions. In these cases, the solutions do not satisfy the equation in the classical sense. However, M. G. Crandall and P.-L. Lions introduced the notion of viscosity solutions to resolve this problem [15]. In order to be able to work in modes of regular solutions, and to have the hope to manage equation (6), it would be necessary for us to work in the framework corresponding to convex situations which are regular modes. Thus, it is necessary for us to work with functions u which are monotone, since the gradient of a convex function is a monotone operator and the derivative of a convex function is increasing. Therefore, to analyze the solutions of Eq. (6), it is advisable to make the following assumption on the function u and the initial data u_0.

Assumption 21 *The function* $u : [0,\infty[\times\mathbb{R}^2 \to \mathbb{R}^2$ *is such that* $u = \nabla\varphi$ *and* $u_0 = \nabla g$.

With the above hypotheses, taking the gradient on (6), one gets an equivalent (at least for smooth solutions) form of the Hamilton-Jacobi equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x_i}\{H(u)\} = 0 \\ u|_{t=0} = u_0(x). \end{cases} \tag{7}$$

Thus, we are led to solving the following Cauchy problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (h(u)\cdot\nabla)u = 0 \\ h(u) = H'(u) \\ u|_{t=0} = \nabla g(x) := u_0(x) \end{cases} \tag{8}$$

Here, $h = (h_j)$ is the velocity vector with n components; $h\cdot\nabla$ is a first order differential operator with coefficients h_j. If one had used the cost function (3) instead of (2), one would be confronted with the equation

$$\begin{cases} \partial_t\varphi + H(t,x,\nabla_x\varphi) = 0 \\ \varphi(0,x) = g(x). \end{cases} \tag{9}$$

Differentiation of Eq. (9) with respect to x and making the choice

$$h(u) = \frac{\partial H}{\partial p} \quad \text{and} \quad \mathscr{G}(u) = -\frac{\partial H}{\partial x}$$

give the following system

$$\begin{cases} \dfrac{\partial u}{\partial t} + (h(t,x,u)\cdot\nabla)u = \mathscr{G}(t,x,u) \\ u|_{t=0} = \nabla g(x) := u_0(x) \end{cases} \tag{10}$$

which, despite some technical difficulties in the calculations would not change the problem.

The equations are nonlinear with quadratic nonlinearity. Because of the nonlinearity of h, one cannot generally expect the global smoothness of solutions of (10). That is singularities appear. The central issue of interest here is the well-posedness of the problem (10). In particular, one is interested to know when the solution u will lose its initial regularity at finite-time.

It is well known that one of the main difficulty when dealing with hyperbolic first-order equations is the appearing, at finite time, of discontinuities in u when H is non-linear. It is easy to verify that global C^1-solution does not exist for (6) in the generic situation, regardless of the smoothness of the initial condition $g(x)$. Singularities in the form of discontinuities in the derivatives of φ would appear at a finite time in most situations, thus the solutions would be Lipschitz continuous but no longer C^1. The simplest example revealing this pathology is, in the one dimensional case, if we identify $u = \varphi_x$. In fact, the simplest example revealing this pathology is, in dimension 1, the equation:

$$\partial_t\varphi + \frac{1}{2}(\partial_x\varphi)^2 = 0, \quad \varphi(0,x) = \varphi_0(x). \tag{11}$$

With $u(t,x) = \partial_x\varphi(t,x)$, by taking the derivative of Eq.(11), one obtains, the equation known as "Burger's equation":[1]

$$\partial_t u + u\partial_x u = 0, \quad u(0,x) = u_0(x)$$

Within this framework the hamiltonian is not other than $H(t,x,p) = \frac{1}{2}p^2$. It is thus independent of x and the hamiltonian system is written:

$$\dot{x}(t) = p(t), \quad \dot{p}(t) = 0, \quad x(0) = X, \quad p(0) = u_0(X).$$

It is integrated at sight to give:

$$x(t) = u_0(X), \quad x(t) = X + tu_0(X).$$

One observes that projection

$$(x(X(t), p(X,t)) \mapsto x(t) = X + tu_0(X)$$

is a bijection as long as remains strictly increasing, that is as long as

$$t < \frac{1}{\max(-\partial_x u_0(x))}.$$

To simplify the exposition, we confine ourselves to the most important special case, where $\mathscr{G} = 0$ in (10) and the functions H do not depend explicitly on t.

3. Global Well-posedness

Since these equations share many properties with the potential initial condition, such as a form of maximum principle and the translation invariance, uniqueness and monotone behavior for the solution, as in the usual case, are expected. We present an elementary proof of this monotone

[1] For examples, one can take a smooth bounded initial condition $u_0(x) = \mathrm{Arctan}(x)$ where shocks can occur even with very regular initial conditions u_0 and $u_0 = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$ where the solution is not defined everywhere.

behavior. The question is to determine the class of initial data for which there is no appearance of shocks.

First, we describe below the appropriate classes of linear operators that we will use in this paper.

Definition 1.*Let Ω be a subset of $\mathbb{R}^2$ and $f : \Omega \to \mathbb{R}^2$.*

(i) *f is monotone if for all $x, y \in \Omega$*

$$(f(y) - f(x), y - x) \geqslant 0.$$

(ii) *An operator $\mathscr{A} : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be monotone if*

$$(\mathscr{A}(x) - \mathscr{A}(y), x - y) \geqslant 0, \quad \forall x, y.$$

(iii) *The map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be ν-monotone, if there exists $\nu > 0$ such that $\forall \xi \in \mathbb{R}^2$ and $y \in \mathbb{R}^2$ the following holds:*

$$(\Phi'(y) \cdot \xi, \xi) \geqslant \nu |\Phi' \cdot \xi|^2.$$

Convexity of a function and monotonicity of its gradient are equivalent. Actually, the following proposition holds, see [16].

Proposition 1.*Let f be a differentiable function on an open convex set Ω of $\mathbb{R}^2$. Then:*

(i) *f is convex on Ω if and only if ∇f is monotone on Ω.*
(ii) *f is strictly convex on Ω if and only if ∇f is strictly monotone on Ω.*

Equipped with these definitions of monotonicity, we now turn to the assumption which allows to obtain smooth global solution of our equation by standing:

Assumption 31. *Assume that h and u_0 are monotone and u_0 is not necessary a gradient function.*

Under the assumption 31, we will show that the problem (8) has a global smooth solution. The strategy for proving our regularity result is based on direct construction following the characteristic method. The following result holds.

Theorem 1.*Assume that the function $h \in C^2(\mathbb{R}^2)$ and $u_0 \in C^2(\mathbb{R})$ are smooth and monotone. Then,*

(1) *The class of solutions of initial condition for which there is no appearance of shocks, is the class of u_0 such that $(I + th(u_0))$ is invertible for all t;*
(2) *There exists an unique monotone and smooth solution of the system (8) on $[0, +\infty[$ provided that*

$$\textbf{(I)} \qquad Spec\,(D(h \circ u_0)(x)) \cap \,]-\infty, 0[= \emptyset, \qquad \forall x,$$

where $Spec(A)$ denotes the spectrum of the matrix A.

Proof of Theorem 1.

Our proofs rely on new ideas introduced by P-L. Lions [14]. We proceed in two steps to prove the global solution of system (8). Firstly, we give necessary conditions for existence of a piecewise smooth global solution by the method of characteristics [17]. We no longer have the explicit representation solution u of the system (8). However, we can cleverly use the characteristic curves to solve the Cauchy problem. Secondly, we prove the well-posedness of our problem by showing the injectivity of the underlying application. Essentially, the idea is to prove that the map $x \mapsto x + th(u_0(x))$ is a smooth diffeomorphism. The whole amounts to invert an operator. This is a finite dimensional nonlinear problem.

• **First Step**. Construction of solutions

One of the classical methods for solving first order non-linear equations is the characteristic one. However, this method has a weak point from the fact that a smooth mapping cannot uniquely have the inverse at a point where its jacobian vanishes, i.e., that is inverse becomes many-valued there. Our aim is to show how to put a reasonable condition in order to obtain smooth solution. We develop the method of characteristics explaining how the smooth solutions of (8) can be obtained. Denoting by X the characteristic that passes through the origin and with a dot the derivative with respect to t, then characteristic lines corresponding to the Cauchy problem (8) are determined by following equations:

$$\begin{cases} \dot{X} = h(u) \\ \dot{u} = 0 \\ X|_{t=0} = x \\ u|_{t=0} = u_0(x). \end{cases} \tag{12}$$

Solving equations (12) yield the solution (while it is smooth) forward in time: at each point $(X|_{t=0}, u_0(x))$ a characteristic curve is the following:

$$X = x + th(u_0(x)). \tag{13}$$

Each of the coordinates satisfies the transport equation (8). Let us assume that problem is resolved; and if it is case, there is a vector field. The equation (12) means that u is constant along the trajectories of this vector field and along this trajectory; if u is constant, it means that the h is constant. Therefore, if one starts from a point x, one has a vector $u_0(x)$, and if one looks on line (14) generated by this field of vector, u does not move. Consider the map $X : [0, +\infty) \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by

$$X(t, x) = x + th(u_0(x)). \tag{14}$$

We want to define u at the point $(X(x,t), t)$. Let us first remark that u is well defined by (14) if and only if the map $(x \mapsto X(x,t))$, for each t, is bijective. This will be in general true for t small (and thus will yield local existence results) and in general false for every t. There exists $T > 0$ such that X is a diffeomorphism of class C^1 (i.e. bijective and with a C^1 inverse) on $\mathbb{R}^2$ for all $t \in [0, T]$. Indeed, the jacobian matrix of the map $(x \to X(t,x))$ is

$$\frac{DX}{Dx}(t, x) = I + tDh(u_0(x)) \cdot Du_0(x), \tag{15}$$

and $\det\left(\frac{DX}{Dx}(0,x)\right) > 0$. Therefore this map is a C^1 diffeomorphism for $0 \leqslant t < T$. For t small, this determinant is obviously positive and the assertion follows from the theorem of local inversion. Nevertheless, the fact that the application of $X(t,x)$ is a local diffeomorphism for each positive time is not in itself sufficient to obtain the existence of global solutions. For this to happen, more properties on the data of the problem are needed.

We will next prove that $X(t,x)$ is a global diffeomorphism for each $t \geqslant 0$ by using the fact that h and u_0 are monotone. In fact, as $Dh(u_0(x))$ and $Du_0(x)$ are symmetric positive matrices, semi-definite positive for each x, we will prove that the eigenvalues of matrix $Dh(u_0)\cdot Du_0(x)$ are real and nonnegative for all x. Roughly speaking, using the method of characteristics, formally, it is expected that

$$u(X(t),t) = u_0(x), \tag{16}$$

such that, at least formally, we have

$$u = u_0(I + th\circ u_0)^{-1}, \tag{17}$$

which remains true until the time of first shock. Since $h\circ u_0$ is the composition of two monotone maps, this implies that $(I+th\circ u_0)^{-1}$ exists.

Proof of (16). To show that (16) is a solution of (8), we differentiate the relation (16)) with respect to t. Clearly, we have

$$\frac{\partial u}{\partial t} + \frac{\partial X}{\partial t}\cdot D_y u = 0.$$

Since $\frac{\partial X}{\partial t} = h(u_0)$ and $u_0 = u$, we deduce the relation

$$\frac{\partial u}{\partial t} + (h(u_0)\cdot\nabla)\,u = 0,$$

in other words

$$\frac{\partial u}{\partial t} + (h(u)\cdot\nabla)\,u = 0,$$

proving that as long as we have the injectivity, we solve the equation on the domain reached on all X of the form (14).

Proof of (17). Assume that

$$\forall t \qquad u(t) : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{is invertible.}$$

Let V be defined by

$$V(u(t,x),t) = x$$

then we write the equation satisfies by $V = u^{-1}$. Taking the derivative of Eq. (8) with respect to t, we obtain

$$\frac{\partial V}{\partial t} - h(y) = 0. \tag{18}$$

Consequently,

$$V(t,y) = V_0(y) + th(y) = u_0^{-1}(y) + th(y).$$

Thus, provided that u is invertible, we obtain

$$u = u_0(I + th\circ u_0)^{-1}.$$

We deduce that the class of solutions of initial condition for which there is no appearance of shocks, is the class of u_0 such that $(I+th(u_0))$ is invertible for all t or, equivalently, the derivative of $(I+th(u_0))$ must be strictly positive. It is also the same thing to say that the Jacobian of the map $x \mapsto x + th(u_0(x))$ is invertible at any point. This is equivalent to saying that $I + t(D(h\circ u_0)(x))$ is invertible for any x and t. Strictly speaking, the condition

$$\mathrm{Spec}\,(D(h\circ u_0)(x)) \cap \,]-\infty, 0[= \emptyset, \qquad \forall x \tag{19}$$

must be verified, where u_0 is assumed to be bounded, Lipschitz, smooth. An equivalent expression of this fact is

$$\inf_{x\in\mathbb{R}^2} \mathrm{dist}\,(\mathrm{Spec}\,(D(h\circ u_0)(x))\,,]-\infty,0[) > 0.$$

This completes the proof of Theorem 1. □

We have then proved that, the system (8) admits a global smooth solution forward in time, if and only if the condition (19) is fulfilled. This kind of results can be compared with spectral method used by Liu *et al.* [18] to obtain similarly results of this kind of equation.

The linear case is very instructive. In this case, one can have shocks. One states that a maximal solution exists under the criterion 19 on the initial data. Indeed, consider the case where

$$h(u) = u \quad \text{ and } \quad u_0 = A_0 x \quad \text{ with } \; A_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

so that A_0 is symmetric matrix. Bearing in mind that in our case $A = DU$. Now we seek the solution of Eq. (8) in the form

$$u = Ax,$$

or equivalently $u = a_{ij}x_j$. This implies that

$$\dot{A} + A^2 = 0 \tag{20}$$

or equivalently $\dot{a}_{ij}x_j + a_{i\alpha}a_{\alpha k}x_k = 0$. It is clear that $\frac{Du}{\partial x} \leqslant 0$. So the spectrum of A consists of negative real. Thus, we can not obtain the propagation of regularity of smooth solutions. But if $A_0 \geqslant 0$, it is possible to find a solution which does not blow up in the infinity (think for example in case of one space dimension). Another case where the propagation of regularity remains true is the case where A_0 is invertible. In this time, by writing the equation given by $P := A^{-1}$. We get

$$\dot{A}P + A\dot{P} = 0,$$

such that

$$\dot{P} = -A^{-1}\dot{A}P = -A^{-1}\dot{A}A^{-1};$$

from (20), we have

$$-\dot{A} = A^2.$$

Moreover, notice further that

$$\dot{P} = -A^{-1}\dot{A}^2A^{-1} = I,$$

which suggests that

$$P = P_0 + tI.$$

Consequently, there exists a maximal solution to Eq. (20):

$$A = (A_0^{-1} + tI)^{-1}.$$

Finally, if we factored the following property holds:

$$A = (I + tA_0)^{-1}A_0, \tag{21}$$

then, the class of solutions of initial condition for which there is no appearance of shocks, is the class where we have

$$\forall x \quad \mathrm{Spec}\,(A_0(x)) \cap]-\infty, 0[= \emptyset.$$

This means that, $\frac{1}{t}$ will never been eigenvalue of A_0. In summary, we have proved the existence and global smooth solutions to the transport equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u\cdot\nabla)u = 0 \\ u|_{t=0} = u_0(x) \end{cases} \tag{22}$$

provided $I + tu_0$ is invertible on $\mathbb{R}^2$, $\forall t \geqslant 0$ and if and only if the condition (19) is conserved.

*Remark.*In order to better understand our main ideas in the next section, we first give a simple example of the situation where there is appearance of shocks. In fact we have shown that roughly speaking, if h and u_0 are monotone or more generally if $h(u_0)$ is monotone, then there exist smooth solutions. Let h and u_0 defined by $h = Au$, $u_0 = Bx$ with A the rotation of angle $-\frac{\pi}{2}$ and $B = A$. We have

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad AB = A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One recovers a rotation of angle $-\frac{\pi}{2}$ and Eq. (14) becomes

$$(1-t)x = X.$$

Thus for the $t < 1$ everything goes well, but in $t = 1$ one loses the invertibility property and shocks appear.

*Remark.*If h is quadratic, $h(u) = u$, to have smooth solution is equivalent to the criterion (19). That means Du_0 is a symmetric matrix and $\mathrm{Spec}(Du_0) \subset \mathbb{R}^+$. In fact, for Eq (8), $g(x)$ convex implies ∇g monotone; since u_0 is monotone then $h(u_0)$ is the composite of two monotone maps. φ_0 is the only case of regularity for all times and this remains also valid when $H'' > 0$. In fact, in that case, we must look at a condition on $H'(\nabla g)$ satisfying the criterion **(I)**, which implies g convex because the following holds:

$$Du_0(x) = H''(\nabla g)\cdot D^2 g$$

The above matrix is the product of two symmetric matrix, that means, product of two matrices whose eigenvalues do not intercept the negative real which suggest that $Dg \geqslant 0$ gives the propagation of the regularity of solutions.

The following lemma contains the key observation needed to solve our problem.

Lemma 1.*The necessary and sufficient condition for the propagation of regularity to the solution for Eq (8) is that $H'(\nabla g)$ must satisfies criterion* **(I)**.

• **Second Step**. *Regularity or one-to-one of the map X.*

Observe that, since the map

$$X \longmapsto x + th(u_0(x))$$

is $\mathscr{C}^\infty$, then X is invertible. The inverse is also of class $\mathscr{C}^\infty$. We only need to verify that the solution we construct, is a smooth solution. The result follows, if we can establish that the application X is invertible. This reduces to study the Jacobian matrix of X. We now proceed to show that this is the case. Calling the jacobian matrix (15) and having in the mind that the matrix $Du_0 \geqslant 0$ (u_0 is monotone) and $Dh \geqslant 0$ (h is monotone). Instead of working with matrix (15), it would be more convenient to work with vectors. We will look at the Jacobian matrix of the transformation (15) in a direction η, which we denote ξ. When the differentiation of the map X is carried out in the direction η, linearized equation results:

$$\eta = \xi + tDh(u_0)\cdot Du_0\cdot\xi. \tag{23}$$

Next, we take the dot product of (23) with $Du_0\cdot\xi$ in order to reveal the monotonous assumptions on u_0 and Dh. We clearly have

$$(Du_0\xi, \eta) = (\xi, Du_0\cdot\xi) + t(Dh\cdot Du_0\cdot\xi, Du_0\cdot\xi). \tag{24}$$

Observe that, in Eq. (24) we are faced with product of two matrices, and the goal is to reverse this product; so be sure that the kernel is non-zero, i.e. that the product is invertible. So the study of injectivity is equivalent to looking at the kernel. We resort to

$$0 = (\xi, Du_0\cdot\xi) + t(Dh\cdot Du_0\cdot\xi, Du_0\cdot\xi). \tag{25}$$

Moreover, by assumptions regarding u_0 and h, we have

$$Du_0\cdot\xi \geqslant 0, \quad \text{and} \quad (Dh\cdot Du_0\cdot\xi, Du_0\cdot\xi) \geqslant 0.$$

We deduce that each term of the equality (24) is zero. Therefore, ξ is in the kernel of the symmetric part. Strictly speaking,

$$\xi \in \mathrm{Ker}(Du_0 + (Du_0)^*),$$

where $*$ means the transposition of a vector. This also results that

$$Du_0\cdot\xi \in \mathrm{Ker}(Dh + (Dh)^*).$$

If h is the derivative of a convex function, meaning that, it is not only monotone, but a symmetric operator monotone, in which case $h' = H''$, i.e $Dh = D^2H$ and so it becomes symmetric. Here D^2H denotes the Hessian matrix of H. Thus

$$Du_0 \cdot \xi \in \mathrm{Ker}(Dh) \quad \text{if } h = H',$$

and this belongs in the kernel of Dh since

$$Dh.Du_0 \cdot \xi = 0,$$

resulting $\xi = 0$. We then have proved injectivity. Summing up the above results yields the theorem. □

• **Third Step:** *a priori estimate.*

One way to get rid the situation in Remark 3, would be to impose more regularity to h and Du_0. We make the following regularity criterion.

Assumption 32. *For all $z \in \mathbb{R}^2$, $\xi \in \mathbb{R}^2$, there exists $\nu > 0$, independent of z and ξ, such that*

(H1) $(Dh(z) \cdot \xi, \xi) \geqslant \nu |h'(z) \cdot \xi|^2$;

(H2) $(Du_0(x)\xi, \xi) \geqslant \nu |u_0'(x) \cdot \xi|^2$.

These assumptions immediately call for some few comments. Both assumptions are assumptions about the inverse of invertible functions. Roughy speaking, this means that, one has a matrix A such that

$$(A\xi, \xi) \geqslant \nu |A\xi|^2. \tag{26}$$

Indeed, let's write $\eta = A\xi$, then (26) is equivalent to

$$(\eta, A^{-1}\eta) \geqslant \nu |\xi|^2,$$

if A is invertible. Formally, one could translated theses assumptions as follows

$$A^{-1} \geqslant \nu I''.$$

The assumption (H1) allows to eliminate the case of the rotations of angle $-\frac{\pi}{2}$. This assumption ensures that u is smooth solution of the problem. In addition, if h' is symmetric, assumption (H2) remains true.

To prove that the map $x \mapsto X$ is a diffeomorphism and bi-lipschitz[2] we must derive also *a priori* estimate.

- *First case: using the estimate on u_0.*

  Taking the dot product of (23) with $Du_0(x) \cdot \xi$, we get:

$$(Du_0(x) \cdot \xi, \xi) + t(h'(Du_0) \cdot \xi \cdot Du_0 \xi) = (Du_0(x) \cdot \xi, \eta). \tag{27}$$

  On the one hand, observe that

$$h'(Du_0) \cdot \xi \cdot Du_0 \xi \geqslant 0.$$

[2] We say that a function $f : \mathbb{R}^d \to \mathbb{R}^d$ is bi-lipschitz if both f and f^{-1} are Lipschitz.

On the other hand we have

$$(Du_0(x) \cdot \xi, \xi) \leqslant \nu |Du_0(x) \cdot \xi|^2,$$

while

$$(Du_0(x) \cdot \xi, \eta) \leqslant |\eta||Du_0(x) \cdot \xi|.$$

From the above estimates and assumptions (H1), we deduce

$$|Du_0(x) \cdot \xi| \leqslant \frac{1}{\nu}|\eta|$$

that controls the term $Du_0 \cdot \xi$. Next, writing

$$\xi = \eta - th' \cdot Du_0 \cdot \xi \tag{28}$$

we get

$$|\xi| = |\eta| + \frac{tC_0}{\nu}|\eta|,$$

where C_0 is Lipschitz constant of h, by taking into account the estimate

$$Du_0 \cdot \xi \leqslant \frac{1}{\nu}|\eta|.$$

Therefore, when assumption (H1) is used,

$$\exists \nu > 0, \qquad \forall x, \qquad |\xi| \leqslant \left(1 + \frac{C_0 t}{\nu}\right)|\eta|$$

which is an a priori estimate giving the existence of a global smooth solution.

- *Second case: using the estimate on h.*

  We begin with the relation (27). On the one hand we get

$$\begin{aligned} t(h'(Du_0) \cdot \xi, Du_0 \cdot \xi) &\leqslant \nu t |h' \cdot Du \cdot \xi|^2 \\ &\leqslant |\eta||Du_0(x) \cdot \xi| \\ &\leqslant C_1 |\eta||\xi|, \end{aligned} \tag{29}$$

where C_1 is now, the Lipschitz constant of u_0. The estimate (29) can be expressed as

$$|h' \cdot Du_0 \cdot \xi| \leqslant \left(\frac{C_1}{\nu t}|\eta||\xi|\right)^{1/2}.$$

On the other hand, the relation (28) suggests that

$$|\xi| \leqslant |\eta| + t\left(\frac{C_1}{\nu t}|\eta||\xi|\right)^{1/2}. \tag{30}$$

With the help of Cauchy-Schwartz inequality, the relation (30) can be rephrased as

$$\begin{aligned} |\xi| &\leqslant |\eta| + \left(\frac{C_1 t}{\nu}\right)^{1/2} |\eta|^{1/2}|\xi|^{1/2} \\ &\leqslant \frac{a}{2}|\xi| + \frac{1}{2a}\left(\frac{C_1 t}{\nu}|\eta|\right), \qquad 0 < a < 2. \end{aligned}$$

Let us consider the following change of variable:

$$x = \left(\frac{|\xi|}{|\eta|}\right)^{1/2}$$

we are led to solve the inequality

$$x^2 - \left(\frac{C_1 t}{\nu}\right)^{1/2} x - 1 \leqslant 0$$

from which we deduce the estimate

$$\frac{|\xi|}{|\eta|} \leqslant \left(\frac{1}{4}\left[\left(\frac{C_1 t}{\nu}\right)^{1/2} + \sqrt{\frac{C_1 t}{\nu} + 4}\right]\right)^2.$$

So we find a behavior near the origin. Consequently, we obtain an estimate depending on t and η:

$$|\xi| \leqslant C(t)|\eta|.$$

The proof of theorem is complete. □

3.1. Some Examples

We now turn to consider some examples which demonstrate our procedure. For simplicity of notations we shall mostly concentrate on the one and two dimensional cases, even if the results of the theorem remains true also in the higher dimensional space.

*Example 1.*It is useful to have in mind what happens in the special case $n = 1$. In one space dimension, Eq. (6) becomes

$$\begin{cases} \partial_t \varphi + H(\varphi_x) = 0 & \text{in } (0,\infty)\times\mathbb{R}, \\ \varphi(0,x) = g(x) & \text{in } \mathbb{R}. \end{cases} \tag{31}$$

This is relatively easy case, because (31) is equivalent to the conservation laws

$$\begin{cases} \partial_t u + (H(u))_x = 0 & \text{in } (0,\infty)\times\mathbb{R}, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R} \end{cases} \tag{32}$$

if we identify $u = \varphi_x$ which can be written as a quasilinear equation

$$\partial_t u + a(u)u_x = 0 \qquad u(0,x) = u_0(x) \quad \text{in } (0,\infty)\times\mathbb{R} \tag{33}$$

where $a(u)$ denotes $H'(u)$. We shall require that (33) be genuinely nonlinear, i.e., that a, the coefficient of u_x, should vary with u in the sense that $a'(u)$ is not zero. Since $a = f'$, this means that $f'' \neq 0$, i.e., the function $f(u)$ is either strictly convex or strictly concave. Here, we will assume that f is strictly convex. As a system of characteristic differential equations is written by

$$\dot{x} = a(u), \quad \dot{z} = 0$$

whose initial condition is

$$x(0) = y, \qquad z(0) = u_0(x),$$

then, one has

$$u(t,x(t)) = z(t), \tag{34}$$

and a family of the characteristic curves is given by

$$x = y + tf'(u_0(x)), \qquad y \in \mathbb{R}. \tag{35}$$

Therefore, it follows that

$$\frac{\partial x}{\partial y} = 1 + tf''(u_0(x))u_0'(x).$$

In that case, we are in a regime of regularity of solutions, because as u_0 and $a(u) = f'$ are increasing functions, the composition of two increasing functions $f'(u_0(x))$ is increasing. We are thus faced with an application $x \mapsto y + tf'(u_0(x))$ which, in term of growth is undervalued by x and which is known to be injective and hence surjective. To be more precise, we have the monotony of the operator $x \mapsto x(t)$ defined in (34) as long as one keeps a diffeomorphism of application (35). We know how to define the value of u at any point in the expression (34). We thus found a regular solution of the problem Eq. (32).

*Example 2.*Consider the following Cauchy problem with convex Hamiltonian:

$$\begin{cases} \partial_t \varphi + \frac{1}{2}(\varphi_{\mathbf{x}})^2 = 0, & \mathbf{x} \in \mathbb{R}^2,\ t > 0, \\ \varphi(0,x) = \frac{1}{2}\mathbf{x}^2, & \mathbf{x} \in \mathbb{R}^2, \end{cases} \tag{36}$$

interpreted through the Eikonal equation (a special case of a HJ equation) which is of an main interest in optimal control theory. With this choice, we immediately obtain

$$X(t,x) = (1+t)x,$$

whereby

$$\frac{DX}{Dx} = 1 + t \geqslant 1 > 0.$$

Therefore

$$X^{-1}(t,x) = \frac{x}{1+t},$$

that is well defined and is a diffeomorphism for each $t \geqslant 0$. As before, we conclude that the Eq (36) has a global classical solution.

*Example 3.*Consider the following Cauchy problem with non-convex Hamiltonian:

$$\begin{cases} \partial_t \varphi = \frac{1}{2}(\varphi_{\mathbf{x}})^2, & \mathbf{x} \in \mathbb{R}^2,\ t > 0, \\ \varphi(0,x) = \frac{1}{2}\mathbf{x}^2, & \mathbf{x} \in \mathbb{R}^2. \end{cases} \tag{37}$$

We immediately obtain

$$X(t,x) = (1-t)x, \qquad \frac{DX}{Dx} = 1 - t$$

whereby

$$X^{-1}(t,x) = \frac{x}{1-t}.$$

This function is a diffeomorphism in $[0,1)$ but is not defined at $t=1$. The corresponding solution is then a classical local solution explodes in finite time.

*Example 4.*Consider the following initial data

$$u_0(x) = A\cdot\mathbf{x}+c$$

where $A\in\mathbb{R}^2$ and c is a fixed real number. In this case,

$$X(t,x) = x+th(A)$$

where we have made use of relation (14). Then,

$$\det\left(\frac{DX(t,x)}{Dx}\right)\equiv 1, \quad \forall t\geqslant 0.$$

*Example 5.*Consider the following nonlinear example

$$\begin{cases}\partial_t\varphi+\varphi_x\varphi_y = 0, & x\in\mathbb{R},\ y\in\mathbb{R}\\ \varphi(0,x,y) = xy.\end{cases}$$

One has $u_0(x,y) = \begin{pmatrix} y\\ x\end{pmatrix}$ and $Du_0(x,y) = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}$ whose eigenvalues intercept the negative real axis. Therefore, we have no regularity for our equation.

*Example 6.*Consider the following scalar Hamilton-Jacobi equation

$$\varphi_t+H(\varphi_x,\varphi_y) = 0, \qquad \varphi(0,x,y) = \varphi_0(x,y)$$

which in a certain sense is equivalent to the following conservation system:

$$\begin{cases} u_t+H(u,v)_x = 0\\ v_t+H(u,v)_y = 0\\ (u,v)(0,x,y) = (u,v)_0(x,y)\end{cases} \tag{38}$$

if we identify

$$(u,v) = (\varphi_x,\varphi_y).$$

Next, we take the hamiltonian and the initial data respectively as

$$H(u,v) = \frac{1}{2}u^2-\frac{1}{2}v^2, \qquad (u,v)_0(x,y) = \varphi(x)-\psi(y)$$

where H is convex in u and concave in v, with φ and ψ are convex functions. In this case we can recast Eq (38) as

$$\begin{pmatrix} u\\ v\end{pmatrix}_t+\begin{pmatrix} H(u,v)\\ 0\end{pmatrix}_x+\begin{pmatrix} 0\\ H(u,v)\end{pmatrix}_y = 0$$

which is solved separately and the solution is of the form

$$(u,v)(t,x,y) = \varphi(t,x)-\psi(t,y).$$

To be more precise, φ and ψ solve the following decoupled system:

$$\frac{\partial\varphi}{\partial t}+\frac{1}{2}\varphi_x^2 = 0 \qquad \frac{\partial\psi}{\partial t}+\frac{1}{2}\psi_y^2 = 0.$$

Then, we agree with the previous case of regularity of solutions. Next, observe that the condition $H'(\nabla g)$ satisfies criterion **(I)** is equivalent to look

$$H' : \mathbb{R}^2\to\mathbb{R}^2 \qquad H'(u,v) = (u,-v)$$

that is

$$H'(\nabla\varphi_0) = \varphi_x-\varphi_y, \qquad \varphi := \varphi_0.$$

In other words, we must look the eigenvalues of the matrix

$$D\begin{pmatrix}\varphi_x\\ -\varphi_y\end{pmatrix} = \begin{pmatrix}\varphi_{xx} & \varphi_{xy}\\ -\varphi_{xy} & -\varphi_{yy}\end{pmatrix}.$$

From the computation of the characteristic polynomial of the above matrix, one can prove that, we have the regularity of the solution provided that the condition

$$|\varphi_{xx}+\varphi_{yy}| < 2|\varphi_{xy}|$$

is fulfilled. In this case, we have complex eigenvalues which imaginary part is nonzero. Another case is when we take the function

$$\varphi(t,x,y) = \phi(x)-\psi(y).$$

It follows the condition

$$\phi_{xx}+\psi_{yy}\geqslant|\phi_{xx}-\psi_{yy}| \quad \text{with} \quad \phi_{xx}\geqslant 0,\quad \psi_{yy}\geqslant 0.$$

since the eigenvalues corresponding to the jacobian matrix are real and positive. Imagine that done; we then deduce the solution of Eq. (38).

*Example 7.*Consider the following 2D Eikonal equation which arises in geometric optics [19,20]:

$$\begin{cases}\partial_t\varphi+\sqrt{\varphi_x^2+\varphi_y^2} = 0, & x\in\mathbb{R},\ y\in\mathbb{R},\\ \varphi(0,x,y) = \varphi_0(x,y).\end{cases}$$

In this equation

$$H(u,v) = \sqrt{u^2+v^2}.$$

Let us write

$$\nabla\varphi_0 = \sqrt{\Phi_x^2+\Phi_y^2}, \qquad \Phi := \varphi_0.$$

Then

$$u_0 = H'(\nabla\Phi) = \frac{\nabla\Phi}{|\nabla\Phi|} = \frac{\Phi_x+\Phi_y}{\sqrt{\Phi_x^2+\Phi_y^2}}.$$

Therefore

$$Du_0(x) = \operatorname{div}\left(\frac{\nabla\Phi}{|\nabla\Phi|}\right) = \frac{\Phi_y^2\Phi_{xx} - 2\Phi_x\Phi_y\Phi_{xy} + \Phi_x^2\Phi_{yy}}{(\Phi_x^2+\Phi_y^2)^{3/2}}.$$

It is more instructive to make the computation when the dimension is greater than two. In this case, carrying out the differentiations yields, in compact form

$$Du_0(x) = \frac{1}{|\nabla\Phi|}\left(I - \frac{\nabla\Phi\otimes\nabla\Phi_0}{|\nabla\Phi|^2}\right) : D^2\Phi = \frac{1}{|\nabla\Phi|}a_{ij}(p)D^2\Phi$$

where $\otimes$ denotes a tensor product of vectors in $\mathbb{R}^n$, $n \geqslant 2$ and

$$a_{ij}(p) = \delta_{ij} - \frac{p_ip_j}{|p|^2}, \qquad p := \nabla\Phi,$$

if $|p| \neq 0$ and δ_{ij} is the Kronecker delta function and $\frac{\nabla\Phi\otimes\nabla\Phi}{|\nabla\Phi|^2}$ is the matrix of the mean curvature levels of hypersurfaces in $\mathbb{R}^2$ passing by $\Phi(x)$ in the region where $|\nabla\Phi| \neq 0$. This show that the condition of the regularity of the solution is equivalent to $H'(\nabla g)$ satisfies criterion **(I)**. This means that, the levels areas are convex.

4. A Priori Estimates

The aim of this section is to investigate further regularity for the solution obtained through Theorem 1. What we want to show in this section is the following: By eliminating the method of characteristics, we wish to obtain a uniform estimate on the gradient that will ensure the spread of the monotony in Eq. (8). To do this, since we do not require our equation to have H convex, our technique consists in using an auxiliary function, denoted Ψ with an initial condition on the equation obtained, so that this one is positive or null. One thus obtains a transport equation whose vector fields are at most linear growth. By exploiting this last equation, one deduces the desired estimate. Interestingly, the ideas we present in this paper provide a different proofs of the talk of P-L. Lions [14]. It well-known that, gradient estimates are easy to obtain in the case of one space dimension but not obvious at all higher dimension.

The following result is in the heart of matter in this section.

Theorem 2. *Let u be the solution of Eq. (8) and let assumptions (H1)-(H2) are fulfilled. Then, the monotonicity properties are propagated in Eq. (8). Moreover, one has the following a priori estimate:*

$$|D_xu| \leqslant C(T), \quad \text{on } [0,T]\times\mathbb{R}^2.$$

Proof of Theorem 2. First we recall that there exists $\alpha > 0$ such that $\forall z \in \mathbb{R}^2$, $\xi \in \mathbb{R}^2$:

$$(h'(z)\cdot\xi,\xi) \geqslant \alpha|h'(z)\cdot\xi|^2.$$

A key tool here is the introduction of the following function:

$$W(x,\xi,t) = u(x,t)\cdot\xi, \quad \xi \in \mathbb{R}^2$$

which implies that

$$u(t,x) = \nabla_\xi W(t,x,\xi)$$

and corresponding to taking the dot product of function u in the direction ξ. Formally, from Eq. (8), one has

$$\begin{cases} \dfrac{\partial W}{\partial t} + (h(\nabla_\xi W)\cdot\nabla_x W = 0, \\ W|_{t=0} = u_0(x)\cdot\xi \end{cases} \tag{39}$$

which is a scalar equation, i.e. a first-order Hamilton-Jacobi equation, equivalent to Eq. (8). Correspondingly, we define

$$Z = \xi\cdot\nabla_\xi W.$$

Next, we express Eq. (39) in the term of Z and obtain:

$$\frac{\partial Z}{\partial t} + h\cdot\nabla_x Z + (h'\nabla_x W, \nabla_\xi Z) = (h'\nabla_x W, \nabla_x W). \tag{40}$$

• **First case**. We make the assumptions on h and u_0.

Assume that u_0 is β-monotone. From (18), this is equivalent to saying that

$$(u^0)^{-1} \geqslant \frac{1}{\alpha}$$

so that provided h is monotone, this implies

$$u^{-1} \geqslant \frac{1}{\alpha}I;$$

and consequently u is β-monotone. We deduce the estimate

$$(Du(x)\cdot\xi,\xi) \geqslant \alpha|Du(x)\cdot\xi|^2$$

and by Cauchy-Schwartz, we get

$$|Du(x)\cdot\xi||\xi| \geqslant (Du(x)\cdot\xi,\xi) \geqslant \alpha|Du(x)\xi|^2. \tag{41}$$

From the formula (41) it follows that

$$|Du(x)\xi| \leqslant \frac{1}{\alpha}|\xi| \tag{42}$$

giving an *a priori* estimate on the size of the gradient. The quantity which propagates in time is

$$(Du(t,x)\xi,\xi) - \alpha|D_xu(t,x)\cdot\xi|^2 \geqslant 0. \tag{43}$$

The main ingredient in Krylov techniques is to translate the term $(Du(t,x)\xi,\xi)$ into $\xi\cdot\nabla_x W$ and $|Du(x,t)\cdot\xi|^2$ into $|\nabla_x W|^2$. This leads us to handle the following expression (much simpler in form):

$$Z_\beta = \xi\cdot\nabla_x W - \beta|\nabla_x W|^2, \qquad \beta > 0.$$

If we can show that Z_β is positive and bounded below, we will derive a Lipschitz estimate allowing to obtain a priori estimates.
On the one hand, in connection with the Hamilton-Jacobi equation, we obtain

$$\begin{cases} \dfrac{\partial}{\partial t}\nabla_x W + (h\cdot\nabla_x)\nabla_x W + h'(W)\cdot\nabla_\xi\nabla_x W = 0, \\ W|_{t=0} = |Du_0| \leqslant C_0. \end{cases} \tag{44}$$

Note that, since ∇W is solution of a transport equation, the norm $|\nabla W|$ also verifies a transport equation. Then, taking the squared norm, it follows

$$\frac{\partial}{\partial t}|\nabla_x W|^2 + (h\cdot\nabla_x)|\nabla_x W|^2 + h'(W)\cdot\nabla_\xi|\nabla_x W|^2 = 0.$$

On the other hand, writing Eq. (44) in term of Z gives

$$\frac{\partial Z_\beta}{\partial t} + h\cdot\nabla_x Z_\beta + (h'\cdot\nabla_x W)\cdot\nabla_\xi Z_\beta = (h'\cdot\nabla W, \nabla W) \geqslant 0.$$

One thus has a first-order transport equation with terms that are at most linear growth. We deduce *a priori* estimate, under the assumptions regarding the monotonicity, provided that

$$Z_\beta|_{t=0} \geqslant 0 \qquad \text{then} \qquad Z_\beta \geqslant 0,$$

i.e. we had thus shown *a priori* estimate. So no need to go in reverse and we understand that it is a very simple proof that the method of characteristics.

• **Second case**. We remove assumption (H2), that is, we have any assumption on U_0 but only that u_0 is regular bounded derivatives. In order to get the corresponding bound, we must use Eq. (44) in term of Z. Thus, from assumptions (H1)-(H2), we deduce

$$\frac{\partial Z_\beta}{\partial t} + h\cdot\nabla_x Z_\beta + (h'\cdot\nabla_x W)\cdot\nabla_\xi Z_\beta \geqslant \alpha|h'\cdot\nabla_x W|^2.$$

The main difficulty here is that we did not have any assumption on u_0, except for u_0 with bounded regular derivatives. If one looks at what occurs in $t=0$, and writing Z_β in u_0, one obtains

$$Z_\beta = (D_x u_0\cdot\xi, \xi) - \beta|D_x u_0\cdot\xi|^2. \tag{45}$$

On the other hand, we have

$$|D_x u_0| \leqslant C_0. \tag{46}$$

The only information we have is that u_0 is monotone; therefore we get the following estimate:

$$Z_\beta \geqslant -C_0\beta|\xi|^2, \qquad T\in(0,+\infty)$$

following from the fact that, $D_x u_0$ is bounded and it grows linearly with ξ. We take the modulus squared, so a priori it is quadratic, in which one loses positivity because of the presence of the term $-C_0|\xi|^2$. It is a bit embarrassing. We want to add a constant multiplied by $|\xi|^2$, to retrieve positive terms. Therefore we introduce a new auxiliary function

$$\Psi = \xi\cdot\nabla_x W - \beta|\nabla_x W|^2 + \gamma|\xi|^2, \qquad \beta>0,\ \ \gamma>0, \tag{47}$$

in order to get a priori estimate. Next, we write the equation in the term of Ψ and estimate the latter. We get:

$$\begin{aligned}\frac{\partial\Psi}{\partial t} + (h^i_j\partial_{x_i}W)\partial_{\xi_j}\Psi + (h\cdot\nabla_x)Z_\beta &\geqslant -\dot\beta|\nabla_x W|^2 + \dot\gamma|\xi|^2 \\ &+ \alpha|h'\nabla_x W|^2 + 2\gamma(h'\cdot D_x W, \xi),\end{aligned} \tag{48}$$

with the notation $h^i_j = \dfrac{\partial h^i}{\partial z_j}$. At time $t=0$, we have $\Psi\geqslant 0$ provided that

$$\gamma(0) \geqslant C_0\beta(0)$$

which is a condition on the initial values, natural to intervene. We choose the applications $t\mapsto\gamma(t)$ to be increasing and $t\mapsto\beta(t)$ decreasing $\gamma>0$ and $\beta>0$. Using Cauchy-Schwarz and estimation (42), it follows

$$2\gamma(h'\cdot D_x W, \xi) \geqslant -\alpha|h'\cdot\nabla_x W|^2 - \frac{\gamma^2}{\alpha}|\xi|^2.$$

We fix T and try to have a bound on $T\in(0,\infty)$. The inequality (48) becomes, after simplification

$$\begin{aligned}\frac{\partial\Psi}{\partial t} + (h^i_j\partial_{x_i}W)\partial_{\xi_j}\Psi + (h\cdot\nabla_x)Z_\beta &\geqslant \\ -\dot\beta|\nabla_x W|^2 + \left(\dot\gamma - \frac{\gamma^2}{\alpha}\right)|\xi|^2.\end{aligned}$$

Observe that, h is bounded increasing function; h^i_j is bounded and $\partial_{x_i}W$ increases at most linearly with ξ (since u is bounded). Thus, we have a vector field at most linear growth, such that if we start with something positive, so we keep the positivity. Therefore,

$$\frac{\partial\Psi}{\partial t} + (h^i_j\partial_{x_i}W)\partial_{\xi_j}\Psi + (h\cdot\nabla_x)Z_\beta \geqslant 0 \tag{49}$$

provided that

$$\dot\gamma \geqslant \frac{\gamma^2}{\alpha}.$$

If we choose β to be small, then we are led to solve the equation

$$\begin{cases} \dot\gamma = \dfrac{\gamma^2}{\alpha} \\ \gamma(0) = C_0\beta. \end{cases} \tag{50}$$

Thus

$$\forall\beta>0, \qquad \Psi\geqslant 0 \quad \text{on} \quad [0, t(\gamma)),$$

where $[0,t(\gamma))$ is the existence interval of γ. By doing this, we get a transport equation which is positive or zero and we find the expression (49). This completes the proof of the Theorem 2. □

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