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Iterative Methods for Solving Nonconvex Equilibrium Variational Inequalities

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Abstract: n this paper, we introduce and consider a new class of equilibrium problems and variational inequalities which is called the nonconvex equilibrium variational inequality. We suggest and analyze some iterative methods for solving the nonconvex equilibrium variational inequalities using the auxiliary principle technique. We prove that the convergence of implicit method requires only monotonicity. Some special cases are also considered. Our proof of convergence is very simple. Results proved in this paper may stimulate further research in this dynamic field.

Keywords: Variational inequalities; nonconvex functions; auxiliary principle technique, convergence.

Introduction

Variational inequalities theory, which was introduced by Stampacchia [22], can be viewed as an important and significant extension of the variational principles. This theory combines the theory of extremal problems and monotone operators under a unified viewpoint. Related to the variational inequalities, we have the equilibrium problems, which was introduced and studied by Blum and Oettli [1] in 1994. It has been shown that the variational inequalities and fixed point problems are special cases of the equilibrium problems. However, the variational inequalities and equilibrium problems are quite two problems. Noor et al 18] have considered a general and unified class, which is called the equilibrium variational inequality. They have also discussed the numerical methods for solving such type of equilibrium variational inequalities using the auxiliary principle technique.

We would like to mention that all the work carried out in this direction assumed that the underlying set is a convex set. In many practical situations, a choice set may not be convex so that the existing results may not be applicable. In recent years, Noor [10,12,13,14] and Bounkhel et al [2] have considered variational inequality in the context of uniformly prox-regular sets. Note that the prox-regular sets are nonconvex sets, see [2,3,21].

In this paper, we introduce and consider a new class of equilibrium variational inequalities on the prox-regular sets, which is called the nonconvex equilibrium variational inequality. This class is quite general and unifying one. One can easily show that the several classes of equilibrium problems and variational inequalities are special cases of this new class. There are a substantial number of numerical methods including projection technique and its variant forms, Wiener-Hopf equations, auxiliary principle and resolvent equations methods for solving variational inequalities. However, it is known that projection, Wiener-Hopf equations, proximal and resolvent equations techniques cannot be extended and generalized to suggest and analyze similar iterative methods for solving variational inequalities. This fact has motivated to use the auxiliary principle technique, which is mainly due to mainly due to Glowinski, Lions and Tremolieres [4]. Noor et al[15,16] has used this technique to develop some iterative schemes for solving various classes of variational inequalities. We point out that this technique does not involve the projection of the operator and is flexible. In this paper, we show that the auxiliary principle technique can be used to suggest and analyze a class of iterative methods for solving the nonconvex equilibrium variational inequalities. We also prove that the convergence of this new implicit method requires

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only the monotonicity. Our method of proof is very simple. Results proved in this paper continue to hold for all the special cases.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\cdot, \cdot$ and $\|.\|$ respectively. Let K be a nonempty and convex set in H. We, first of all, recall the following well-known concepts from nonlinear convex analysis and nonsmooth analysis [3,21]. Poliquin et al. [21] and Clarke et al [3] have introduced and studied a new class of nonconvex sets, which are called uniformly prox-regular sets.

Definition 2.1. The proximal normal cone of K at $u \in H$ is given by

$$N_K^P(u) := \{\xi \in H : u \in P_K[u + \alpha\xi]\},$$

where $\alpha > 0$ is a constant and

$$P_K[u] = \{u^* \in K : d_K(u) = \|u - u^*\|\}.$$

Here $d_K(.)$ is the usual distance function to the subset K, that is

$$d_K(u) = \inf_{v \in K} \|v - u\|.$$

The proximal normal cone $N_K^P(u)$ has the following characterization.

Lemma 2.1. Let K be a nonempty, closed and convex subset in H. Then $\zeta \in N_K^P(u)$, if and only if, there exists a constant $\alpha > 0$ such that

$$\zeta, v - u \leq \alpha\|v - u\|^2, \quad \forall v \in K.$$

Definition 2.2. For a given $r \in (0, \infty]$, a subset K_r is said to be normalized uniformly r-prox-regular if and only if every nonzero proximal normal cone to K_r can be realized by an r-ball, that is,$\forall u \in K_r$ and $0 \neq \xi \in N_{K_r}^P(u)$, one has

$$(\xi)/\|\xi\|, v - u \leq (1/2r)\|v - u\|^2, \quad \forall v \in K_r.$$

It is clear that the class of normalized uniformly prox-regular sets is sufficiently large to include the class of convex sets, p-convex sets, $C^{1,1}$submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets; see [3,21]. It is well-known [2,3,21] that the union of two disjoint intervals [a,b] and [c,d] is a prox-regular set with $r = \frac{c-b}{2}$. For other examples of prox-regular sets, see Noor [12,14]. Obviously, for $r = \infty$, the uniformly prox-regularity of K_r is equivalent to the convexity of K. This class of uniformly prox-regular sets have played an important part in many nonconvex applications such as optimization, dynamic systems and differential inclusions.

For the sake of simplicity, we take $\gamma = \frac{1}{r}$. Then it is clear that for $r = \infty$, we have $\gamma = 0$.

For given bifunction $F(.,.)H \times H \Longrightarrow R$ and nonlinear operator T, we consider the problem of finding $u \in K_r$ such that

$$F(u, v) + Tu, v - u + \gamma\|v - u\|^2 \geq 0,$$

$\forall v \in K_r$, which is called the *nonconvex equilibrium variational inequality.*

We note that, if $K_r \equiv K$, the convex set in H, then problem (2.1) is equivalent to finding $u \in K$ such that

$$F(u, v) + Tu, v - u \geq 0, \forall v \in K. \tag{1}$$

Inequality of type (2.1) is called the *equilibrium variational inequality,* considered and studied by Noor et al [18], Takahashi and Takahashi [23] and Yao et al [24].

For a suitable and appropriate choice of the bifunction and the spaces, one can obtain several new classes of equilibrium and variational inequalities in recent years, see [1-24] and the references therein.

3. Main Results

In this section, we use the auxiliary principle technique of Glowinski, Lions and Tremolieres [4] as developed by Noor et al[15-17] to suggest and analyze a some iterative methods for solving the nonconvex equilibrium variational inequality (2.1). We would like to mention that this technique does not involve the concept of the projection and the resolvent, which is the main advantage of this technique.

For a given $u \in K_r$ satisfying (2.1), consider the problem of finding $w \in K_r$ such that

$$\rho F(w, v) + \rho Tw, v - w + w - u,$$

$$v - w + \rho\gamma\|v - w\|^2 \geq 0, \quad \forall v \in K_r,$$

where $\rho > 0$ and $\gamma > 0$ are constants. Inequality of type (3.1) is called the auxiliary nonconvex equilibrium variational inequality. Note that if $w = u$, then w is a solution of (2.1). This simple observation enables us to suggest the following iterative method for solving the nonconvex equilibrium variational inequalities (2.1).

Algorithm 3.1. For a given $u_0 \in K_r$, compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned} &\rho F(u_{n+1}, v) + \rho Tu_{n+1}, v - u_{n+1} \\ &+u_{n+1} - u_n, v - u_{n+1} \end{aligned} \tag{2}$$

$$+\rho\gamma\|v - u_{n+1}\|^2 \geq 0,$$
$$\forall v \in K_r.$$

Algorithm 3.1 is called the proximal point algorithm for solving nonconvex equilibrium variational inequality (2.1). In particular, if $\gamma = 0$, then the uniformly prox-regular set K_r becomes the convex set K, and consequently Algorithm 3.1 reduces to:

Algorithm 3.2. For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_{n+1}, v) + \rho Tu_{n+1}, v - u_{n+1}$$

$$+u_{n+1} - u_n, v - u_{n+1} \geq 0, \quad \forall v \in K,$$

which is known as the proximal point algorithm for solving equilibrium variational inequalities (2.2) and has been studied extensively.

For the convergence analysis of Algorithm 3.1, we recall the following concepts and results.

Definition 3.1. A bifunction $F(.,.) : H \times H \to H$ is said to be *monotone*, iff

$$F(u, v-u) + F(v, u-v) \le 0, \quad \forall u, v \in H.$$

Definition 3.2. An operator $T : H \to H$ is said to be *monotone*, iff

$$Tu - Tv, u - v \ge 0, \quad \forall u, v \in H.$$

We now consider the convergence criteria of Algorithm 3.1. The analysis is in the spirit of Noor et al[15-19].

Theorem 3.1. Let the bifunction $F(.,.) : K_r \times K_r \longrightarrow H$ and the operator $T : H \longrightarrow H$ be monotone. If u_{n+1} is the approximate solution obtained from Algorithm 3.1 and $u \in K_r$ is a solution of (2.1), then

$$(1 - 4\gamma\rho)\|u - u_{n+1}\|^2 \le \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2.$$

Proof. Let $u \in K_r$ be a solution of (2.1). Then

$$-F(v, u) + Tv, u - v + \gamma\|v - u\|^2 \ge 0,$$

$$\forall v \in K_r,$$

since T and $F(.,.)$ are monotone operators.

Taking $v = u_{n+1}$ in (3.5), we have

$$-F(u_{n+1}, u) + Tu_{n+1}, u - u_{n+1} + \gamma\|u - u_{n+1}\|^2 \ge 0.$$

Setting $v = u$ in (3.2), and using (3.6), we have

$$\begin{aligned} &u_{n+1} - u_n, u - u_{n+1} \\ &\ge \rho F(u_{n+1}, u) - \rho Tu_{n+1}, u_{n+1} - u) \\ &-2\rho\gamma\|u_{n+1} - u\|^2 \ge 0. \end{aligned} \tag{3}$$

From this, one can easily obtain

$$\begin{aligned} &(1 - 4\rho\gamma)\|u - u_{n+1}\|^2 \\ &\le \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2, \end{aligned}$$

the required result (3.3).

Theorem 3.2. Let H be a finite dimension subspace and let u_{n+1} be the approximate solution obtained from Algorithm 3.1. If $u \in K_r$ is a solution of (2.1) and $\rho < \frac{1}{4\gamma}$, then $\lim_{n\longrightarrow\infty} u_n = u$.

Proof. Let $u \in K_r$ be a solution of (2.1). Then it follows from (3.4) that the sequence $\{u_n\}$ is bounded and

$$\sum_{n=0}^{\infty} \|u_n - u_{n+1}\|^2 \le \|u_0 - u\|^2,$$

which implies that

$$\lim_{n\longrightarrow\infty} \|u_n - u_{n+1}\| = 0.$$

Let $\hat{u}$ be a cluster point of the sequence $\{u_n\}$ and let the subsequence $\{u_j\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in K_r$. replacing u_n by u_{n_j} in (3.2) and taking the limit $n_j \longrightarrow \infty$ and using (3.8), we have

$$F(\hat{u}, v) + \hat{u}, v - \hat{u}$$

$$+\gamma\|v - \hat{u}\|^2 \ge 0, \quad \forall \quad v \in K_r,$$

which implies that $\hat{u}$ solves the nonconvex equilibrium variational inequality (2.1) and

$$\|u_n - u_{n+1}\|^2 \le \|\hat{u} - u_n\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and $\lim_{n\longrightarrow\infty} u_n = \hat{u}$. the required result.

We note that for $r = \infty$, the r-prox-regular set K becomes a convex set and the nonconvex equilibrium variational inequality (2.1) collapses to the equilibrium variational inequality (2.2). Thus our results include the previous known results as special cases.

It is well-known that to implement the proximal point methods, one has to calculate the approximate solution implicitly, which is itself a difficult problem. To overcome this drawback, we suggest another iterative method, the convergence of which requires only partially relaxed strongly monotonicity, which is a weaker condition that cocoercivity.

For a given $u \in K_r$ satisfying (2.1), consider the problem of finding $w \in K_r$ such that

$$\rho F(u, v) + \rho Tu, v - w + w - u,$$

$$v - w + \gamma\|v - w\|^2 \ge 0, \quad \forall \quad v \in K_r,$$

which is also called the auxiliary variational inequality. Note that problems (3.1) and (3.10) are quite different. If $w = u$, then clearly w is a solution of the nonconvex equilibrium variational inequality (2.1). This fact enables us to suggest and analyze the following iterative method for solving the nonconvex equilibrium variational inequality (2.1).

Algorithm 3.3. For a given $u_0 \in K_r$, compute the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_n, v) + \rho Tu_n, v - u_{n+1} + u_{n+1} - u_n,$$

$$v - u_{n+1} + \gamma\|v - u_{n+1}\|^2 \ge 0, \quad \forall v \in K_r.$$

Note that for $r = \infty$, the uniformly prox-regular set K_r becomes a convex set K and Algorithm 3.3 reduces to:

Algorithm 3.4. For a given $u_0 \in K$, calculate the approximate solution u_{n+1} by the iterative scheme

$$\rho F(u_n, v) + \rho Tu_n, v - u_{n+1}$$

$$+u_{n+1} - u_n, v - u_{n+1} \ge 0, \quad \forall v \in K,$$

which is known as the projection iterative method for solving equilibrium variational inequalities (2.2).

For appropriate and suitable choice of the operators and the spaces, one can suggest and analyze several iterative methods for solving the nonconvex bifunction equilibrium variational inequalities. This shows that Algorithms suggested in this paper are more general and unifying one.

Using essentially the technique of Theorem 3.1 and Theorem 3.2, one can consider the convergence analysis of Algorithm 3.3.

Conclusion and Future outlook. In this paper, we have introduced and consider the nonconvex equilibrium variational inequalities. We have used the auxiliary principle technique to suggest and analyze some iterative methods for solving theses problems. It is worth mentioning that this technique does not involve the projection or the resolvent operator technique. This is the main advantage of this method. The comparison of this technique with other methods is an interesting problem for future research. One can easily prove the similar results for the multivalued(set-valued) nonconvex equilibrium variational inequalities using the ideas and techniques of this paper. Using the technique of Noor [9], one suggest and analyze three-step iterative methods for solving the nonconvex equilibrium variational inequalities, which is another aspect of future research. We hope that the interested readers are encouraged to find the applications of the nonconvex equilibrium problems in different areas of pure and applied sciences.

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