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On the Mathematical Formulation of Fractional Derivatives

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Abstract: Fractional Calculus is a robust mathematical tool with many applications in science and physics. Nevertheless fractional derivatives fail to fulfill the properties of the common derivatives since they do not correspond to differentials. Hence, their use in geometrical and physical problems is questionable. In the present article, a new fractional derivative, the $\Lambda-$fractional derivative ($\Lambda-$FD) is proposed in order to tackle all these problems.

Keywords: Fractional calculus, fractional integral, fractional derivative, L-fractional derivative, $\Lambda-$fractional derivative.

1 Introduction

Fractional analysis is a very popular and relatively novel theory, which is applied in various fields such as mechanics (Drapaca et al. [1], Di Paola et al. [2] ,Carpinteri et al. [3]), physics (Hilfer [4], West et.al [5]), engineering, (Tarasov [6, 7], Baleanu [8], Sumelka [9]), biology Magin [10], control theory etc. Its popularity is based on the belief that Fractional Calculus may describe various physical phenomena in a better and more flexible way, especially when those phenomena are connected to fractal geometry Tatom [11]. Furthermore some studies exist involving fractional calculus in variation procedures that are necessary for working in physical problems (Agrawal [12], Muslih et.al [13]). Moreover, variational problems might be raised with the use of fractional derivatives, since Noether's variational theory might be questionable and the adoption of those derivatives might not satisfy geometrical and physical demands (Atanackovic et al. [14, 15], Frederico et al. [16]). However, all those procedures may be questionable, since fractional derivatives do not correspond to differentials and in fact they are not real derivatives but mathematical operators. It is clear that fractional analysis fails to perform immediate operations, like Leibniz rule for derivatives and various composition rules,(see Tarasov [17]). These operations are demanded by differential topology in order for a derivative to exist Chillingworth [18]. Trying to present a fractional derivative that conforms with the rules demanded by differential topology, Lazopoulos [19] proposed the L-fractional derivative. Nevertheless, that derivative did not completely serve that purpose. In the present work, a new fractional derivative is defined, along with a new space, the Λ-fractional space. That fractional derivative with its global character in the initial space, becomes local in the fractional Λ-space. Therefore, everything concerning differential geometry, field theories, equations, potentials, etc may be developed in the fractional Λ-space. Solving the problem in the Λ-space, the results may be pulled back to the initial space. Let us notice that the tangent spaces in the Λ-space correspond to surfaces that are tangent surfaces in the initial space. Applications are presented discussing the tangent spaces of a simple curve and solving a simple differential equation as well. In those applications the game between the initial and the Λ- fractional space will be clarified.

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2 Fractional calculus: the L-fractional derivative

The L-FD was first defined in Lazopoulos et al. [19]as the derivative that occurs from Adda's [20] definition of a fractional differential (see Lazopoulos et al. [21,22]):

$$d^{\gamma}f(x) = {}_{a}^{L}D_{x}^{\gamma}f(x)d^{\gamma}x, \tag{1}$$

where $d^{\gamma}f(x)$ is the fractional differential of function f(x) and $d^{\gamma}x$ the fractional differential of x. The first definition of the L-FD was stated as the ratio of the corresponding Caputo derivatives:

$${}_{a}^{L}D_{x}^{\gamma}f(x) = \frac{{}_{a}^{C}D_{x}^{\gamma}f(x)}{{}_{a}^{C}D_{x}^{\gamma}x}. \tag{2}$$

Nevertheless, in this article we revised the L-FD, adopting the version of Caputo derivatives with the use of Riemann-Liouville derivatives:

$${}_{a}^{L}D_{x}^{\gamma}f(x) = \frac{{}_{a}^{RL}D_{x}^{\gamma}(f(x)-f(a))}{{}_{a}^{RL}D_{x}^{\gamma}(x-a)}. \tag{3}$$

3 The proposed fractional space

The main disadvantage of the existing fractional derivatives is that they fail to satisfy Leibniz and composition rules. Therefore, they cannot correspond to a differential. As a consequence, the use of the various fractional derivatives in geometrical and physical problems is questionable. On the other hand, the proposed formulation yields derivatives corresponding to differentials, fulfilling all the necessary conditions demanded by a derivative. Hence, it could be feasible to create a fractional differential geometry theory, not in the original space but in a new space which will be defined below. Let us consider the Caputo derivative, of a function f(x):

$${}_{a}^{C}D_{x}^{\gamma}f(x) = \frac{d}{dx}({}_{a}I_{x}^{1-\gamma}(f(x)-f(a)), \tag{4}$$

where

$${}_{a}I_{x}^{1-\gamma}f(x) = \frac{1}{\Gamma(1-\gamma)}\int_{a}^{x}\frac{f(s)}{(x-s)^{\gamma}}ds. \tag{5}$$

The Caputo fractional derivative is expressed as a common derivative of the integral ${}_{a}I_{x}^{1-\gamma}(f(x)-f(a))$. The L-fractional derivative introduced by Lazopoulos [21], might be adopted without the problems that Caputo derivative exhibits. The Λ-fractional derivative is thus defined by:

$${}_{a}^{\Lambda}D_{x}^{\gamma}f(x) = \frac{{}_{a}^{C}D_{x}^{\gamma}f(x)}{{}_{a}^{C}D_{x}^{\gamma}x} = \frac{d({}_{a}I_{x}^{1-\gamma}(f(x)-f(a)))}{d({}_{a}I_{x}^{1-\gamma}(x-a))}. \tag{6}$$

Introducing the new variable

$$X = {}_{a}I_{x}^{1-\gamma}(x-a) \tag{7}$$

and imposing $F(X) = {}_{a}I_{x}^{1-\gamma}f((x(X))-f(a))$, the $\Lambda-$fractional derivative, (6) may be expressed as a common derivative of the function F(X) with respect to X. Hence the analysis has been transferred to plane (X, F(X)) with the conventional differential analysis. That space is called the $\Lambda-$space where in fact the fractional derivatives with their global character in the initial space have been transformed in conventional derivatives in the $\Lambda-$space with local character. Then the results may be transferred back to the initial space. Following that procedure the fractional differential geometry and fractional field theory may be established. The proposed Λ-fractional derivative satisfies Leibniz's rule for the product of the ${}_{a}I_{x}^{1-\gamma}(f(x)-f(a))$ and ${}_{a}I_{x}^{1-\gamma}(y(x)-y(a))$.Indeed,

$$\frac{d({}_{a}I_{x}^{1-\gamma}[f(x)].{}_{a}I_{x}^{1-\gamma}[y(x)])}{d({}_{a}I_{x}^{1-\gamma}[x])} = \frac{d({}_{a}I_{x}^{1-\gamma}[f(x)])}{d({}_{a}I_{x}^{1-\gamma}[x])}{}_{a}I_{x}^{1-\gamma}[y(x)] + {}_{a}I_{x}^{1-\gamma}[f(x)]\frac{d({}_{a}I_{x}^{1-\gamma}[y(x)])}{d({}_{a}I_{x}^{1-\gamma}[x])}. \tag{8}$$

Here, from the present point ${}_aI_x^{1-\gamma}[g(x)]$ we mean

$$ {}_aI_x^{1-\gamma}[g(x)] = {}_aI_x^{1-\gamma}(g(x)-g(a)). \tag{9}$$

Furthermore, the fractional derivative of a composite function is defined by

$$\frac{d({}_aI_x^{1-\gamma}[f(y(x))]}{d({}_aI_x^{1-\gamma}[x])} = \frac{d({}_aI_x^{1-\gamma}[f(y))]}{d({}_aI_x^{1-\gamma}[y])} \cdot \frac{d({}_aI_x^{1-\gamma}[y(x)])}{d({}_aI_x^{1-\gamma}[x])}. \tag{10}$$

The linearity properties along with Leibniz and composition rules, (8,9), satisfy all the Differential Topology conditions, Chillingworth [18], for the existence of a differential and further the existence of Fractional Differential Geometry. The results in the Λ-space are pulled back through the well known identity,

$$ {}_a^CD_x^{\gamma}({}_aI_x^{1-\gamma}f(x)) = f(x). \tag{11}$$

It is quite striking that the tangent spaces in the Λ-space turn out to be transferred in the initial space as tangent surfaces. Yet, some of the most established books in fractional analysis are referred [24,25,26], just to compare the present version of the Fractional Analysis to the existing one. The application that is presented in the following paragraph clarifies the procedure.

4 Application

Let us consider the function

$$f(x) = x^4. \tag{12}$$

Following the procedure, we get

$$X = {}_0I_x^{1-\gamma}x = \frac{1}{\Gamma(1-\gamma)}\int_a^x \frac{s}{(x-s)^{\gamma}}ds = \frac{1}{\Gamma(1-\gamma)}\frac{x^{2-\gamma}}{2-3\gamma+\gamma^2} \tag{13}$$

and hence,

$$x = ({}_0I_x^{1-\gamma})^{(-1)}(X). \tag{14}$$

In fact, taking into consideration (13,14) yields

$$x = ((2-3\gamma+\gamma^2)\Gamma(1-\gamma)X)^{\frac{1}{2-\gamma}}. \tag{15}$$

Further, we have

$$F(X) = {}_0I_x^{1-\gamma}f(x(X)) = \frac{1}{\Gamma(1-\gamma)}\int_a^x \frac{s^4}{(x-s)^{\gamma}}ds = \frac{24}{\Gamma(6-\gamma)}x^{5-\gamma}. \tag{16}$$

Introducing (15) into (16) we get

$$F(X) = \frac{24}{\Gamma(1-\gamma)}((2-3\gamma+\gamma^2)\Gamma(1-\gamma)X)^{\frac{5-\gamma}{2-\gamma}}. \tag{17}$$

Therefore, from the original space (x, f(x)), we are transferred to the fractional Λ-space (X, F(X)) , where the fractional Λ-derivative although in the initial space has a non-local character, in the Λ-space preserves the local character as a conventional one. Performing the analysis on that space following conventional approaches, we are transferred back to the original space (x,f(x) through the relation (11).

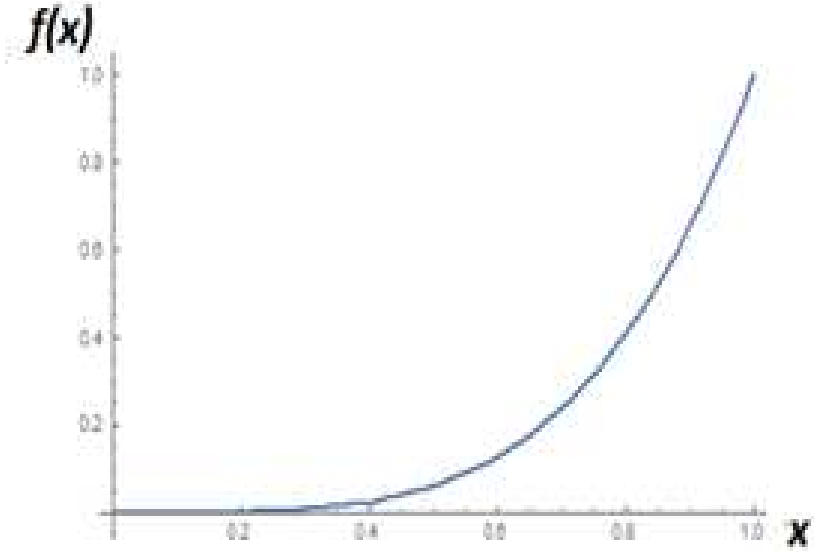

Figure 1.The curve in the original (x, f(x)) plane..

However, in the fractional space (X,F(X)) the curve is shown in Fig.2 for fractional dimension γ=0.6 and a=0.

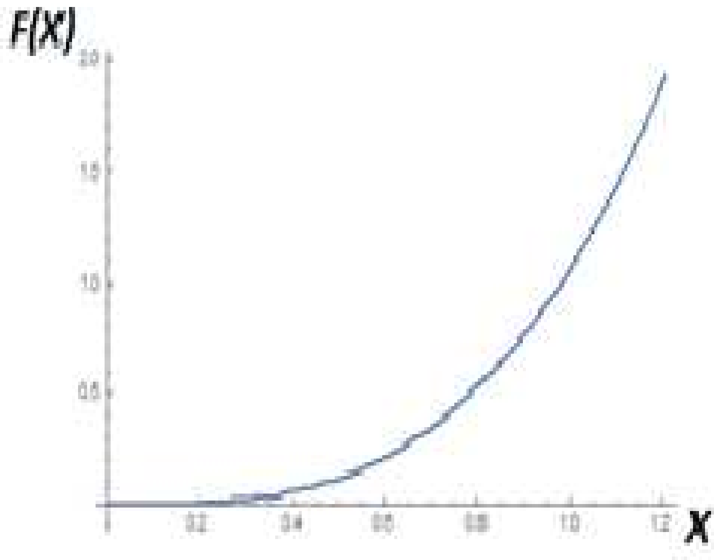

Figure 2. The curve in the Fractional Λ-space (X,F(X)) for γ=0.6 and a=0.

Further, the derivative in the fractional Λ -space is defined by

$$D(F(X)) = \frac{dF(X)}{dX}. \tag{18}$$

With the help of Mathematica we may find that

$$D(F(X)) = \frac{24(5-\gamma)(2-3\gamma+\gamma^2)((2-3\gamma+\gamma^2)\Gamma(1-\gamma)X)^{\frac{\gamma-1}{2-\gamma}}((2-3\gamma+\gamma^2)\Gamma(1-\gamma)X)^{\frac{4-\gamma}{2-\gamma}}}{(2-\gamma)\Gamma(6-\gamma)}. \tag{19}$$

For X_0=0.6 and γ =0.6 and a=0, the derivative in the fractional plane is equal to D(F(X_0))= 1.1190. Hence, the tangent space Y(X) of the curve at a point X_0 is defined by the line,

$$Y(X) = F(X_0) + D(F(X_0))(X - X_0) \tag{20}$$

and according to (17), F(0.6)=0.2136, the tangent upon at the point (0.6, 0.2136) of the curve in the fractional Λ-plane is defined by,

$$Y(X) = 0.2136 + 1.1190(X - 0.6). \tag{21}$$

The curve with its tangent space in the Fractional plane is shown in Fig.3.

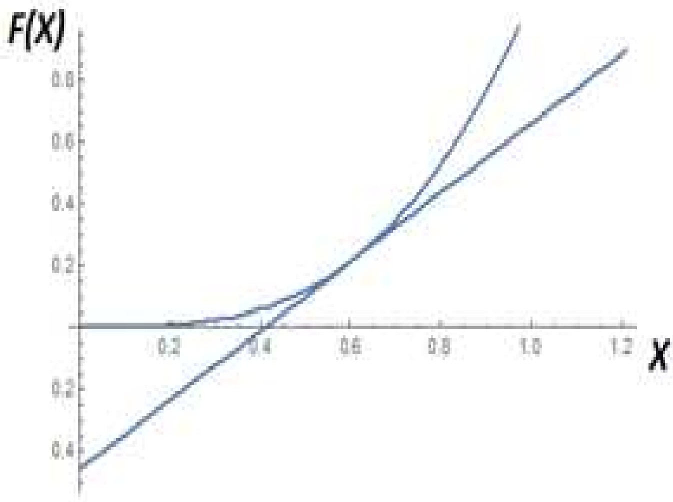

Figure 3. The curve with its tangent space in the fractional Λ-plane.

Now trying to transfer back the tangent space of the curve in the initial space (x,f(x)), we may define the corresponding curve in the initial plane through (20), simply substituting X by the RHS of(12). In the present case with γ=0.6 and a=0,

$$X = 0.8050x^{1.4}. \tag{22}$$

Taking into consideration the correspondence between x and X, (15), the point X_0=0.6 for γ=0.6 corresponds to the point x_0=0.8106 in the initial axis x. Therefore, configurating the tangent space Y(X) into the original space we find the curve,

$$y(x) = f(x)_{x=0.8106} + ({}_a^C D_{0.8106}^{1-\gamma}(\frac{dF(X)}{dX})_{\gamma=0.6})(\frac{x^{2-\gamma}}{(2-3\gamma+\gamma^2)\Gamma(1-\gamma)} - 0.6)_{\gamma=0.6}. \tag{23}$$

Performing the necessary calculus with the help of the Mathematica pack we define the curve y(x) by the equation,

$$y(x) = 0.4317 + 1.9645(0.8050x^{1.4} - 0.6). \tag{24}$$

That curve is the transferred back to the initial space of the tangent space in the Λ-space at X=0.6 that for γ=0.6 corresponds to x=0.8106,(21).

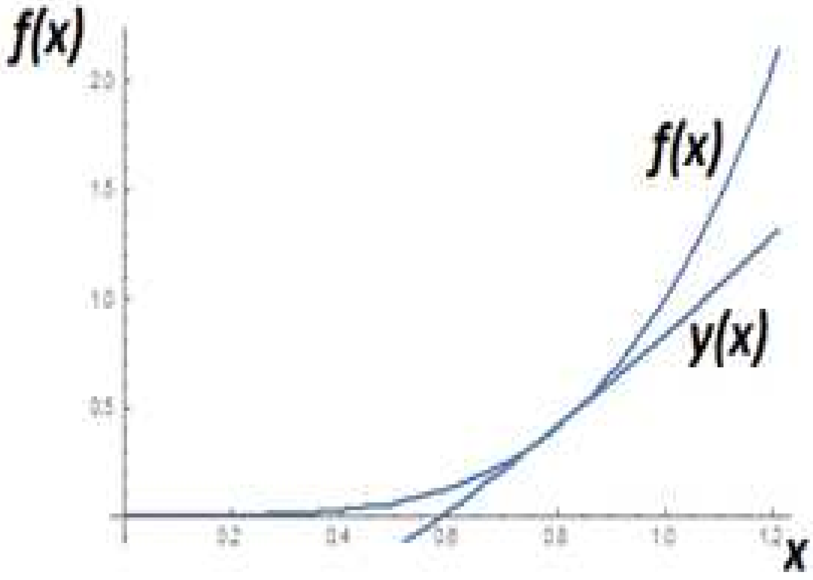

Figure 4. The curve f(x) and the tangent curve y(x) at the initial space.

It is not only clear but also quite striking that the tangent space in the Λ-space is transferred as tangent curve to the original curve.

5 Conclusion

A new fractional derivative (Λ-fractional derivative) is introduced, that has all the derivative properties demanded by the differential topology. Those derivatives are defined in a new space, the fractional Λ-space, where fractional differential geometry and fractional field theory as well may be established. Working on the Λ-fractional space, where the Λ-fractional derivative behaves in the conventional way, the results may be pulled back to the original space. The present formulation of fractional derivative yields reliable results in various variation procedures too.

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