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Fixed Point Theorem for Cyclic Maps on Partial Metric spaces

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Abstract: In this paper, a class of cyclic contractions on partial metric spaces is introduced. A fixed point theorem for cyclic contractions on partial metric spaces satisfying (ψ, ϕ) contractive condition, and illustrative examples are given.

Keywords: Partial metric space, Fixed point theorem, Cyclic map

1. Introduction and Preliminaries

Fixed point theory has been rapidly developing field since the pioneering work of Banach in 1922 [5]. A great number of studies concerning fixed points of contractions on different spaces have been reported. Among these spaces are the metric spaces, quasi-metric spaces [6,9] cone metric spaces [10,11], Menger (statistical) spaces [20], fuzzy metric spaces [17]. In 1992 Matthews [18],[19] introduced a relatively new space called Partial metric space (PMS) and proved the analog of the Banach fixed point theorem on this space. The wide application potential of PMS resulted in immediate publications in the area [21], [26], [3], [4].

Cyclic maps and best proximity points have been introduced by Kirk-Srinavasan-Veeramani [16] in 2003. Various results on cyclic maps have been obtained since then (See e.g. [2,8,7,13,14,23,22,24]).

The purpose of this study is to investigate existence and uniqueness of fixed points of cyclic maps on Partial metric spaces. Therefore, we first define cyclic maps on Partial metric spaces. Then we give a fixed point theorem for cyclic maps satisfying (ψ, ϕ) contractive conditions, where ψ and ϕ are the so-called altering distance functions introduced by Khan et all [15]. This theorem is an analog of the theorem given recently by Shatanawi [25] on metric spaces.

Partial metric space is defined by Matthews as follows (See [18])

Definition 1. *Let X be a nonempty set and let $p : X \times X \to [0,\infty)$ satisfy*

$$\begin{array}{l} (PM1)\ x = y \Leftrightarrow p(x,x) = p(y,y) = p(x,y) \\ (PM2)\ p(x,x) \le p(x,y) \\ (PM3)\ p(x,y) = p(y,x) \\ (PM4)\ p(x,y) \le p(x,z) + p(z,y) - p(z,z) \end{array} \tag{1}$$

for all x, y and $z \in X$. Then the pair (X,p) is called a partial metric space and p is called a partial metric on X.

It can be easily verified that the function $d_p : X \times X \to R^+$ defined by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{2}$$

satisfies the conditions of a metric on X. On the other hand each partial metric p on X generates a T_0 topology τ_p on X, whose base is a family of open p-balls

$$\{B_p(x,\epsilon) : x \in X, \epsilon > 0\}$$

where $B_p(x,\epsilon) = \{y \in X : p(x,y) \le p(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

Definitions of convergence, Cauchy sequence, completeness and continuity on partial metric spaces are given as follows [18].

Definition 2.

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1. *A sequence $\{x_n\}$ in the PMS (X,p) converges to the limit x if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.*
2. *A sequence $\{x_n\}$ in the PMS (X,p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.*
3. *A PMS (X,p) is called complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p, to a point $x\in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.*
4. *A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $F(B_p(x_0,\delta)) \subseteq B_p(Fx_0,\epsilon)$.*

The following lemma is one of the basic results in Partial metric spaces [18,19,4]).

Lemma 1.

1. *A sequence $\{x_n\}$ is a Cauchy sequence in the PMS (X,p) if and only if it is a Cauchy sequence in the metric space (X,d_p).*
2. *A PMS (X,p) is complete if and only if the metric space (X,d_p) is complete. Moreover*

$$\begin{aligned}&\lim_{n\to\infty} d_p(x,x_n) = 0 \Leftrightarrow \\ &p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m).\end{aligned} \tag{3}$$

Next, we give two lemmas stated and proved in [12,1] which will be used in the proofs of our main results.

Lemma 2. *Assume that $x_n \to z$ as $n\to\infty$ in a PMS (X,p) such that $p(z,z) = 0$. Then $\lim_{n\to\infty} p(x_n,y) = p(z,y)$ for every $y\in X$.*

Lemma 3. *Let (X,p) be a complete PMS. Then*

(A) If $p(x,y) = 0$ then $x = y$,
(B) If $x \neq y$, then $p(x,y) > 0$.

Cyclic maps and best proximity points defined in [16] have been studied thoroughly on various spaces.

Definition 3. *Let A and B be non-empty subsets of a metric space (X,d) and $T : A\cup B \to A\cup B$. T is called cyclic map if $T(A) \subset B$ and $T(B) \subset A$.*

A point $x \in A\cup B$ is called a best proximity point if $d(x,Tx) = d(A,B)$ where $d(A,B) = \inf\{d(a,b) : a\in A, b\in B\}$.

Altering distance functions have been introduced by Khan et all [15].

Definition 4. *The function $\phi : [0,\infty) \longrightarrow [0,\infty)$ is called an altering distance function if it satisfies the following conditions:*

1. *ϕ is continuous and nondecreasing.*
2. *$\phi(t) = 0$ if and only if $t = 0$.*

2. Main Results

In this section we define cyclic contractions satisfying so-called (ψ,ϕ) conditions on partial metric spaces and state the fixed point theorem for these maps.

Definition 5. *Let A and B be non-empty subsets of a partial metric space (X,p) and let ψ and ϕ be altering distance functions. A cyclic map $T : A\cup B \to A\cup B$ is said to be (ψ,ϕ) contractive if it satisfies*

$$\psi(p(Tx,Ty)) \le \psi(m) - \phi(m), \tag{4}$$

where

$$m = \max\{p(x,y), p(Tx,x), p(Ty,y)\} \tag{5}$$

for all $x\in A$ and $y \in B$.

Theorem 1. *Let A and B be non-empty closed subsets of a complete partial metric space (X,p). Assume that $T : A\cup B \to A\cup B$ is a (ψ,ϕ) contractive map. Then T has a unique fixed point in $A\cap B$.*

Proof. We first prove the existence part. Take an arbitrary $x_0 \in A$ and define the sequence $\{x_n\}$ as

$$x_n = Tx_{n-1}, \quad n = 1,2,3,\ldots. \tag{6}$$

Since T is cyclic, the subsequence $\{x_{2k}\} \subset A$ and the subsequence $\{x_{2k+1}\} \subset B$. If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbf{N}$, then obviously, the fixed point of T is x_{n_0}. Assume that $x_{n+1} \neq x_n$ for all $n\in\mathbf{N}$.

Suppose that n is even, i.e., $n = 2k$. Upon substitution $x = x_{2k}$ and $y = x_{2k+1}$ in (4) we obtain

$$\begin{aligned}\psi(p(x_{2k+1},x_{2k+2})) &= \psi(p(Tx_{2k},Tx_{2k+1})) \\ &\le \psi(m_{2k}) - \phi(m_{2k}),\end{aligned} \tag{7}$$

where m_n is defined as

$$\begin{aligned}m_n &= \max\{p(x_n,x_{n+1}), p(Tx_n,x_n), p(Tx_{n+1},x_{n+1})\} \\ &= \max\{p(x_n,x_{n+1}), p(x_{n+1},x_{n+2})\}.\end{aligned} \tag{8}$$

Suppose that

$$m_{2k} = p(x_{2k+1},x_{2k+2}), \tag{9}$$

which implies

$$\begin{aligned}\psi(p(x_{2k+1},x_{2k+2})) \le\ &\psi(p(x_{2k+1},x_{2k+2})) \\ &- \phi(p(x_{2k+1},x_{2k+2})).\end{aligned} \tag{10}$$

From this inequality it follows that $\phi(p(x_{2k+1},x_{2k+2})) = 0$, and hence, $p(x_{2k+1},x_{2k+2}) = 0$ since ϕ is an altering distance function. Thus, from Lemma 3, we have $x_{2k+1} = x_{2k+2} = Tx_{2k+1}$ which contradicts our assumption that $Tx_n \neq x_n$. Therefore, we must have

$$m_{2k} = p(x_{2k},x_{2k+1}), \tag{11}$$

and hence,

$$\begin{aligned}\psi(p(x_{2k+1},x_{2k+2})) \le\ &\psi(p(x_{2k},x_{2k+1})) \\ &- \phi(p(x_{2k},x_{2k+1})).\end{aligned} \tag{12}$$

In addition we have,

$$p(x_{2k+1}, x_{2k+2}) \leq p(x_{2k}, x_{2k+1}), \tag{13}$$

for every $k \in \mathbf{N}$.

Assume now that n is odd, i.e., $n = 2k+1$. Then the inequality (4) with $x = x_{2k+1}$ and $y = x_{2k+2}$ becomes

$$\begin{aligned}\psi(p(x_{2k+2}, x_{2k+3})) &= \psi(p(Tx_{2k+1}, Tx_{2k+2})) \\ &\leq \psi(m_{2k+1}) - \phi(m_{2k+1}).\end{aligned} \tag{14}$$

If

$$m_{2k+1} = p(x_{2k+2}, x_{2k+3}), \tag{15}$$

then

$$\begin{aligned}\psi(p(x_{2k+2}, x_{2k+3})) \leq \psi(p(x_{2k+2}, x_{2k+3})) \\ - \phi(p(x_{2k+2}, x_{2k+3})).\end{aligned} \tag{16}$$

From (16) it follows that $\phi(p(x_{2k+2}, x_{2k+3})) = 0$, thus, $p(x_{2k+2}, x_{2k+3}) = 0$. From Lemma 3, we have $x_{2k+2} = x_{2k+3} = Tx_{2k+2}$ which is a contradiction with the assumption $Tx_n \neq x_n$. Therefore,

$$m_{2k+1} = p(x_{2k+1}, x_{2k+2}), \tag{17}$$

and hence,

$$\begin{aligned}\psi(p(x_{2k+2}, x_{2k+3})) \leq \psi(p(x_{2k+1}, x_{2k+2})) \\ - \phi(p(x_{2k+1}, x_{2k+2})).\end{aligned} \tag{18}$$

We also have

$$p(x_{2k+2}, x_{2k+3}) \leq p(x_{2k+1}, x_{2k+2}), \tag{19}$$

for every $k \in \mathbf{N}$. From (13) and (19) we deduce that the sequence $\{p(x_n, x_{n+1})\}$, is nonincreasing and nonnegative. Thus,

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = s \tag{20}$$

for some $s \geq 0$.

On the other hand, from (12) and (18) we have

$$\psi(p(x_{n+2}, x_{n+1})) \leq \psi(p(x_{n+1}, x_n)) - \phi(p(x_{n+1}, x_n)) \tag{21}$$

for all $n \in \mathbf{N}$. Taking limit as $n \to \infty$ of both sides of (21) and regarding the continuity of the functions ψ and ϕ we obtain

$$\psi(s) \leq \psi(s) - \phi(s) \tag{22}$$

which cannot hold unless $s = 0$. As a result, we get

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0 \tag{23}$$

and using $PM2$,

$$0 \leq p(x_n, x_n) \leq p(x_n, x_{n+1}) \tag{24}$$

we have

$$\lim_{n\to\infty} p(x_n, x_n) = 0. \tag{25}$$

We will prove now that $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_p), where d_p is defined in (2). We first show that $\{x_{2n}\}$ is a Cauchy sequence in (X, d_p). Suppose the contrary, that is $\{x_{2n}\}$ is not Cauchy. Then, for some $\varepsilon > 0$ there exist subsequences $\{x_{2n(i)}\}$ and $\{x_{2m(i)}\}$ of $\{x_{2n}\}$ such that

$$d_p(x_{2n(i)}, x_{2m(i)}) \geq \varepsilon, \quad n(i) > m(i) > i, \tag{26}$$

where we take $n(i)$ as the smallest index satisfying (26). Then we have

$$d_p(x_{2n(i)-2}, x_{2m(i)}) < \varepsilon. \tag{27}$$

It is easy to see that

$$\lim_{n\to\infty} d_p(x_n, x_{n+1}) = 0 \tag{28}$$

by taking limit as $n \to \infty$ in

$$\begin{aligned}d_p(x_n, x_{n+1}) = 2p(x_n, x_{n+1}) - p(x_n, x_n) \\ - p(x_{n+1}, x_{n+1}).\end{aligned} \tag{29}$$

Using the triangle inequality we get

$$\begin{aligned}\varepsilon &\leq d_p(x_{2m(i)}, x_{2n(i)}) \\ &\leq d_p(x_{2m(i)}, x_{2n(i)-2}) + d_p(x_{2n(i)-2}, x_{2n(i)-1}) \\ &+ d_p(x_{2n(i)-1}, x_{2n(i)}) \\ &\leq \varepsilon + d_p(x_{2n(i)-2}, x_{2n(i)-1}) + d_p(x_{2n(i)-1}, x_{2n(i)}).\end{aligned} \tag{30}$$

We obtain

$$\lim_{i\to\infty} d_p(x_{2n(i)}, x_{2m(i)}) = \lim_{i\to\infty} 2p(x_{2n(i)}, x_{2m(i)}) = \varepsilon \tag{31}$$

upon taking limit $i \to \infty$ in (30). On the other hand, using triangle inequality we have,

$$\begin{aligned}\varepsilon &\leq d_p(x_{2m(i)}, x_{2n(i)} \\ &\leq d_p(x_{2m(i)}, x_{2n(i)+1}) + d_p(x_{2n(i)+1}, x_{2n(i)}) \\ &\leq d_p(x_{2m(i)}, x_{2n(i)}) + 2d_p(x_{2n(i)+1}, x_{2n(i)})\end{aligned} \tag{32}$$

and also

$$\begin{aligned}\varepsilon &\leq d_p(x_{2n(i)}, x_{2m(i)}) \\ &\leq d_p(x_{2n(i)}, x_{2m(i)-1}) + d_p(x_{2m(i)-1}, x_{2m(i)}) \\ &\leq d_p(x_{2m(i)}, x_{2n(i)}) + 2d_p(x_{2m(i)-1}, x_{2m(i)}).\end{aligned} \tag{33}$$

Taking limit as $i \to \infty$ in the inequalities (32) and (33) and using (28), and (31), we get

$$\begin{aligned}&\lim_{i\to\infty} d_p(x_{2m(i)}, x_{2n(i)+1}) \\ &= \lim_{i\to\infty} 2p(x_{2m(i)}, x_{2n(i)+1}) = \varepsilon\end{aligned} \tag{34}$$

and

$$\begin{aligned}&\lim_{i\to\infty} d_p(x_{2m(i)-1}, x_{2n(i)}) \\ &= \lim_{i\to\infty} 2p(x_{2m(i)-1}, x_{2n(i)}) = \varepsilon.\end{aligned} \tag{35}$$

Now, upon substitution $x = x_{2m(i)-1}$ and $y = x_{2n(i)}$ in (4) we have

$$\begin{aligned}\psi(p(x_{2m(i)}, x_{2n(i)+1})) &= \psi(p(Tx_{2m(i)-1}, Tx_{2n(i)})) \\ &\leq \psi(m) - \phi(m).\end{aligned} \tag{36}$$

where

$$\begin{aligned}m = \max\{&p(x_{2m(i)-1}, x_{2n(i)}), p(x_{2m(i)}, x_{2m(i)-1}), \\ &p(x_{2n(i)+1}, x_{2n(i)})\}\end{aligned}$$

Letting $i \to \infty$ in the above inequality and regarding (23), (31), (34) and (35) we obtain

$$\psi(\frac{\varepsilon}{2}) \le \psi(\frac{\varepsilon}{2}) - \phi(\frac{\varepsilon}{2}) \tag{37}$$

which clearly implies $\varepsilon = 0$. However, this contradicts the assumption that $\{x_{2n}\}$ is not Cauchy in (X, d_p). Then, $\{x_{2n}\}$ is a Cauchy sequence. Since (X, p) is complete, then (X, d_p) is also complete by Lemma 1 and hence, $\{x_{2n}\} \subset A$ converges to a limit, say $x \in A$. Using similar arguments, we can prove that $\{x_{2n+1}\}$ is a Cauchy sequence in B. Therefore, $\{x_{2n+1}\} \subset B$ converges to the a limit, say $y \in B$. Then,

$$\lim_{n\to\infty} d_p(x_{2n}, x) = 0, \quad \text{and} \quad \lim_{n\to\infty} d_p(x_{2n+1}, y) = 0. \tag{38}$$

It is clear that

$$\begin{aligned} 0 &\le d_p(x,y) \\ &\le d_p(x, x_{2n}) + d_p(x_{2n}, x_{2n+1}) + d_p(x_{2n+1}, y). \end{aligned} \tag{39}$$

Taking limit as $n \to \infty$ in and using (28) and (38) we obtain $d_p(x, y) = 0$, that is $x = y$. Thus, both subsequences $\{x_{2n}\} \subset A$ and $\{x_{2n+1}\} \subset B$ converge to the same limit x and moreover,

$$\{x_{2n}\} \cup \{x_{2n+1}\} = \{x_n\}.$$

Hence, the sequence $\{x_n\} \in X$ converges to $x \in X$.

On the other hand, by Lemma 1 we observe that

$$\begin{aligned} &\lim_{n\to\infty} d_p(x, x_n) = 0 \quad \text{if and only if} \\ p(x,x) &= \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m) \end{aligned} \tag{40}$$

From (4) with $x = x_n$ and $y = x_m$, we have

$$\begin{aligned} &\psi(p(x_{n+1}, x_{m+1})) \\ &\le \psi(\max\{p(x_n, x_m), p(x_{n+1}, x_n), p(x_{m+1}, x_m)\}) \\ &-\phi(\max\{p(x_n, x_m), p(x_{n+1}, x_n), p(x_{m+1}, x_m)\}). \end{aligned} \tag{41}$$

Letting $n, m \to \infty$, and using (23) and (40) we get

$$\psi(p(x,x)) \le \psi(p(x,x)) - \phi(p(x,x)) \tag{42}$$

which immediately implies $p(x, x) = 0$.

Consider now (4) with $x = x_n$, $y = x$. Then we have,

$$\begin{aligned} &\psi(p(Tx_n, Tx)) \\ &\le \psi(\max\{p(x_n, x), p(x_{n+1}, x_n), p(Tx, x)\}) \\ &-\phi(\max\{p(x_n, x), p(x_{n+1}, x_n), p(Tx, x)\}). \end{aligned} \tag{43}$$

Letting $n \to \infty$, we obtain

$$\psi(p(x, Tx)) \le \psi(p(Tx, x)) - \phi(p(Tx, x)) \tag{44}$$

which implies $\psi(p(x, Tx)) = 0$, hence $p(x, Tx) = 0$. According to Lemma (3), $x = Tx$, that is, $x \in A \cap B$ is the fixed point of T.

To prove the uniqueness, we assume that $z \in X$ is another fixed point of T such that $z \neq x$. Then from (4) with $x = x$ and $y = z$ we have

$$\begin{aligned} &\psi(p(Tx, Tz)) = \psi(p(x, z)) \\ &\le \psi(\max\{p(x, z), p(Tx, x), p(Tz, z)\}) \\ &-\phi(\max\{p(x, z), p(Tx, x), p(Tz, z)\}), \end{aligned} \tag{45}$$

which leads to

$$\psi(p(x,z)) \le \psi(p(x,z)) - \phi(p(x,z)), \tag{46}$$

and hence to $p(x, z) = 0$ implying $x = z$ by Lemma 3. Thus, the fixed point of T is unique.

We next give the following examples of cyclic maps satisfying the conditions of the Theorem 1.

*Example 1.*Let $X = [0, 1]$ and $A = B = [0, 1]$. Define $T : X \to X$ as $Tx = \frac{x}{3}$. For $p(x, y) = \max\{x, y\}$ the space (X, p) is a partial metric space. Define $\psi(t) = 3t$ and $\phi(t) = t$. Then the conditions of the Theorem 1 are satisfied and therefore T has a unique fixed point. Indeed, $x = 0$ is the unique fixed point of T. Observe that, in $[0, 1]$, $Tx \le x$ and if $x \le y$, then

$$\begin{aligned} p(x,y) &= y \\ p(Tx,x) &= x \\ p(Ty,y) &= y \\ p(Tx,Ty) &= Ty, \end{aligned}$$

which implies

$$\max\{p(x,y), p(Tx,x), p(Ty,y)\} = y.$$

Therefore,

$$\begin{aligned} &\psi(p(Tx,Ty)) = \psi(Ty) = 3\tfrac{y}{3} \\ &= y \le \psi(y) - \phi(y) = 3y - y = 2y. \end{aligned}$$

As a second example we give a piecewise continuous cyclic map.

*Example 2.*Let $X = [-1, 1]$ and $A = [-1, 0]$ and $B = [0, 1]$. Define $T : X \to X$ as follows:

$$Tx = \begin{cases} -\dfrac{x}{3} & \text{if } x \in [-1, 0] \\[2ex] -\dfrac{x}{2} & \text{if } x \in [0, 1] \end{cases}$$

Define $p(x, y) = \max\{|x|, |y|\}$ and $\psi(t) = t$, $\phi(t) = \frac{t}{2}$. The map T satisfies the conditions of Theorem 1 and has a unique fixed point. Indeed, let $x \in A$ and $y \in B$. Then, all possible cases can be listed as follows:

If $|x| \le |y|$, then

$$\begin{aligned} &\max\{p(x,y), p(Tx,x), p(Ty,y)\} \\ &= \max\{|y|, |x|, |y|\} = |y|. \end{aligned}$$

Since $p(Tx, Ty) = \max\{\frac{|x|}{3}, \frac{|y|}{2}\} = \frac{|y|}{2}$ then we have,

$$\frac{|y|}{2} \le |y| - \frac{|y|}{2} = \frac{|y|}{2}.$$

Now, if $|y| \le |x|$, then

$$\begin{aligned} &\max\{p(x,y), p(Tx,x), p(Ty,y)\} \\ &= \max\{|x|, |x|, |y|\} = |x|. \end{aligned}$$

If $p(Tx,Ty) = \max\{\frac{|x|}{3}, \frac{|y|}{2}\} = \frac{|y|}{2}$ then we get,

$$\frac{|y|}{2} \leq |x| - \frac{|x|}{2} = \frac{|x|}{2}.$$

If $p(Tx,Ty) = \max\{\frac{|x|}{3}, \frac{|y|}{2}\} = \frac{|x|}{3}$ then we get,

$$\frac{|x|}{3} \leq |x| - \frac{|x|}{2} = \frac{|x|}{2}.$$

One can easily see that the fixed point of the map T is $x = 0$ and $0 \in A \cap B$.

Last, we state some particular cases of the Theorem 1 by choosing the altering distance functions ψ and ϕ in special ways.

Corollary 1. *Let A and B be non-empty closed subsets of a complete partial metric space (X,p). Assume that $T : A \cup B \to A \cup B$ is cyclic map satisfying*

$$p(Tx,Ty) \leq k \max\{p(x,y), p(Tx,x), p(Ty,y)\} \tag{47}$$

for all $x \in A$ and $y \in B$ where $0 \leq k < 1$. Then T has a unique fixed point in $A \cap B$.

Proof. Define the altering distance functions as

$$\psi(t) = t \quad \text{and} \quad \phi(t) = (1-k)t.$$

Then by Theorem 1 T has a unique fixed point.

Corollary 2. *Let A and B be non-empty closed subsets of a complete partial metric space (X,p). Assume that $T : A \cup B \to A \cup B$ is cyclic map satisfying*

$$p(Tx,Ty) \leq k\{p(x,y) + p(Tx,x) + p(Ty,y)\} \tag{48}$$

for all $x \in A$ and $y \in B$ where $0 \leq k < 1$. Then T has a unique fixed point in $A \cap B$.

Proof. Since

$$\begin{aligned} p(x,y) &\leq \max\{p(x,y), p(Tx,x), p(Ty,y)\} \\ p(Tx,x) &\leq \max\{p(x,y), p(Tx,x), p(Ty,y)\} \\ p(Ty,y) &\leq \max\{p(x,y), p(Tx,x), p(Ty,y)\} \end{aligned}$$

then

$$\begin{aligned} &p(x,y) + p(Tx,x) + p(Ty,y) \\ &\leq 3\max\{p(x,y), p(Tx,x), p(Ty,y)\} \end{aligned}$$

Define

$$\psi(t) = t \quad \text{and} \quad \phi(t) = (1-3k)t.$$

Then

$$\begin{aligned} &\psi(p(Tx,Ty)) = p(Tx,Ty) \\ &\leq k\{p(x,y) + p(Tx,x) + p(Ty,y)\} \\ &\leq 3k\max\{p(x,y), p(Tx,x), p(Ty,y)\} \\ &= \psi(\max\{p(x,y), p(Tx,x), p(Ty,y)\}) \\ &-\phi(\max\{p(x,y), p(Tx,x), p(Ty,y)\}) \end{aligned} \tag{49}$$

and hence T has a unique fixed point by Theorem 1.

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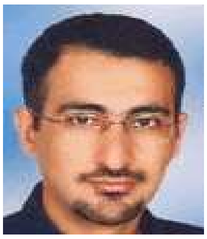

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