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Impulsive Mixed Fractional Differential Equations with Delay

Esma Kenef[1,2] *and Assia Guezane-Lakoud*[1,*]

[1] Laboratory of Advanced Materials, Department of Mathematics, Badji Mokhtar-Annaba University, P.O. Box 12, 23000 Annaba, Algeria
[2] ENSET Skikda, Algeria

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Abstract: We investigate, using the Krasnoselskii's fixed point theorem, the existence of solutions for delay impulsive differential equations involving left and right Caputo fractional derivatives with multi-base points. The results are new and complete the existing ones.

Keywords: Impulsive differential equation, fixed point theorem, existence of solution, delay differential equations, fractional derivative.

1 Introduction, motivation and preliminaries

Recently, fractional differential equations have been investigated in many papers in literature since they have several applications in various fields. Following this development, more attentions is given to impulsive differential equations of fractional order, [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17] where the questions on existence, uniqueness and stability of solutions are treated. Moreover, when these processes involve hereditary phenomena or delay argument that may cause undesirable performance in the system, it is necessary to analyze delay effects on the dynamical behaviors of the impulsive fractional differential equations. For more results on impulsive fractional differential equations with delay, we refer to [1,2,3,7].

Different methods have been applied to investigate the solvability of boundary value problems for impulsive differential equations, such as fixed point theorems, upper and lower solutions method, variational methods and critical point theorems.

Recently, fractional differential equations containing the left and right fractional derivatives have been considered in [18,19,20,21,22,23,24,23,26]. The left and right fractional derivatives may arise naturally as in some physical situations, where the state of the process depends on all its previous states and on the results of its future development, for more details see [20,22].

In [14], the authors studied the following impulsive fractional problem

$$\begin{cases} {}^{C}D^{\upsilon}_{T^-}\left({}^{C}D^{\upsilon}_{0^+}u(t)\right)=f(t,u), \quad 0\leq t\leq T, t\neq t_j, \\ \Delta\left(D^{\upsilon}_{T^-}\left({}^{C}D^{\upsilon}_{0^+}u\right)\right)(t_j)=I_j\left(u(t_j)\right), j=1,2...n, \\ u(0)=u(T)=0, \end{cases}$$

where $\upsilon\in(\frac{1}{2},1]$, $0=t_0<t_1<\cdots<t_n<t_{n+1}=T$. Using critical point theory and variational method, the authors proved the existence of at least one solution or infinitely many solutions. Using Banach fixed point theorem and the nonlinear alternative of Leray–Schauder, the authors in [7], explored the existence of solutions for the following initial value problems of fractional order functional differential equations with infinite delay:

* Corresponding author e-mail: a_guezane@yahoo.fr, as kenef16@yahoo.fr

$$\begin{cases} {}^{C}D_{0^+}^{\upsilon}u(t) = f(t,u_t), \quad 0 \le t \le b, 0 < \upsilon < 1, \\ u(t) = \varphi(t), t \in [-\infty, 0]. \end{cases}$$

In [29], the following multi-base points fractional initial value problems with impulses on the half lines are investigated:

$$\begin{cases} \left({}^{C}D_{*}^{\upsilon}u(t)\right) = f\left(t, u, {}^{C}D_{*}^{\varsigma}u\right), \quad t > 0, \\ u(0) = 0 \\ \Delta u(t_j) = I_j\left(t_j^-, u(t_j^-)\right), j = 1, 2 \ldots p, \end{cases}$$

where $0 < \varsigma < \upsilon < 1$, $0 = t_0 < t_1 < \cdots$, $\lim_{k\to\infty} t_k = \infty$, ${}^{C}D_{*}^{\upsilon}$ is the standard Caputo fractional derivative at the base points t_k. That is for all $t \in (t_k, t_{k+1}]$, we have ${}^{C}D_{*}^{\upsilon}u(t) = {}^{C}D_{t_k}^{\upsilon}u(t)$. The authors discussed the existence of solutions on the half line by means of Schauder fixed point theorem.

Our main objective is to discuss the existence of solutions for the following boundary value problem for impulsive fractional differential equations with delay and involving multi-base points right and left Caputo derivatives (P):

$$ {}^{C}D_{t_{j+1}^-}^{\upsilon}\left({}^{C}D_{t_j^+}^{\varsigma}u(t)\right) = f(t, u_t), 0 < t < 1, t \ne t_j; j = 0, 1 \ldots p \tag{1.1}$$

$$u(t) = \varphi(t), t \in [-r, 0] \tag{1.2}$$

$$u'(0) = 0 \tag{1.3}$$

$$\left({}^{C}D_{t_j^+}^{\varsigma}u\right)|_{t=t_{j+1}^-} = g_j\left(t_{j+1}^-, u\left(t_{j+1}^-\right) - u\left(t_j^+\right)\right), j = 0, 1, \ldots, p \tag{1.4}$$

$$\Delta u(t_j) = h_j\left(t_j^-, u(t_j^-)\right), j = 1, 2 \ldots p \tag{1.5}$$

$$\Delta u'(t_j) = \widetilde{h}_j\left(t_j^-, u(t_j^-)\right), j = 1, 2 \ldots p \tag{1.6}$$

where $0 < \upsilon < 1$, $1 < \varsigma < 2$, such that $\upsilon + \varsigma > 2$, ${}^{C}D_{t_{j+1}^-}^{\upsilon}$, ${}^{C}D_{t_j^+}^{\varsigma}$ are respectively the left and the right Caputo fractional derivatives, u is the unknown function and the history of state is $u_t(\theta) = u(t+\theta)$, for $\theta \in [-r, 0]$, $f : [0,1] \times D \to \mathbb{R}$, $D = \{u : [-r, 0] \to \mathbb{R}, u$ is continuous everywhere except for a finite number of points θ at which $u(\theta)$ and the right limit $u(\theta^-)$ exist and $u(\theta^-) = u(\theta)\}$, $f(t, u_t)$ is measurable on $[0,1]$ according to t for any $u_t \in D$.

The functions $h_j, \widetilde{h}_j, g_j : [0,1] \times \mathbb{R} \to \mathbb{R}$, for $j = 1, \ldots, p$, are given. The initial function $\varphi : \mathbb{R} \to \mathbb{R}$, satisfies $\varphi(0) = 0$. The impulsive moments t_k are such $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1$, $\Delta u(t_j) = u\left(t_j^+\right) - u\left(t_j^-\right)$, $u\left(t_j^+\right) = \lim_{h\to 0^+} u(t_j + h)$, and $u\left(t_j^-\right) = \lim_{h\to 0^-} u(t_j + h)$, $u\left(t_j^+\right)$ and $u\left(t_j^-\right)$ are the right and the left limits of $u(t)$ at the point $t = t_j$, $j = 1, \ldots, p$ respectively. $\Delta u'(t_j) = u'\left(t_j^+\right) - u'\left(t_j^-\right)$, $u'\left(t_j^+\right) = \lim_{h\to 0^+} u'(t_j + h)$, and $u'\left(t_j^-\right) = \lim_{h\to 0^-} u'(t_j + h)$ and $\left(D_{t_j^+}^{\varsigma}u\right)_{|t=t_{j+1}^-} = \lim_{t\to t_{j+1}^-} D_{t_j^+}^{\varsigma}u(t)$.

Our approach is based on the Krasnoselskii fixed point theorem. To our best knowledge, no work has been reported on the impulsive fractional differential equations involving both left and right fractional derivatives and in the presence of a delay in the literature. Thus these results are considered as a contribution to this emerging field.

2 Preliminaries

In this section, we first introduce some necessary definitions and properties of fractional calculus which will be used in this sequel. We refer the reader to [18,28,30] for more details.

Definition 1. *Let $J=[a,b]$ be a finite interval of $\mathbb{R}$. The left and the right Riemann-Liouville fractional integral $I_{a^+}^{\upsilon}f(t)$ and $I_{b^-}^{\upsilon}f(t)$ of order $\upsilon\in\mathbb{R}^+$ are defined respectively by*

$$I_{a^+}^{\upsilon}f(t)=\frac{1}{\Gamma(\upsilon)}\int_a^t(t-s)^{\upsilon-1}f(s)\,ds, t>a.$$

$$I_{b^-}^{\upsilon}f(t)=\frac{1}{\Gamma(\upsilon)}\int_t^b(s-t)^{\upsilon-1}f(s)\,ds, t<b,$$

Provided the right-hand sides are pointwise defined on $[a,b]$.

Definition 2. *The left and the right Caputo fractional derivatives ${}^{C}D_{a^+}^{\upsilon}f(t)$ and ${}^{C}D_{b^-}^{\upsilon}f(t)$ of order $\upsilon\in\mathbb{R}^+$ are defined respectively by*

$${}^{C}D_{a^+}^{\upsilon}f(t)=\frac{1}{\Gamma(n-\upsilon)}\int_a^t(t-s)^{n-\upsilon-1}f^{(n)}(s)\,ds,$$
$${}^{C}D_{b^-}^{\upsilon}f(t)=\frac{(-1)^n}{\Gamma(n-\upsilon)}\int_t^b(s-t)^{n-\upsilon-1}f^{(n)}(s)\,ds,$$

where $n=[\upsilon]+1,[\upsilon]$ means the integer part of υ.

Proposition 1. *Let $\upsilon\in\mathbb{R}^+$ and let $n=[\upsilon]+1$, if $f\in AC^n([a,b])$, then*

$$I_{a^+}^{\upsilon C}D_{a^+}^{\upsilon}f(t)=f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k$$

and

$$I_{b^-}^{\upsilon C}D_{b^-}^{\upsilon}f(t)=f(t)-\sum_{k=0}^{n-1}\frac{(-1)^k f^{(k)}(b)}{k!}(b-t)^k.$$

Next, we state Krasnoselskii's fixed point theorem.

Theorem 1. *Let M be closed bounded convex nonempty subset of a Banach space E. Suppose that A and B map M into E such that*
i) A is completely continuous,
ii) B is a contraction mapping,
iii) $x,y\in M$ implies $Ax+By\in M$.
Then there exists $z\in M$ such that $z=Az+Bz$.

3 Main results

In this section, we transform the problem (P) to an integral equation, then we apply Krasnoselski fixed point theorem.

Lemma 1. *The boundary value problem (P) is equivalent to the following integral equation:*

$$\begin{aligned}u(t)=&\frac{1}{\Gamma(\upsilon)\Gamma(\varsigma)}\int_0^t\left(\int_0^{\mu}(t-s)^{\varsigma-1}(\mu-s)^{\upsilon-1}ds\right)f(\mu,u_\mu)\,d\mu\\&+\frac{1}{\Gamma(\upsilon)\Gamma(\varsigma)}\int_t^{t_1}\left(\int_0^{t}(t-s)^{\varsigma-1}(\mu-s)^{\upsilon-1}ds\right)f(\mu,u_\mu)\,d\mu\\&+\frac{t^\varsigma}{\Gamma(\varsigma+1)}g_0\left(t_1^-,u(t_1^-)-u(0^+)\right),\qquad t\in[0,t_1),\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{\Gamma(\upsilon)\Gamma(\varsigma)} \int_{t_1}^{t} \left(\int_{t_1}^{\mu} (t-s)^{\varsigma-1} (\mu-s)^{\upsilon-1} ds \right) f(\mu, u_\mu)\, d\mu \\
&\quad + \tfrac{1}{\Gamma(\upsilon)\Gamma(\varsigma)} \int_{t}^{t_2} \left(\int_{t_1}^{t} (t-s)^{\varsigma-1} (\mu-s)^{\upsilon-1} ds \right) f(\mu, u_\mu)\, d\mu \\
&\quad + \tfrac{1}{\Gamma(\upsilon)\Gamma(\varsigma)} \int_{0}^{t_1} \left(\int_{0}^{\mu} (t_1-s)^{\varsigma-1} (\mu-s)^{\upsilon-1} ds \right) f(\mu, u_\mu)\, d\mu \\
&\quad + h_1\left(t_1^-, u(t_1^-)\right) + \frac{t_1^{\varsigma}}{\Gamma(\varsigma+1)} g_0\left(t_1^-, u\left(t_1^-\right) - u\left(0^+\right)\right) \\
&\quad + \frac{(t-t_1)^{\varsigma}}{\Gamma(\varsigma+1)} g_1\left(t_2^-, u\left(t_2^-\right) - u\left(t_1^+\right)\right) \\
&\quad + (t-t_1)\left(\tfrac{1}{\Gamma(\upsilon)\Gamma(\varsigma-1)} \int_{0}^{t_1} \left(\int_{0}^{\mu} (t_1-s)^{\varsigma-2} (\mu-s)^{\upsilon-1} ds \right) f(\mu, u_\mu)\, d\mu \right. \\
&\quad \left. + \widetilde{h}_1\left(u(t_1^-)\right) + \frac{t_1^{\varsigma-1}}{\Gamma(\varsigma)} g_0\left(t_1^-, u\left(t_1^-\right) - u\left(0^+\right)\right) \right),
\end{aligned}
$$

$t \in [t_1, t_2)$,

$$
\begin{aligned}
&= \int_{t_j}^{t_{j+1}} G_j(t,\mu) f(\mu, u_\mu)\, d\mu + \sum_{k=1}^{j} \left(\int_{t_{k-1}}^{t_k} F_k(\mu) f(\mu, u_\mu)\, d\mu + h_k\left(t_k^-, u\left(t_k^-\right)\right) \right. \\
&\quad \left. + \frac{(t_k - t_{k-1})^{\varsigma}}{\Gamma(\varsigma+1)} g_{k-1}\left(t_k^-, u\left(t_k^-\right) - u\left(t_{k-1}^+\right)\right) \right) \\
&\quad + \sum_{k=1}^{j-1} (t_j - t_k) \left(\int_{t_{k-1}}^{t_k} H_k(\mu) f(\mu, u_\mu)\, d\mu + \widetilde{h}_k\left(t_k^-, u\left(t_k^-\right)\right) \right. \\
&\quad \left. + \frac{(t_k - t_{k-1})^{\varsigma-1}}{\Gamma(\varsigma)} g_{k-1}\left(t_k^-, u\left(t_k^-\right) - u\left(t_{k-1}^+\right)\right) \right) \\
&\quad + (t-t_j) \sum_{k=1}^{j} \left(\int_{t_{k-1}}^{t_k} H_k(\mu) f(\mu, u_\mu)\, d\mu + \widetilde{h}_k\left(t_k^-, u\left(t_k^-\right)\right) \right. \\
&\quad \left. + \frac{(t_k - t_{k-1})^{\varsigma-1}}{\Gamma(\varsigma)} g_{k-1}\left(t_k^-, u\left(t_k^-\right) - u\left(t_{k-1}^+\right)\right) \right),
\end{aligned}
$$

$t \in (t_j, t_{j+1}], \quad j = 2, \cdots, p$

$$= \varphi(t), \quad t \in [-r, 0],$$

where

$$
G_j(t,\mu) = \frac{1}{\Gamma(\upsilon)\Gamma(\varsigma)} \begin{cases} \int_{t_j}^{\mu} (t-s)^{\varsigma-1} (\mu-s)^{\upsilon-1} ds, t_j \le \mu \le t \le t_{j+1} \\ \int_{t_j}^{t} (t-s)^{\varsigma-1} (\mu-s)^{\upsilon-1} ds, t_j \le t \le \mu \le t_{j+1}, \end{cases}
$$

$\forall j = 0, \cdots, p.$

and

$$F_j(\mu) = \frac{1}{\Gamma(\upsilon)\Gamma(\varsigma)} \int\limits_{t_{j-1}}^{\mu} (t_j - s)^{\varsigma-1} (\mu - s)^{\upsilon-1} ds,$$

$$H_j(\mu) = \frac{1}{\Gamma(\upsilon)\Gamma(\varsigma-1)} \int\limits_{t_{j-1}}^{\mu} (t_j - s)^{\varsigma-2} (\mu - s)^{\upsilon-1} ds,$$

$$t_{j-1} \leq \mu \leq t_j, \quad \forall j = 1, \cdots, p.$$

*Proof.*Since the history condition $u(t) = \varphi(t), t \in [-r,0]$ is known, so we will investigate the existence of solution in $[0,1]$. Let $t \in [0,t_1)$ and rewrite the equation (1.1) as

$$^{C}D^{\upsilon}_{t_1^-}\left({}^{C}D^{\varsigma}_{0+}u(t)\right) = f(t,u_t), 0 < t < t_1, \tag{3.1}$$

applying the fractional integral $I^{\upsilon}_{t_1^-}$ to the equation (3.1), we get

$$^{C}D^{\varsigma}_{0+}u(t) = I^{\upsilon}_{t_1^-} f(t,u_t) + c_0, \tag{3.2}$$

using condition (1.4), we obtain

$$c_0 = g_0\left(t_1^-, u\left(t_1^-\right) - u\left(0^+\right)\right),$$

substituting c_0 in (3.2), it yields

$$^{C}D^{\varsigma}_{0+}u(t) = I^{\upsilon}_{t_1^-} f(t,u_t) + g_0\left(t_1^-, u\left(t_1^-\right) - u\left(0^+\right)\right). \tag{3.3}$$

Now, applying the fractional integral I^{ς}_{0+} to the equation (3.3), we get

$$u(t) = I^{\varsigma}_{0+} I^{\upsilon}_{t_1^-} f(t,u_t) + I^{\varsigma}_{0+}\left(g_0\left(t_1^-, u\left(t_1^-\right) - u\left(0^+\right)\right)\right) + c_1 + c_2 t. \tag{3.4}$$

Taking conditions (1.2) and (1.3) into account, we obtain

$$c_1 = \varphi(0) = 0, c_2 = u'(0) = 0,$$

substituting c_1 and c_2 in (3.4) yields

$$\begin{aligned} u(t) &= I^{\varsigma}_{0+} I^{\upsilon}_{t_1^-} f(t,u_t) + \frac{t^{\varsigma}}{\Gamma(\varsigma+1)} g_0\left(t_1^-, u\left(t_1^-\right) - u\left(0^+\right)\right) \\ &= \frac{1}{\Gamma(\upsilon)\Gamma(\varsigma)} \int\limits_0^t \left((t-s)^{\varsigma-1} \int\limits_{t_1}^{s} (s-\mu)^{\upsilon-1} f(\mu,u_\mu)\, d\mu\right) ds \\ &\quad + \frac{t^{\varsigma}}{\Gamma(\varsigma+1)} g_0\left(t_1^-, u\left(t_1^-\right) - u\left(0^+\right)\right). \end{aligned}$$

Using Fubini Theorem, we get

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\upsilon)\Gamma(\varsigma)} \int\limits_0^t \left(\int\limits_0^{\mu} (t-s)^{\varsigma-1} (\mu-s)^{\upsilon-1} ds\right) f(\mu,u_\mu)\, d\mu \\ &\quad + \frac{1}{\Gamma(\upsilon)\Gamma(\varsigma)} \int\limits_t^{t_1} \left(\int\limits_0^{t} (t-s)^{\varsigma-1} (\mu-s)^{\upsilon-1} ds\right) f(\mu,u_\mu)\, d\mu \\ &\quad + \frac{t^{\varsigma}}{\Gamma(\varsigma+1)} g_0\left(t_1^-, u\left(t_1^-\right) - u\left(0^+\right)\right). \end{aligned}$$

Now, for $t \in [t_1, t_2)$, we apply the fractional integral $I_{t_2^-}^{\upsilon}$ to the equation (1.1) to get

$$ {}^{C}D_{t_1^+}^{\varsigma} u(t) = I_{t_2^-}^{\upsilon} f(t, u_t) + b_0, \tag{3.5} $$

using condition (1.4), gives

$$ b_0 = g_1 \left(t_2^-, u\left(t_2^-\right) - u\left(t_1^+\right)\right). $$

Substituting b_0 in (3.5), we obtain

$$ {}^{C}D_{t_1^+}^{\varsigma} u(t) = I_{t_2^-}^{\upsilon} f(t, u_t) + g_1 \left(t_2^-, u\left(t_2^-\right) - u\left(t_1^+\right)\right). \tag{3.6} $$

Applying the fractional integral $I_{t_1^+}^{\varsigma}$ to the equation (3.6), we get

$$ u(t) = I_{t_1^+}^{\varsigma} I_{t_2^-}^{\upsilon} f(t, u_t) + \frac{(t-t_1)^{\varsigma}}{\Gamma(\varsigma+1)} g_1 \left(t_2^-, u\left(t_2^-\right) - u\left(t_1^+\right)\right) + b_1 + b_2 (t - t_1). \tag{3.7} $$

Taking conditions (1.5) and (1.6) into account, we obtain

$$ \begin{aligned} b_1 &= u\left(t_1^+\right) = h_1 \left(t_1^- u(t_1^-)\right) + u\left(t_1^-\right) \\ &= h_1 \left(t_1^-, u(t_1^-)\right) + + \frac{t_1^{\varsigma}}{\Gamma(\varsigma+1)} g_0 \left(t_1^-, u\left(t_1^-\right) - u\left(0^+\right)\right) \\ &\quad + \frac{1}{\Gamma(\upsilon)\Gamma(\varsigma)} \int_0^{t_1} \left(\int_0^{\mu} (t_1 - s)^{\varsigma-1} (\mu - s)^{\upsilon-1} ds \right) f(\mu, u_\mu) \, d\mu \end{aligned} $$

and

$$ \begin{aligned} b_2 &= u'\left(t_1^+\right) = \widetilde{h}_1 \left(u(t_1^-)\right) + u'\left(t_1^-\right) \\ &= \widetilde{h}_1 \left(u(t_1^-)\right) + \frac{t_1^{\varsigma-1}}{\Gamma(\varsigma)} g_0 \left(u\left(t_1^-\right) - u\left(0^+\right)\right) \\ &\quad + \frac{1}{\Gamma(\upsilon)\Gamma(\varsigma-1)} \int_0^{t_1} \left(\int_0^{\mu} (t_1 - s)^{\varsigma-2} (\mu - s)^{\upsilon-1} ds \right) f(\mu, u_\mu) \, d\mu. \end{aligned} $$

Substituting b_1 and b_2 in (3.7) and using Fubini Theorem, we get

$$ \begin{aligned} u(t) &= \frac{1}{\Gamma(\upsilon)\Gamma(\varsigma)} \int_{t_1}^{t} \left(\int_{t_1}^{\mu} (t - s)^{\varsigma-1} (\mu - s)^{\upsilon-1} ds \right) f(\mu, u_\mu) \, d\mu \\ &\quad + \frac{1}{\Gamma(\upsilon)\Gamma(\varsigma)} \int_{t}^{t_2} \left(\int_{t_1}^{t} (t - s)^{\varsigma-1} (\mu - s)^{\upsilon-1} ds \right) f(\mu, u_\mu) \, d\mu \\ &\quad + \frac{1}{\Gamma(\upsilon)\Gamma(\varsigma)} \int_{0}^{t_1} \left(\int_{0}^{\mu} (t_1 - s)^{\varsigma-1} (\mu - s)^{\upsilon-1} ds \right) f(\mu, u_\mu) \, d\mu \\ &\quad + h_1 \left(t_1^-, u(t_1^-)\right) + \frac{t_1^{\varsigma}}{\Gamma(\varsigma+1)} g_0 \left(t_1^-, u\left(t_1^-\right) - u\left(0^+\right)\right) \\ &\quad + \frac{(t-t_1)^{\varsigma}}{\Gamma(\varsigma+1)} g_1 \left(t_2^-, u\left(t_2^-\right) - u\left(t_1^+\right)\right) \\ &\quad + (t - t_1) \left(\frac{1}{\Gamma(\upsilon)\Gamma(\varsigma-1)} \int_0^{t_1} \left(\int_0^{\mu} (t_1 - s)^{\varsigma-2} (\mu - s)^{\upsilon-1} ds \right) f(\mu, u_\mu) \, d\mu \right. \\ &\quad \left. + \widetilde{h}_1 \left(u(t_1^-)\right) + \frac{t_1^{\varsigma-1}}{\Gamma(\varsigma)} g_0 \left(t_1^-, u\left(t_1^-\right) - u\left(0^+\right)\right) \right). \end{aligned} $$

Continuing this process for $t \in [t_p, 1]$, the following integral equation is obtained

$$u(t) = \frac{1}{\Gamma(\upsilon)\Gamma(\varsigma)} \int_{t_p}^{t} \left(\int_{t_p}^{\mu} (t-s)^{\varsigma-1} (\mu-s)^{\upsilon-1} ds \right) f(\mu, u_\mu) d\mu$$
$$+ \frac{1}{\Gamma(\upsilon)\Gamma(\varsigma)} \int_{t}^{1} \left(\int_{t_p}^{t} (t-s)^{\varsigma-1} (\mu-s)^{\upsilon-1} ds \right) f(\mu, u_\mu) d\mu$$

$$+ \sum_{k=1}^{p} \left(\frac{1}{\Gamma(\upsilon)\Gamma(\varsigma)} \int_{t_{k-1}}^{t_k} \left(\int_{t_{k-1}}^{\mu} (t_k-s)^{\varsigma-1} (\mu-s)^{\upsilon-1} ds \right) f(\mu, u_\mu) d\mu \right.$$
$$\left. + h_k(t_k^-, u(t_k^-)) + \frac{(t_k - t_{k-1})^{\varsigma}}{\Gamma(\varsigma+1)} g_{k-1}(t_k^-, u(t_k^-) - u(t_{k-1}^+)) \right)$$

$$+ \sum_{k=1}^{p-1} (t_p - t_k) \left(\frac{1}{\Gamma(\upsilon)\Gamma(\varsigma-1)} \int_{t_{k-1}}^{t_k} \left(\int_{t_{k-1}}^{\mu} (t_k-s)^{\varsigma-2} (\mu-s)^{\upsilon-1} ds \right) f(\mu, u_\mu) d\mu \right.$$
$$\left. + \widetilde{h}_k(t_k^-, u(t_k^-)) + \frac{(t_k - t_{k-1})^{\varsigma-1}}{\Gamma(\varsigma)} g_k(t_k^-, u(t_k^-) - u(t_{k-1}^+)) \right)$$

$$+ (t - t_p) \sum_{k=1}^{p} \left(\frac{1}{\Gamma(\upsilon)\Gamma(\varsigma-1)} \int_{t_{k-1}}^{t_k} \left(\int_{t_{k-1}}^{\mu} (t_k-s)^{\varsigma-2} (\mu-s)^{\upsilon-1} ds \right) f(\mu, u_\mu) d\mu \right.$$
$$\left. + \widetilde{h}_k(t_k^-, u(t_k^-)) + \frac{(t_k - t_{k-1})^{\varsigma-1}}{\Gamma(\varsigma)} g_k(t_k^-, u(t_k^-) - u(t_{k-1}^+)) \right).$$

Conversely, suppose that u satisfies the integral equations given in the Lemma 1. By a direct computation, it follows that u satisfies the problem (P). This achieves the proof.

The functions G_j, F_j and H_j satisfy the following properties.

Lemma 2. *The functions G_j, F_j and H_j are nonnegative and satisfy the following estimates:*

1- $G_j(t,\mu) \leq \frac{1}{(\upsilon+\varsigma-2)\Gamma(\upsilon)\Gamma(\varsigma)}$, for all $t, \mu \in [t_j, t_{j+1}]$, $j = 0, \cdots, p$,

2- $F_j(\mu) \leq \frac{1}{(\upsilon+\varsigma-2)\Gamma(\upsilon)\Gamma(\varsigma)}$ and $H_j(\mu) \leq \frac{1}{(\upsilon+\varsigma-2)\Gamma(\upsilon)\Gamma(\varsigma)}$ for all $\mu \in [t_{j-1}, t_j]$, $j = 1, \cdots, p$.

Proof. It is obvious that G_j, F_j and H_j are nonnegative. Let $t_j \leq \mu \leq t \leq t_{j+1}$, then

$$G_j(t,\mu) = \frac{1}{\Gamma(\upsilon)\Gamma(\varsigma)} \int_{t_j}^{\mu} (t-s)^{\varsigma-1} (\mu-s)^{\upsilon-1} ds$$
$$\leq \frac{1}{\Gamma(\upsilon)\Gamma(\varsigma)} \int_{t_j}^{\mu} (\mu-s)^{\upsilon-1} ds$$
$$= \frac{(\mu-t_j)^{\upsilon}}{\upsilon\Gamma(\upsilon)\Gamma(\varsigma)} \leq \frac{1}{(\upsilon+\varsigma-2)\Gamma(\upsilon)\Gamma(\varsigma)}.$$

For $t_j \le t \le \mu \le t_{j+1}$, we have

$$\begin{aligned} G_j(t,\mu) &= \frac{1}{\Gamma(\upsilon)\Gamma(\varsigma)} \int_{t_j}^{t} (t-s)^{\varsigma-1} (\mu-s)^{\upsilon-1} ds \\ &\le \frac{1}{\Gamma(\upsilon)\Gamma(\varsigma)} \int_{t_j}^{t} (\mu-s)^{\upsilon-1} ds \\ &= \frac{(\mu-t_j)^{\upsilon} - (\mu-t)^{\upsilon}}{\upsilon\Gamma(\upsilon)\Gamma(\varsigma)} \le \frac{1}{(\upsilon+\varsigma-2)\Gamma(\upsilon)\Gamma(\varsigma)}. \end{aligned}$$

Consequently, $G_j(t,\mu) \le \frac{1}{(\upsilon+\varsigma-2)\Gamma(\upsilon)\Gamma(\varsigma)}$, for all $t,\mu \in [t_j, t_{j+1}]$, $j = 0,\cdots,p$. Similarly, we prove that $F_j(\mu) \le \frac{1}{(\upsilon+\varsigma-2)\Gamma(\upsilon)\Gamma(\varsigma)}$ and $H_j(\mu) \le \frac{1}{(\upsilon+\varsigma-2)\Gamma(\upsilon)\Gamma(\varsigma)}$ for all $\mu \in [t_{j-1}, t_j]$, $j = 1,\cdots,p$.

Define the Banach space $E = PC([-r,1],\mathbb{R}) \cap PC^1([0,1],\mathbb{R})$, with the norm $\|u\| = \max_{t\in[-r,1]} |u(t)|$, where

$$\begin{aligned} PC([-r,1],\mathbb{R}) = \Big\{ & u : [-r,1] \to \mathbb{R}, u \in C\left((t_j,t_{j+1}]\right) \cup C[-r,0], \\ & u\left(t_j^+\right) \text{ and } u\left(t_j^-\right),\ j = 1,\cdots,p, \text{ exist and } u\left(t_j^+\right) = u(t_j) \Big\}, \end{aligned}$$

$$\begin{aligned} PC^1([0,1],\mathbb{R}) = \Big\{ & u : [0,1] \to \mathbb{R}, u \in C^1\left((t_j,t_{j+1}]\right),\ u'\left(t_j^+\right),\ u'\left(t_j^-\right), \\ & j = 1,\cdots,p, \text{ exist and } u'\left(t_j^-\right) = u'(t_j) \Big\}. \end{aligned}$$

Definition 3. *A function $u \in E$ is said to be a solution for problem (P) if it satisfies the differential equation (1.1) and the conditions (1.2)-(1.6).*

Define the operators A and B on E by

$$Au(t) = \begin{cases} 0, & t \in [-r,0], \\ A_j u(t), & t \in [t_j,t_{j+1}), j = 0,\cdots,p \end{cases}$$

and

$$Bu(t) = \begin{cases} \varphi(t), & t \in [-r,0] \\ B_j u(t), & t \in [t_j,t_{j+1}), j = 0,\cdots,p \end{cases}$$

where

$$A_j u(t) = \int_{t_j}^{t_{j+1}} G_j(t,\mu) f(\mu,u_\mu) d\mu, t \in [t_j,t_{j+1}), j = 0,\cdots,p$$

$$B_j u(t) = \begin{cases} \sum\limits_{k=1}^{j} \left(\int\limits_{t_{k-1}}^{t_k} F_k(\mu) f(\mu,u_\mu) d\mu + h_k\left(t_k^-, u\left(t_k^-\right)\right) \right. \\ \left. + \frac{(t_k-t_{k-1})^{\varsigma}}{\Gamma(\varsigma+1)} g_{k-1}\left(t_k^-, u\left(t_k^-\right) - u\left(t_{k-1}^+\right)\right)\right) \\ + \sum\limits_{k=1}^{j-1} (t_j - t_k) \left(\int\limits_{t_{k-1}}^{t_k} H_k(\mu) f(\mu,u_\mu) d\mu + \widetilde{h}_k\left(t_k^-, u\left(t_k^-\right)\right) \right. \\ \left. + \frac{(t_k-t_{k-1})^{\varsigma-1}}{\Gamma(\varsigma)} g_{k-1}\left(t_k^-, u\left(t_k^-\right) - u\left(t_{k-1}^+\right)\right)\right) \\ + (t-t_j) \sum\limits_{k=1}^{j} \left(\int\limits_{t_{k-1}}^{t_k} H_k(\mu) f(\mu,u_\mu) d\mu + \widetilde{h}_k\left(t_k^-, u\left(t_k^-\right)\right) \right. \\ \left. + \frac{(t_k-t_{k-1})^{\varsigma-1}}{\Gamma(\varsigma)} g_{k-1}\left(t_k^-, u\left(t_k^-\right) - u\left(t_{k-1}^+\right)\right)\right), \\ t \in [t_j,t_{j+1}), j = 0,\cdots,p \end{cases}$$

Since the solution u is known on $[-r,0]$, we will investigate the existence of solution on $[0,1]$. Obviously, problem (P) has a solution if and only if $A+B$ has a fixed point, i.e.

$$Au(t)+Bu(t)=u(t),t\in[0,1].$$

The following hypotheses will be needed:

(H_1) The function $f(.,0)$ is continuous and not identically null on $[0,1]$, and there exists a nonnegative function $k\in L_1(0,1)$, such that

$$|f(t,x)-f(t,y)|\leq k(t)|x-y|,0\leq t\leq 1,x,y\in\mathbb{R}$$

and

$$\|k\|_{L_1}<\frac{(\upsilon+\varsigma-2)\Gamma(\upsilon)\Gamma(\varsigma)}{24p} \tag{3.8}$$

(H_2) The functions $h_j(.,0)=0$, for all $j=1,\cdots,p$ and there exist nonnegative continuous functions $a_j\in C([0,1],\mathbb{R}_+)$, $j=1,\cdots,p$ such that

$$\left|h_j(t,x)-h_j(t,y)\right|\leq a_j(t)|x-y|,0\leq t\leq 1,x,y\in\mathbb{R},j=1,\cdots,p$$

and

$$a=\max_{j=1,\cdots,p}\left(\|a_j\|_{C[0,1]}\right)<\frac{1}{8p}. \tag{3.9}$$

(H_3) $\widetilde{h}_j(.,0)=0$,for all $j=1,\cdots,p$ and there exist nonnegative functions $b_j\in C[0,1]$, $j=1,\cdots,p$, such that

$$\left|\widetilde{h}_j(t,x)-\widetilde{h}_j(t,y)\right|\leq b_j(t)|x-y|,0\leq t\leq 1,x,y\in\mathbb{R},j=1,\cdots,p$$

$$b=\max_{j=1,\cdots,p}\left(\|b_j\|_{C[0,1]}\right)<\frac{1}{16p}. \tag{3.10}$$

(H_4) There exist nonnegative functions $c_j\in C[0,1]$, $j=0,\cdots,p$, such that

$$\left|g_j(t,x)-g_j(t,y)\right|\leq c_j(t)|x-y|,0\leq t\leq 1,x,y\in\mathbb{R},j=0,\cdots,p$$

$$c=\max_{j=0,\cdots,p}\left(\|c_j\|_{C[0,1]}\right)<\frac{\Gamma(\varsigma)}{48p}. \tag{3.11}$$

Let $M=\{u\in E\in,\|u\|\leq R\}$, where R is chosen such that

$$R\geq\max\left(\frac{24pL}{(\upsilon+\varsigma-2)\Gamma(\upsilon)\Gamma(\varsigma)-24p\|k\|_{L_1}},\frac{24pd}{\Gamma(\varsigma)}\right), \tag{3.12}$$

where $L=\max_{t\in[0,1]}|f(.,0)|$ and $d=\max_{j=0,\cdots,p}\left|g_j(.,0)\right|$. Clearly, M is a nonempty, bounded and convex subset of E.

Theorem 2. *Under the hypotheses $(H_1)-(H_4)$, the problem (P) has at least one nontrivial solution in M.*

*Proof.*We will demonstrate that all the assumptions of the Krasnoselskii fixed point theorem are verified, so, the proof will be done in a few steps. **First,** A is continuous on M. In fact, consider the sequence $(u_n)_n\subset M$ such that $u_n\to u$ in M, then thanks to Lemma 2 and the Hypothesis (H_1), it yields $t\in\left[t_j,t_{j+1}\right]$, $j=0,\cdots,p$

$$|Au_n(t)-Au(t)|=\left|A_ju_n(t)-A_ju(t)\right|$$

$$\leq\int_{t_j}^{t_{j+1}}G_j(t,\mu)\left|f\left(\mu,u_{n\mu}\right)-f\left(\mu,u_\mu\right)\right|d\mu\leq\frac{\|k\|_{L_1}}{(\upsilon+\varsigma-2)\Gamma(\upsilon)\Gamma(\varsigma)}\|u_n-u\|.$$

Second, the family (Au) is uniformly bounded on M. Let $u \in M$, then hypothesis (H_1) implies for $t \in [t_j, t_{j+1}]$, $j = 0, \cdots, p$

$$\begin{aligned}
|Au(t)| = |A_j u(t)| &\leq \int_{t_j}^{t_{j+1}} G_j(t,\mu) \left| f(\mu, u_\mu) \right| d\mu \\
&\leq \int_{t_j}^{t_{j+1}} G_j(t,\mu) \left| f(\mu, u_\mu) - f(\mu, 0) \right| d\mu + \int_{t_j}^{t_{j+1}} G_j(t,\mu) \left| f(\mu, 0) \right| d\mu \\
&\leq \frac{\|u\| . \|k\|_{L_1} + L}{(\upsilon + \varsigma - 2)\Gamma(\upsilon)\Gamma(\varsigma)} \leq \frac{R\|k\|_{L_1} + L}{(\upsilon + \varsigma - 2)\Gamma(\upsilon)\Gamma(\varsigma)}
\end{aligned} \tag{3.13}$$

Third, the family (Au) is equicontinuous on M. Let $u \in M$, and $t_j < \mu_1 < \mu_2 < t_{j+1}$, $j = 0, \cdots, p$ then

$$\begin{aligned}
|Au(\mu_2) - Au(\mu_1)| &= |A_j u(\mu_2) - A_j u(\mu_1)| \\
&\leq \int_{t_j}^{\mu_1} \left| G_j(\mu_2,\mu) - G_j(\mu_1,\mu) \right| \left| f(\mu, u_\mu) \right| d\mu \\
&+ \int_{\mu_1}^{\mu_2} \left| G_j(\mu_2,\mu) - G_j(\mu_1,\mu) \right| \left| f(\mu, u_\mu) \right| d\mu \\
&+ \int_{\mu_2}^{t_{j+1}} \left| G_j(\mu_2,\mu) - G_j(\mu_1,\mu) \right| \left| f(\mu, u_\mu) \right| d\mu \\
&\leq \left(\frac{R\|k\|_{L_1} + L}{\Gamma(\upsilon)\Gamma(\varsigma)} \right) \left(\int_{t_j}^{\mu_1} \int_{t_j}^{\mu} \left((\mu_2 - s)^{\varsigma - 1} - (\mu_1 - s)^{\varsigma - 1} \right) (\mu - s)^{\upsilon - 1} ds d\mu \right. \\
&+ \int_{\mu_1}^{t_{j+1}} \int_{t_j}^{\mu_1} \left((\mu_2 - s)^{\varsigma - 1} - (\mu_1 - s)^{\varsigma - 1} \right) (\mu - s)^{\upsilon - 1} ds d\mu \\
&\left. + \int_{\mu_1}^{\mu_2} \int_{\mu_1}^{\mu} (\mu_2 - s)^{\varsigma - 1} (\mu - s)^{\upsilon - 1} ds d\mu + \int_{\mu_2}^{t_{j+1}} \int_{\mu_1}^{\mu_2} (\mu_2 - s)^{\varsigma - 1} (\mu - s)^{\upsilon - 1} ds d\mu \right) \\
&\leq 3(\mu_2 - \mu_1) \left(\frac{R\|k\|_{L_1} + L}{\Gamma(\upsilon + 1)\Gamma(\varsigma + 1)} \right).
\end{aligned}$$

Hence, $|Au(\mu_2) - Au(\mu_1)|$ tends to zero when $\mu_1 \to \mu_2$.

This proves the equicontinuity in the case $t \neq t_j$, $j = 1, \ldots, p+1$, it remains to examine the points $t = t_j$. First, we prove the equicontinuity at $t = t_j^-$, let us fix $\delta_1 > 0$ such that $\{t_k,\ k \neq j\} \cap [t_j - \delta_1, t_j + \delta_1] = \emptyset$, for $0 < h < \delta_1$, it yields

$$\begin{aligned}
&\left| Au(t_j) - Au(t_j - h) \right| = \left| A_{j-1}u(t_j) - A_{j-1}u(t_j - h) \right| \\
&\leq \left(2(t_j - t_{j-1})^{\varsigma} - 2(t_j - t_{j-1} - h)^{\varsigma} - h^{\varsigma} \right) \left(\frac{R\|k\|_{L_1} + L}{\Gamma(\upsilon + 1)\Gamma(\varsigma + 1)} \right),
\end{aligned}$$

so, the right-hand side tends to zero as $h \to 0$.

Next, we prove the equicontinuity at $t = t_j^+$, fix $\delta_2 > 0$ such that $\{t_k,\ k \neq j\} \cap [t_j - \delta_2, t_j + \delta_2] = \emptyset$, for $0 < h < \delta_2$, we get

$$\begin{aligned}
\left| Au(t_j + h) - Au(t_j) \right| &= \left| A_j u(t_j + h) - A_j u(t_j) \right| \\
&\leq h^{\varsigma} \left(\frac{R\|k\|_{L_1} + L}{\Gamma(\upsilon + 1)\Gamma(\varsigma + 1)} \right) \to 0, \text{ as } h \to 0.
\end{aligned}$$

Hence, we conclude that A is completely continuous on M by Arzela-Ascoli theorem.

Fourth, the mapping B is a contraction on M. Let $u,v\in M$, then by hypothesis $(H_1)-(H_4)$, and for $t\in[t_j,t_{j+1}]$, $j=1,\cdots,p$, it yields

$$\begin{aligned}|Bu(t)-Bv(t)| &= |B_ju(t)-B_jv(t)|\\ &\leq \left[\frac{3p\|k\|_{L_1}}{(\upsilon+\varsigma-2)\Gamma(\upsilon)\Gamma(\varsigma)}+pa\right.\\ &\quad \left.+2pb+\frac{6pc}{\Gamma(\varsigma)}\right]\|u-v\|\\ &\leq \frac{\|u-v\|}{2},\end{aligned}$$

so B is a contraction on M.

Fifth. Let $u,v\in M$. Taking (3.12) and (3.13) into account, we obtain

$$|Au(t)|\leq\frac{R\|k\|_{L_1}+L}{(\upsilon+\varsigma-2)\Gamma(\upsilon)\Gamma(\varsigma)}\leq\frac{R}{24p}\leq\frac{R}{24},u\in M$$

Proceeding as in the second step and taking (3.9)-(3.10)-(3.11) into account, we get for $v\in M$

$$\begin{aligned}|Bv(t)| \leq & \tfrac{Rp}{(\upsilon+\varsigma-2)\Gamma(\upsilon)\Gamma(\varsigma)}\\ &\times\left(3\|k\|_{L_1}+a(\upsilon+\varsigma-2)\Gamma(\upsilon)\Gamma(\varsigma)+2b\Gamma(\varsigma)+6c\right)\\ &+3p\tfrac{L+(\upsilon+\varsigma-2)d\Gamma(\upsilon)}{(\upsilon+\varsigma-2)\Gamma(\upsilon)\Gamma(\varsigma)}\leq\frac{5R}{8},\end{aligned}$$

so

$$|Au(t)+Bv(t)|\leq|Au(t)|+|Bv(t)|=\frac{17R}{24}\leq R,\quad u,v\in M.$$

Consequently, $(Au+Bv)\in M$, for all $u,v\in M$. Thanks to Krasnoselskii fixed point theorem, we deduce that $A+B$ has a fixed point $u\in M$ and then problem (P) has at least one nontrivial solution in M. The proof is complete.

Example: Consider the following impulsive problem with delay that we denote by (P1), with $p=1, t_1=\frac{1}{2}$, $\upsilon=0.75$, $\varsigma=1.75$:

$$\begin{aligned}{}^CD^{\upsilon}_{\frac{1}{2}^-}\left({}^CD^{\varsigma}_{0^+}u(t)\right) &= f(t,u_t), 0\leq t<\frac{1}{2}\\ {}^CD^{\upsilon}_{1^-}\left({}^CD^{\varsigma}_{\frac{1}{2}^+}u(t)\right) &= f(t,u_t), \frac{1}{2}\leq t<1\\ u(t) &= \varphi(t), t\in[-r,0], u'(0)=0\\ \left({}^CD^{\varsigma}_{0^+}u\right)|_{t=\frac{1}{2}^-} &= \left({}^CD^{\varsigma}_{\frac{1}{2}^+}u\right)|_{t=1^-}=0\end{aligned}$$

$$\Delta u\left(\frac{1}{2}\right)=h_1\left(\frac{1}{2}^-,u(\frac{1}{2}^-)\right),\quad \Delta u'\left(\frac{1}{2}\right)=\widetilde{h}_1\left(\frac{1}{2}^-,u(\frac{1}{2}^-)\right),$$

$$\begin{aligned}f(t,x) &= \frac{2\sin t^2}{45}\left(x-\frac{t}{2(1+x^2)}\right),\\ h_1(t,x) &= x\frac{\cos t^2}{8},\ \widetilde{h}_1(t,x)=x\frac{\sin t^2}{16}, t\in[0,1], x\in\mathbb{R},\end{aligned}$$

The assumptions of Theorem 2 are satisfied and hypothesis (H_1) holds:

$$
\begin{aligned}
f(t,0) &= \frac{t\sin t^2}{45} \text{ nonidentical null on } [0,1],\\
|f(t,x)-f(t,y)| &\le \frac{\sin t^2}{15}|x-y| = k(t)|x-y|, t\in[0,1], x,y\in\mathbb{R},\\
\|k\|_{L_1} &= \int_0^1 \frac{\sin t^2}{15}dt = 0.020685 <\\
0.023463 &= \frac{(\upsilon+\varsigma-2)\Gamma(\upsilon)\Gamma(\varsigma)}{24}.
\end{aligned}
$$

Let us check hypothesis (H_2):

$$
\begin{aligned}
h_1(t,0) &= 0,\\
|h_1(t,x)-h_1(t,y)| &= \frac{\cos t^2}{8}|x-y|, t\in[0,1], x,y\in\mathbb{R},\\
a_1(t) &= \frac{\cos t^2}{8}, a = \frac{\cos 1}{8} = 0.067538 < \frac{1}{8} = 0.125.
\end{aligned}
$$

Hypothesis (H_3) holds, in fact:

$$
\begin{aligned}
\widetilde{h}_1(t,0) &= 0\\
\left|\widetilde{h}_1(t,x)-\widetilde{h}_1(t,y)\right| &= \frac{\sin t^2}{6}|x-y|, t\in[0,1], x,y\in\mathbb{R},
\end{aligned}
$$

$$
b = \frac{\sin 1}{16} = 0.052592 < \frac{1}{16} = 0.0625.
$$

Hypothesis (H_4) holds. In fact,

$$
c = \max_{j=0,1}\left\|c_j\right\|_{L_1} = 0 < \frac{\Gamma(1.75)}{24}.
$$

We have

$$
\sup\{|f(t,0)|, 0\le t\le 1\} = \frac{\sin 1}{45} = L = 0.018699 and d = 0,
$$

and by computations we get

$$
\max\left(\frac{24pL}{(\upsilon+\varsigma-2)\Gamma(\upsilon)\Gamma(\varsigma)-24p\|k\|_{L_1}}, \frac{24pd}{\Gamma(\varsigma)}\right) = 6.7311.
$$

Now, if we choose $R=7$, then we conclude by Theorem 2 the existence of at least one nontrivial solution u for problem (P1) such that $\|u\|\le 7$.

4 Conclusion

In this paper, we have proven the existence of solutions to a boundary value problem with delay and involving multi-base points right and left Caputo derivatives. The main tools are Arzela-Ascoli theorem, Banach contraction principle and Krasnoselskii fixed point theorem. The presence of impulsive moments with left and right fractional derivatives in the posed problem makes it more complicated and interesting. Similar problems with different types of fractional derivatives will be studied in future works.

Conflict of Interest

The authors declare that they have no conflict of interest.

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