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Numerical Solution to Initial Value Problems for Fractional Differential Equations

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Abstract: Fractional calculus has achieved a great interest in the last decades since many physical problems are modeled with fractional differential equations. The definition of fractional derivatives involves integral operators, some of them having singular kernel, and its calculation is not easy. For that reason, in addition to theoretical developments, it is important to look for accurate numerical approximations to these operators. In this work we propose a new and simple numerical scheme to approximate the solutions to initial value problems involving Caputo-Fabrizio fractional derivatives. Following some previous results, we choose a wavelet basis with special properties, apply the wavelet decomposition to the data, calculate the fractional derivatives of the wavelet basis and combine them by means of a Galerkin-type scheme to reconstruct the unknown from its wavelet coefficients. The properties of the chosen basis guarantee that the numerical scheme is simple, stable and its accuracy can be easily improved. It could be adapted to solve initial value problems combining other fractional and natural order derivatives and fractional partial differential equations. We present some numerical illustrative examples to show its performance.

Keywords: Fractional calculus, Caputo-Fabrizio fractional derivative, wavelet analysis.

1 Introduction

Fractional calculus has achieved a great interest in the last decades since there is a wide range of problems in physics, chemistry, biology, economics and engineering that are modeled with differential equations involving fractional derivatives. In recent papers the importance and usefulness of fractional differential equations involving Riemann-Liouville, Caputo, Atangana-Baleanu and Caputo-Fabrizio derivative, were shown (see [1,2,3,4,5,6,7,8,9,10,11,12]). These fractional derivatives are nonlocal. Their definition involves integral operators, some of them having singular kernel, and its calculation may not be easy. For that reason, in addition to theoretical developments, it is important to look for accurate numerical approximations to these operators. There are many results on fractional calculus devoted to the solvability of fractional differential equations. Theoretical results concerning existence and uniqueness of solutions appear, for example, in [9,10,13,14,15,16]. Different approaches to effectively calculate solutions to differential equations involving fractional derivatives were proposed in [9,10,17,18,19,20,21,22,23].
In this work we propose a new and simple numerical scheme to approximate the solution to the initial value problem (IVP)

$$\begin{cases} D_0^\alpha f(t) + \lambda f(t) = g(t), \\ f(0) = 0, \end{cases} \tag{1}$$

where $D_0^\alpha f$ is the Caputo-Fabrizio fractional derivative of order $\alpha \in (0,1)$ in $[0,b]$, f is the unknown and g is the data, $g(0) = 0$.
In [15], existence and uniqueness for the solution of (1) are proved for $a \neq -\infty$ and $\lambda = 0$. Explicit formulae are presented in [24] for $\lambda = 0$ and in [1] for any λ. Both of them include the computation of the primitive of the data function. In all cases $a \neq -\infty$.
Our approximation technique is based on some previous works [25,26,27,28]. We choose a wavelet basis with appropriate

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properties: well localized in both, time and frequency domain, smooth, band limited, infinitely oscillating with fast decay; and apply the wavelet decomposition to the data, calculate the fractional derivatives of the wavelet basis and combine them by means of a Galerkin-type scheme to reconstruct the unknown from its wavelet coefficients.
We present the approximation scheme by steps, solving two auxiliary problems. The first one is the construction of an approximate primitive f of a given function g considering $a = -\infty$. The second one is the solution to the fractional differential equation $D^{\alpha}_{-\infty} f + \lambda f = g$. Finally, the IVP (1) is solved. The choice of the basis guarantees that the numerical scheme is simple, stable and its accuracy can be easily improved. We do not need to impose any supplementary properties to the data or to the unknown. It is worth noting that this technique could be extended to solve other differential equations containing fractional derivatives when no explicit formulae for the solution is available.
This work is organized as follows. In the next section we introduce the notation and the definitions of the operators. In Section 3 we describe the wavelet basis and present the approximate primitive. The auxiliary fractional differential equation is solved in Section 4. Section 5 contains the solution to the IVP (1). The error is analyzed. Some numerical examples are presented in Section 6. Finally we state some conclusions.

2 Mathematical background

We denote by $H^1(a,b)$, the Sobolev space $W^{1,2}(a,b)$ of functions $u : (a,b) \to \mathbb{R}$, with (weak) derivative $u' = D^1 u \in L^2(a,b)$.
In [29] the authors defined the new Caputo-Fabrizio fractional derivative (CFFD) of order $0 < \alpha < 1$ of $f \in H^1(a,b)$, as

$$D^{\alpha}_{a} f(t) := \frac{M(\alpha)}{1-\alpha} \int_a^t f'(\tau) e^{-\frac{\alpha(t-\tau)}{1-\alpha}} d\tau, \tag{2}$$

where $-\infty \leq a < b$ and $M(\alpha)$ is a normalized factor. The authors proved some results concerning this new derivative that resemble those of the classical one. For instance, the CFFD of order $0 < \alpha < 1$ of a constant function is zero and

$$\lim_{\alpha \to 1} D^{\alpha}_{a} f(t) = f'(t), \qquad \lim_{\alpha \to 0} D^{\alpha}_{a} f(t) = f(t) - f(a). \tag{3}$$

It is worth noting the CFFD is expressed by an integral operator with causal kernel $k(t) = e^{-\frac{\alpha t}{1-\alpha}}$, $t > 0$ that is not singular, $k \in L^1(-\infty, b) \cap L^2(-\infty, b)$.
If we consider $a = -\infty$ and change variables in (2), CFFD can be associated to the convolution operator:

$$D^{\alpha}_{-\infty} f(t) = \frac{M(\alpha)}{1-\alpha} \int_0^{\infty} f'(t-\tau) e^{-\frac{\alpha}{1-\alpha}\tau} d\tau = \frac{M(\alpha)}{1-\alpha} f' * k(t).$$

Then, we have $D^{\alpha}_{-\infty} f \in L^2(-\infty, b)$ and $\widehat{k}(\omega) = \frac{1-\alpha}{\alpha + i\omega(1-\alpha)}$.
Thus, we rewrite the CFFD of order $\alpha \in (0,1)$ as

$$D^{\alpha}_{-\infty} f(t) = \frac{M(\alpha)}{2\pi(1-\alpha)} \int_{\mathbb{R}} \widehat{f'}(\omega) \widehat{k}(\omega) e^{i\omega t} d\omega$$

or

$$D^{\alpha}_{-\infty} f(t) = \frac{M(\alpha)}{1-\alpha} \int_{\mathbb{R}} \widehat{f}(\omega) h(\omega) e^{i\omega t} d\omega \tag{4}$$

with the smooth kernel $h(\omega) = \frac{1}{2\pi} \frac{i\omega(1-\alpha)}{\alpha + i\omega(1-\alpha)}$ that does not depend on t.

3 Approximate primitives

In [28], we propose a method based on the wavelet transform, to find a primitive f of g, i.e., an approximate solution to

$$D^{\alpha}_{-\infty} f(t) = g(t). \tag{5}$$

Below we summarize the method. We describe the wavelet basis where the data g is decomposed. Afterwards the fractional derivative of the basis is calculated. Finally the approximate primitive is presented.

3.1 The wavelet basis

We choose a smooth, infinitely oscillating wavelet with fast decay, $\psi \in \mathscr{S}$ (the Schwartz class), well localized in both time and frequency domains, [30]. Its Fourier transform verify $\text{supp}|\widehat{\psi}(2^{-j}\omega)| = \Omega_j$, where

$$\Omega_j = \left\{\omega : 2^j(\pi-\beta) \le |\omega| \le 2^{j+1}(\pi+\beta)\right\}$$

with $0 < \beta \le \pi/3$.
The family $\{\psi_{jk} = 2^{j/2}\ \psi(2^j t - k),\ j,k \in \mathbb{Z}\}$, is an orthonormal basis (BON) of $L^2(\mathbb{R})$ associated to a Multiresolution Analysis (MRA).
With the usual notation $W_j = \text{span}\{\psi_{jk}, k \in \mathbb{Z}\}$ and $V_J = \bigoplus_{j<J} W_j$ for the wavelet and scale subspaces respectively, we have $L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j = \bigoplus_{j\ge n} W_j + V_n,\ n \in \mathbb{Z}$, ([31], [32]).
There is also a scale function $\phi \in V_0$ such that $\{\phi(t-k), k \in \mathbb{Z}\}$ is BON of V_0.

Remark. The sets Ω_{j-1}, Ω_j, Ω_{j+1} have little overlap and W_j is nearly a basis for the set of functions whose Fourier transform has support in Ω_i (see Figure 1).

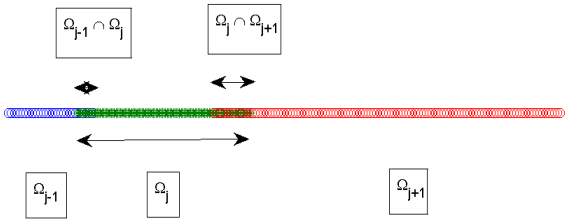

Fig. 1: The sets Ω_{j-1}, Ω_j, Ω_{j+1}

3.2 The data

For any J, the data function $g \in L^2(\mathbb{R})$ can be decomposed as

$$g(t) = \sum_{j\in\mathbb{Z}} \mathscr{Q}_j g(t) = \mathscr{P}_J g(t) + \sum_{j\ge J} \mathscr{Q}_j g(t) = \sum_{n\in\mathbb{Z}} \langle g, \phi_{Jn}\rangle\, \phi_{Jn}(t) + \sum_{j\ge J}\sum_{k\in\mathbb{Z}} \langle g, \psi_{jk}\rangle\, \psi_{jk}(t),$$

where $\mathscr{P}_J g$ and $\mathscr{Q}_j g$ are the orthogonal projections of g in V_J and W_j, respectively. The properties of localization of the wavelets guarantee absolute convergence in each W_j (see [33] for details concerning the basis and its implementation).

We suppose that there exist $J_{min}, J_{max} \in \mathbb{Z}$ such that: $g = \sum_{j=J_{min}}^{J_{max}} g_j + r$ with $||r||_2 < \varepsilon ||g||_2 \cong 0$, i.e., the levels where the energy of g is concentrated. Let $g_j = \sum_{k\in\mathbb{Z}} c_{jk}\psi_{jk} \in W_j$ be the projection on W_j, where $c_{jk} = \langle g, \psi_{jk}\rangle$ are the wavelet coefficients.
We denote by $\tilde{g}_j$ the truncated projection of g on W_j,

$$\tilde{g}_j(t) = \sum_{k\in\mathbb{K}_j} c_{jk}\psi_{jk}(t), \tag{6}$$

where $\mathbb{K}_j \subset \mathbb{Z}$, $|\mathbb{K}_j| = \eta_j < \infty$, that satisfies $\sum_{k\notin\mathbb{K}_j} |\langle g, \psi_{jk}\rangle|^2 < \varepsilon ||g_j||^2$ and $\varepsilon \cong 0$.

3.3 The derivatives of the wavelet basis

Let $v_{jk} = D^{\alpha}_{-\infty}\psi_{jk}$, then

$$v_{jk}(t) = \frac{M(\alpha)}{1-\alpha}\int_{\mathbb{R}} \widehat{\psi}_{jk}(\omega) h(\omega) e^{i\omega t} d\omega. \tag{7}$$

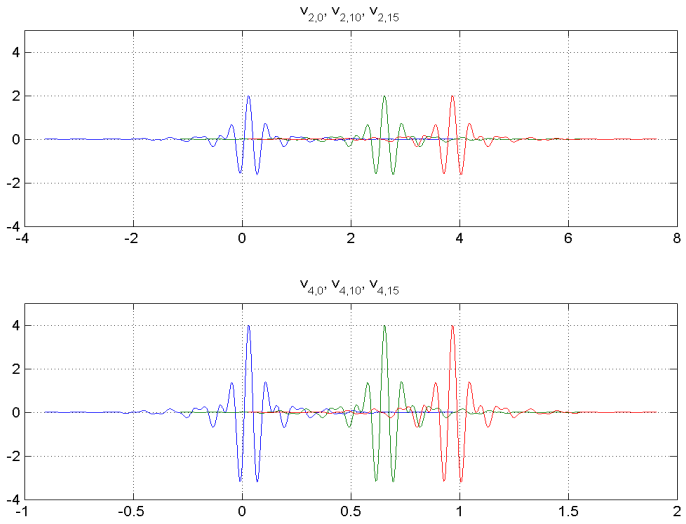

Fig. 2: Some v_{jk} with $\alpha = 0.5$

The functions v_{jk} defined in (7) inherit translation and expansion properties similar to the wavelets ψ_{jk}. We show some of them in Figure 2. Based on the next Lemma we consider $v_{jk} \in W_j$.

Lemma 1. *Let v_{jk} be defined by (7), then $v_{jk} \in W_{j-1} \cup W_j \cup W_{j+1}$. Moreover, the statement $v_{jk} \in W_j$ is nearly true.*

Proof. From (4)

$$v_{jk}(t) = \frac{M(\alpha)}{1-\alpha}\,(\psi'_{jk} * k)(t) \;\Rightarrow\; \widehat{v}_{jk}(\omega) = \frac{M(\alpha)}{(1-\alpha)}\, i\omega\, \widehat{\psi}_{jk}(\omega)\widehat{k}(\omega)$$

then,

$$\mathrm{supp}(\widehat{v}_{jk}) \subseteq \Omega_j.$$

For the chosen wavelet basis, the sets Ω_{j-1}, Ω_j, Ω_{j+1} are nearly non overlapping (see Figure 1), and W_j is a basis for the set of functions whose Fourier transform has support in Ω_j. Thus, disregarding the few wavelets whose Fourier transform has support in the *small* intersections $\Omega_{j-1} \cap \Omega_j$ or $\Omega_j \cap \Omega_{j+1}$, we can consider $v_{jk} \in W_j$.

□

Now we turn back to the calculus of a primitive

$$f(t) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} b_{jk}\psi_{jk}(t). \tag{8}$$

Considering the result of Lemma 1. we work on each level j, $J_{min} \le j \le J_{max}$ and restricted to $\mathbb{K}_j$, i.e., for $\tilde{f}_j = \sum_{k\in\mathbb{K}_j} b_{jk}\psi_{jk}$ it follows

$$D^{\alpha}_{-\infty}\tilde{f}_j(t) = \sum_{k\in\mathbb{K}_j} b_{jk}v_{jk}(t) = \tilde{g}_j(t), \tag{9}$$

then, $\sum_{k\in\mathbb{K}_j} b_{jk}v_{jk} \cong \sum_{k'\in\mathbb{K}_j} c_{j'k'}\psi_{j'k'}$, and we calculate the vector of coefficients $\mathbf{b}^j_k = \{b_{jk}\}_{k\in\mathbb{K}_j}$ from the linear system

$$\left\langle \sum_{l\in\mathbb{K}_j} b_{jl}v_{jl}, \psi_{jm} \right\rangle = \sum_{l\in\mathbb{K}_j} b_{jl}\left\langle v_{jl}, \psi_{jm}\right\rangle = c_{jm},\ m \in \mathbb{K}_j,$$

or, in matrix form

$$\mathscr{M}^j \mathbf{b}^j_k = \mathbf{c}^j_k,\ k \in \mathbb{K}_j, \tag{10}$$

where $\mathscr{M}^j_{lm} = \langle v_{jl}, \psi_{jm} \rangle$.
Based on the properties of the wavelet basis, $\mathscr{M}^j$ is not singular on each level j.

Lemma 2. *$\mathscr{M}^j$ is a band matrix.*

Proof. Since,

$$h(\omega) = \frac{1}{2\pi} \frac{i\omega(1-\alpha)}{\alpha + i\omega(1-\alpha)} = \frac{1}{2\pi} \left(h_{even}(\omega) + i h_{odd}(\omega) \right)$$

there exists $N \in \mathbb{N}$ such that

$$h(\omega) = \frac{1}{2\pi} \left(\sum_{n=0}^{N} a_n \cos(\frac{n\omega}{2^j}) + i \sum_{n=1}^{N} b_n \sin(\frac{n\omega}{2^j}) \right) + \varepsilon(\omega),$$

where $\varepsilon(\omega)$ is an error that is small for large N. Then, from (7),

$$v_{jk}(t) \cong \frac{M(\alpha)}{2\pi(1-\alpha)} \int_{\Omega_j} \widehat{\psi}_{jk}(\omega) \left(\sum_{n=0}^{N} a_n \cos(\frac{n\omega}{2^j}) + i \sum_{n=1}^{N} b_n \sin(\frac{n\omega}{2^j}) \right) e^{i\omega t} d\omega.$$

We observe that,

$$a_n \cos(\frac{n\omega}{2^j}) \widehat{\psi}_{jk}(\omega) = \frac{a_n}{2} (e^{i\frac{n\omega}{2^j}} + e^{-i\frac{n\omega}{2^j}}) \widehat{\psi}_{jk}(\omega) = \frac{a_n}{2} (\widehat{\psi}_{j(k-n)}(\omega) + \widehat{\psi}_{j(k+n)}(\omega))$$

and

$$b_n \sin(\frac{n\omega}{2^j}) \widehat{\psi}_{jk}(\omega) = \frac{b_n}{2i} (e^{i\frac{n\omega}{2^j}} - e^{-i\frac{n\omega}{2^j}}) \widehat{\psi}_{jk}(\omega) = \frac{b_n}{2i} (\widehat{\psi}_{j(k-n)}(\omega) - \widehat{\psi}_{j(k+n)}(\omega)).$$

Then,

$$v_{jk}(t) \cong \frac{M(\alpha)}{1-\alpha} \left[a_0 \psi_{jk}(t) + \sum_{n=1}^{N} \left(\frac{a_n + b_n}{2} \psi_{j(k-n)}(t) + \frac{a_n - b_n}{2} \psi_{j(k+n)}(t) \right) \right]$$

and for $m \in \mathbb{K}_j$, $0 \leq m \leq N$, we can approximate the elements of the matrix

$$\mathscr{M}^j_{km} = \langle v_{jk}, \psi_{jm} \rangle \cong \begin{cases} \frac{M(\alpha)}{1-\alpha} \frac{a_{k-m} + b_{k-m}}{2}, & \text{if } k-m > 0 \\ \frac{M(\alpha)}{1-\alpha} \frac{a_{k+m} - b_{k+m}}{2}, & \text{if } k-m < 0 \end{cases}$$

$$\mathscr{M}^j_{kk} = \langle v_{jk}, \psi_{jk} \rangle \cong \frac{M(\alpha)}{1-\alpha} a_0.$$

The inner products are zero for $m > N$. Thus, $\mathscr{M}^j$ is a band matrix. □

Remark. As expected, in all numerical developments, the matrix $\mathscr{M}^j$ is a diagonal dominant matrix, due to the characteristics of the kernel h and the wavelet basis.

Finally,

$$\tilde{f}(t) = \sum_{j=J_{min}}^{J_{max}} \tilde{f}_j(t) \tag{11}$$

is an approximate solution to (5).

Remark. Note that it is not necessary to calculate v_{jk}. Only the inner product $\mathscr{M}^j_{km} = \langle v_{jk}, \psi_{jm} \rangle$ are needed to solve (10).

4 An auxiliary fractional differential equation

For casual $g \in L^2(-\infty, b]$, $g(0) = 0$, let's consider $f \in H^1(-\infty, b)$, $b \in \mathbb{R}$ such that

$$D^{\alpha}_{-\infty} f(t) + \lambda f(t) = g(t). \tag{12}$$

We construct such f using the scheme described in the previous section.
The relationship between the coefficients of f_j and g_j now reads

$$D^{\alpha}_{-\infty} f_j(t) + \lambda f_j(t) = \sum_{k \in \mathbb{K}_j} b_{jk}(v_{jk}(t) + \lambda \psi_{jk}(t)) = \tilde{g}_j(t)$$

and

$$\sum_{k \in \mathbb{K}_j} b_{jk}(v_{jk}(t) + \lambda \psi_{jk}(t)) \cong \sum_{k' \in \mathbb{K}_j} c_{jk'} \psi_{jk'}(t)$$

or

$$\tilde{\mathscr{M}}^j \mathbf{b}^j_k = \mathbf{c}^j_k,\ k \in \mathbb{K}_j,$$

with $\tilde{\mathscr{M}}^j = (\mathscr{M}^j + \lambda Id)$.

Remark. From Lemma 2., the associated matrix $\tilde{\mathscr{M}}^j$ is a band matrix.
Since,

$$\tilde{\mathscr{M}}^j{}_{km} = \langle v_{jk}, \psi_{jm} \rangle,$$
$$\tilde{\mathscr{M}}^j{}_{kk} = \langle v_{jk}, \psi_{jk} \rangle \cong \frac{M(\alpha)}{1-\alpha} a_0 + \lambda,$$

and the inner products are zero for $m > N$, then the system can be solved efficiently.

Remark. For casual g, the proposed method produces a solution of the auxiliary problem that satisfies $f(0) \cong 0$.
Actually, being $g(t) = 0$ for $t \le 0$, its wavelets coefficients c_{jk} are null, with a few exceptions associated to wavelets with support in a neighborhood of the origin. The coefficients $\mathbf{b}^j_k$, satisfy $\tilde{\mathscr{M}}^j \mathbf{b}^j_k = \mathbf{c}^j_k$, thus $\mathbf{b}^j_k$ are also null.

5 A solution to the IVP

Finally we focus on the IVP (1), i. e.,

$$\begin{cases} D^{\alpha}_0 f(t) + \lambda f(t) = g(t),\ t > 0 \\ f(0) = 0. \end{cases}$$

Note that in this PVI we calculate $D^{\alpha}_0 f$ instead of $D^{\alpha}_{-\infty} f$ as in (12) .
Let's consider the solution f to the equation

$$D^{\alpha}_{-\infty} f(t) + \lambda f(t) = g(t)\chi_{[0,b]}$$

constructed as presented in Section 4., where $\chi_{[0,b]}$ is the characteristic function of the interval $[0,b]$, and define $\overline{f}(t) = f(t)\chi_{[0,b]}$.

Lemma 3. *$\overline{f}$ is a solution to the IVP.*

Proof. We note that:

- for $t < 0$ we have $\overline{f}(t) = 0$ and $\overline{f}'(t) = 0$,
- for $t > 0$, $\overline{f}'(t) = f'(t)$,
- $D^{\alpha}_{-\infty} f(t) + \lambda f(t) = e^{-\frac{\alpha t}{1-\alpha}} D^{\alpha}_{-\infty} f(0) + D^{\alpha}_0 f(t) + \lambda f(t) = e^{-\frac{\alpha t}{1-\alpha}}(g(0) - \lambda f(0)) + D^{\alpha}_0 \overline{f}(t) + \lambda \overline{f}(t)$
  $= D^{\alpha}_0 \overline{f}(t) + \lambda \overline{f}(t)$.

Then, $D^{\alpha}_0 \overline{f}(t) + \lambda \overline{f}(t) = g(t)$ and $\overline{f}(0) = 0$. □

Existence and uniqueness of solution to this IVP has been proved in previous papers (see for example [1] and [24]):

Theorem 1. *If $g \in C(0, +\infty)$ and $g(0) = 0$, there exists a unique continuous solution to*

$$D^{\alpha}_0 f(t) + \lambda f(t) = g(t), \qquad f(0) = 0.$$

Remark. We have constructed an approximation to it.

5.1 The error

While performing the proposed scheme we introduce some approximation errors. Below we give a detailed description of them.

- Approximation of the data g:
  as we mentioned in the Subsection 3.2. The data function $g \in L^2(\mathbb{R})$ can be decomposed as

$$g(t) = \sum_{n\in\mathbb{Z}} \langle g, \phi_{Jn}\rangle \, \phi_{Jn}(t) + \sum_{j\geq J} \sum_{k\in\mathbb{Z}} \langle g, \psi_{jk}\rangle \, \psi_{jk}(t), \tag{13}$$

  then, we introduce an error supposing that there exist $J_{min}, J_{max} \in \mathbb{Z}$ such that: $g = \sum_{j=J_{min}}^{J_{max}} g_j + r$, $g_j \in W_j$, with $||r||_2 < \varepsilon \|g\|_2 \cong 0$. Assumptions on r guarantee that it can be neglected. In addition, g_j is truncated and we actually work with $\tilde{g}_j = \sum_{k\in\mathbb{K}_j} c_{jk}\psi_{jk}(t)$ where $\mathbb{K}_j \subset \mathbb{Z}$, $|\mathbb{K}_j| = \eta_j < \infty$. It satisfies $\sum_{k\notin\mathbb{K}_j} |\langle g, \psi_{jk}\rangle|^2 < \varepsilon ||g_j||^2$ and $\varepsilon \cong 0$. Once more, $\mathbb{K}_j$ is chosen such that the truncation error is small. It can also be reduced considering larger $\mathbb{K}_j$.
- $v_{jk} \in W_j$:
  this statement enables us to consider the equation (9) and it is justified by Lemma 1. The error that this assumption introduces can be reduced considering more levels simultaneously.
- Computation of $\tilde{\mathscr{M}}^j$: some errors are introduced in the calculation of the inner products $\langle v_{jk}, \psi_{jl}\rangle$. The corresponding integrals can be performed in the frequency domain, i.e., on compact subsets and it can be computed with good precision.
- Solving (4):
  matrices $\tilde{\mathscr{M}}^j$ have good condition number (nearly 1). Thus, the linear system can be solved efficiently.

6 Numerical applications

6.1 Example 1

To illustrate the performance of the proposed approximation scheme we solve IVP (1) for $\lambda = 0$, $\alpha = 0.5$ and g a casual function defined as $g(t) = v(t)\sin(t)$ where $v(t)$ is a smooth window in [0,19]. In Figure 3 we show the plots of g and of its *exact* primitive from the formula appearing in [1]. Wavelet analysis indicates that the energy of the data g is concentrated

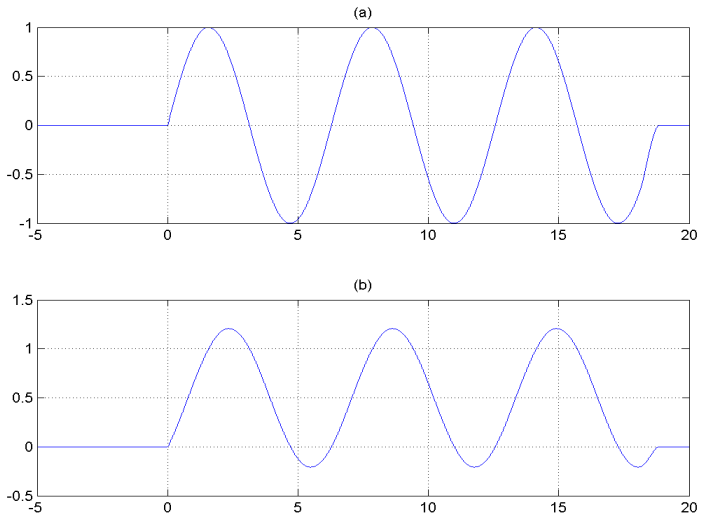

Fig. 3: (a) $g(t) = v(t)\sin(t)$, (b) solution to IVP (1)

in the subspaces W_{-2} and W_{-3} as shown in Table 1.

Table 1: Energy distribution of g

level j	energy of g	frequency
0	0.0009	[3.14, 6.28]
-1	0.0111	[1.57, 3.14]
-2	0.8639	[0.78, 1.57]
-3	0.0996	[0.39, 0.78]
-4	0.0088	[0.19, 0.39]
-5	0.0146	[0.09, 0.19]

For the reconstruction we consider levels $-4 \leq j \leq -1$ and calculate $\tilde{f}_j = \sum_{k\in\mathbb{K}_j} b_{jk}\psi_{jk}$. The approximate solution $\tilde{f}$ to the IVP is obtained adding the components $\sum_{j=-4}^{-1} \tilde{f}_j$. In Figure 4 we plot this solution (green) together with the *exact one* calculated from the formula in [1] (blue).

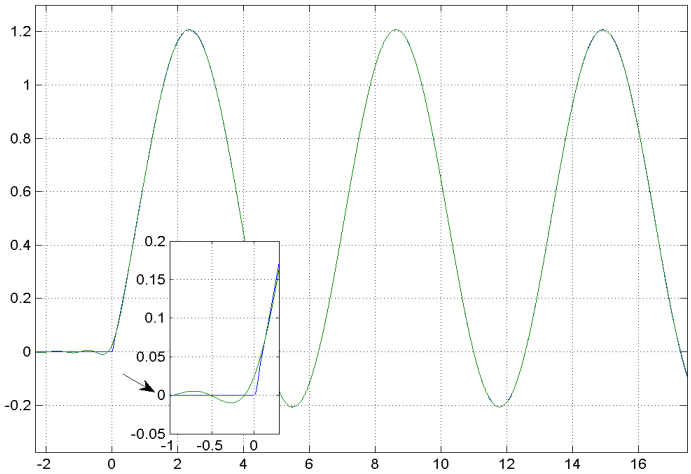

Fig. 4: Approximate primitive of g (green) vs exact solution (blue)

6.2 Example 2

We consider the IVP (1) for $\lambda = 2.1$, $\alpha = 0.5$ and a casual function $g(t) = v(t)e^{-\frac{t^2}{2}}(\sin(12\pi t) - \sin(2.5\pi t))$ where $v(t)$ is a smooth window in [0,4]. In Figure 5 the plots of g and the *exact* solution are shown. Once more we perform the wavelet analysis and observe that 97% of the energy of g is concentrated on the levels $j = 0, 1$ and $j = 3$. We present the energy distribution of g in Table 2.
We consider levels $-1 \leq j \leq 4$, for these levels we calculate $\tilde{f}_j = \sum_{k\in\mathbb{K}_j} b_{jk}\psi_{jk}$ and reconstruct the solution performing $\sum_{j=-1}^{4} \tilde{f}_j$. In Figure 6 we present the approximate solution to the IVP (1) (green) and the *exact* solution (blue).

6.3 Example 3

Finally we consider the IVP (1) for $\lambda = 3.5$, $\alpha = 0.7$ and a casual function $g(t) = v(t)t^2\sin(4.5\pi t)$ where $v(t)$ is a smooth window in [0,5]. In Figure 7 the plots of g and the *exact* solution are shown. In this case the wavelet analysis indicates that 96% of the energy of g is concentrated on the level $j = 2$. We present the energy distribution of g in Table 3.
We consider levels $1 \leq j \leq 2$ and calculate $\tilde{f}_j = \sum_{k\in\mathbb{K}_j} b_{jk}\psi_{jk}$ and reconstruct the solution performing $\sum_{j=1}^{2} \tilde{f}_j$. Finally, in Figure 8 we show the approximate solution $\overline{f}$ of the IVP (1).

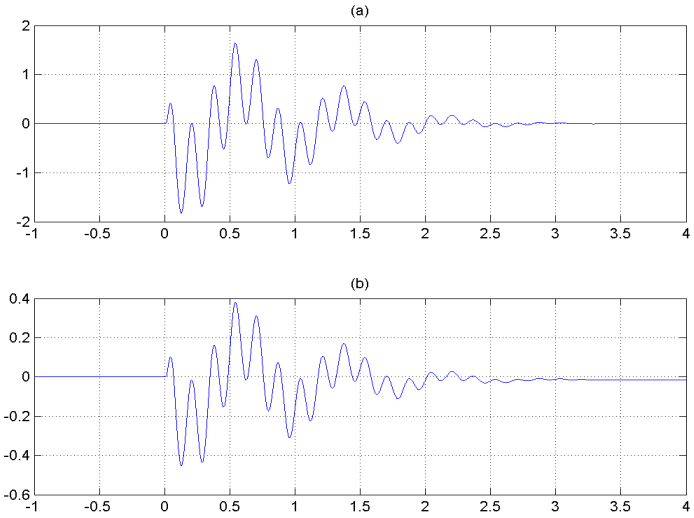

Fig. 5: (a) $g(t) = v(t)\exp(-\frac{t^2}{2})(\sin(12\pi t) - \sin(2.5\pi t))$, (b) solution to the IVP (1)

Table 2: Energy distribution of g

level j	energy of g	frequency
-1	0.0114	[1.57, 3.14]
0	0.0922	[3.14, 6.28]
1	0.3903	[6.28, 12.56]
2	0.0142	[12.56, 25.12]
3	0.4806	[25.12, 50.24]
4	0.0026	[50.24, 100.48]
5	0.0001	[100.48, 200.96]

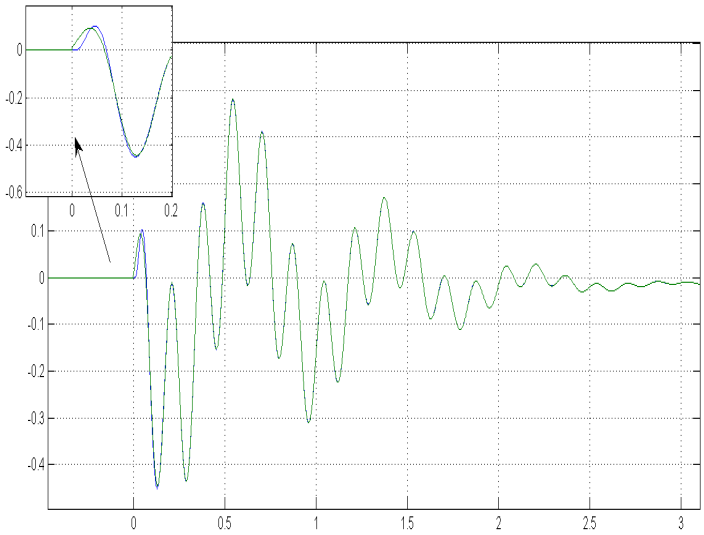

Fig. 6: Approximate solution $\overline{f}$ to the IVP (1) (green) vs *exact* solution (blue)

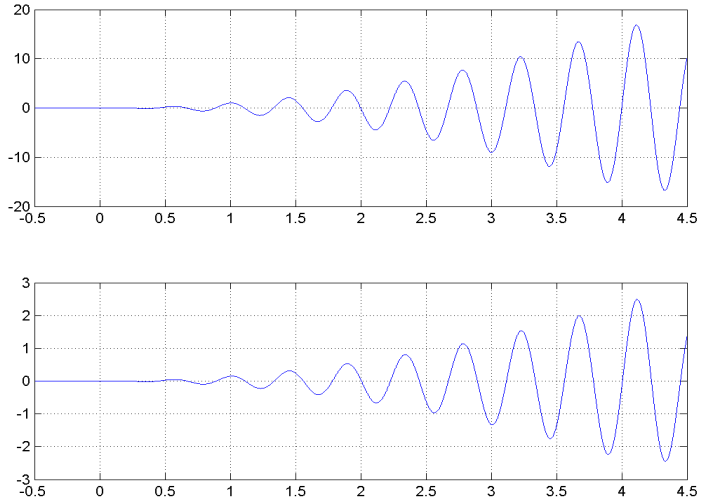

Fig. 7: (a) $g(t) = v(t)t^2 \sin(4.5\pi t)$, (b) solution to the IVP (1)

Table 3: Energy distribution of g

level j	energy of g	frequency
0	0.0003	[3.14, 6.28]
1	0.0300	[6.28, 12.56]
2	0.9693	[12.56, 25.12]
3	0.0004	[25.12, 50.24]

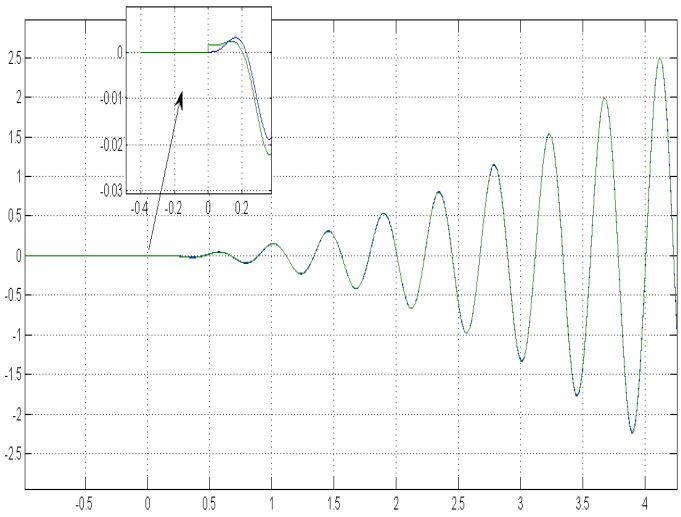

Fig. 8: Approximate solution $\overline{f}$ to the IVP (1) (green) vs *exact* solution (blue)

7 Conclusion

We have presented an approximate solution to an initial value problem involving Caputo-Fabrizio fractional derivatives. Following some previous results we choose a wavelet basis with suitable properties - well localized in both, time and frequency domain, smooth, band limited, infinitely oscillating with fast decay - where the unknown and the data are to be decomposed. In order to solve the IVP (1) we proceed by steps. First we calculate the fractional derivatives of the wavelet basis. Afterwards we solve some auxiliary problems. In all cases the wavelet coefficients of the unknown are the solution to a linear algebraic system of equations. Finally we build the approximate solution to the IVP from its wavelet coefficients. The choice of the basis guarantees that the numerical scheme is simple, stable and its accuracy can be easily improved. We do not need to impose any supplementary properties to the data or to the unknown. If the data is given only at discrete points, we can proceed because we only use its wavelet coefficient. No integration of the data is needed. The matrix to be inverted in each level has shown good condition number in all the numerical examples we have developed. Another approximation we have to carry on is the one involving the images of the wavelet basis and the inner product between them and the wavelets. Once more the properties of the basis enable us to compute them accurately and to be able to refine this calculation if necessary. It is worth noting that this technique could be applied to other fractional derivatives. We exhibit some numerical examples that illustrate the good performance of the proposed scheme.
In future work we are interested in extending this technique to solve IVP combining ordinary and fractional derivatives and boundary value problems for fractional partial differential equations.

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