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New Approximate Solutions to Fractional Smoking Model Using the Generalized Mittag-Leffler Function Method

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Abstract: Smoking is one of the principal drivers of health problems and continues being one of the world's most critical health challenges. So in this work, we addressed the elements of a giving up smoking model containing fractional derivatives. Generalized Mittag-Leffler function method (for short GMLFM) is applied to obtained approximate and analytical solutions of nonlinear fractional differential equation systems such as a smoking model of fractional order. The solution of this model will be acquired in the type of infinite series which converges quickly to its correct esteem. In addition, we compare our outcomes and the outcomes obtained by the Runge-Kutta method to demonstrate the dependability and effortlessness of the technique. Moreover, the solutions obtained are displayed graphically.

Keywords: Fractional calculus, nonlinear fractional dynamics systems, Mittag-Leffler function, smoking model.

1 Introduction, motivation and preliminaries

Fractional calculus (FC) is a branch of mathematics that manages derivatives and integrals whose order might be an arbitrary number, in this way it is considered a generalization of integer-order differentiation and integration. It began more than 300 years ago when the notation for differentiation of non-integer order $1/2$ was argued among Leibniz and L'Hospital. Since then, fractional calculus has been grown step by step, is currently a very active research area of mathematical analysis as attested by the huge number of publications [1,2,3,4,5].

Nowadays, smoking is one of the real medical issues on the planet. In excess of 5 million deaths in the world are caused due to the effect of smoking in different organs of the human body. A shot of heart attack is 70% more in smokers contrasted with the people who are not smoking. Smokers have a 10% higher rate of lung malignancy than that of non-smokers. Bad breath, hypertension, coughing are the primary impacts of short-term smoking. Lately, mouth cancer, throat malignancy, lung tumor, gum infection, coronary illness, stomach ulcers are the main threatening because of long-term smoking. The life of smokers is 10 to 13 years shorter than that of non-smokers. Smoking kills numerous people in their most active life as indicated by the reports of the World Health Organization (WHO). Each researcher, specialist and mathematician attempts to control smoking for securing the future of individuals. To provide a better description of cigarette smoking phenomena, mathematicians tried to make distinctive compelling smoking models.

Since 2000, a lot of efforts have been made by many researchers to understand the dynamics of smoking, in order to predict the effect of smokers on society and decreasing the number of smokers. For example, the first model suggested by Garsow et al. [6], in which they split a total population into three classes: potential smokers P(t), that is, people who do not smoke yet but might become smokers in the future, smokers S(t), and quit smokers Q(t), that is, people who have quit smoking forever. After that, many authors developed the form of this model, such as: Sharami and Gumel [7], modified the basic model and explained a novel category of smokers named chain smokers, Zaman [8] derived and extended a smoking model suggested by Garsow et al. taking into consideration the occasional smokers compartment and showed its qualitative behaviour, Ertürk et al. [9] studied a fractional giving up smoking model and obtain the analytic approximate solution using a multi-step differential transform method, Zeb et al. [10] introduced a fractional smoking

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model in which they studied local and global stability of the model and its general solutions in which the interaction between occasional and potential smokers occurs, Alkhudhari et al. [11] introduced the global dynamics of mathematical equations characterizing smoking, Singh et al. [12] examined a fractional giving up smoking model associated with a new fractional derivative with non-singular kernel, Haq et al. [13] studied a fractional giving up smoking model and find the analytic approximate solution using Laplace Adomian decomposition method, Matintu [14] studied a smoking epidemic model which analyzes the spread of smoking in a population and divided a total population into five population classes, namely, potential, moderate, heavy, temporarily recovered, and permanently recovered class and many others.

Here, we study a modified model that describes a giving up smoking model and this model is developed as follows:

$$\begin{cases} \frac{dP(t)}{dt} = bN(t) - \beta_1(t)L(t)P(t) - (d_1+\mu)P(t) + \tau Q(t), \\ \frac{dL(t)}{dt} = \beta_1(t)L(t)P(t) - \beta_2(t)L(t)S(t) - (d_2+\mu)L(t), \\ \frac{dS(t)}{dt} = \beta_2(t)L(t)S(t) - (\gamma+d_3+\mu)S(t), \\ \frac{dQ(t)}{dt} = \gamma S(t) - (\tau+d_4+\mu)Q(t), \\ \frac{dN(t)}{dt} = (b-\mu)N(t) - (d_1P(t)+d_2L(t)+d_3S(t)+d_4Q(t)), \end{cases} \tag{1}$$

with given initial condition

$$P(0) = m_1,\ L(0) = m_2,\ S(0) = m_3,\ Q(0) = m_4,\ N(0) = m_5, \tag{2}$$

where $P(t)$, $L(t)$, $S(t)$, $Q(t)$ and $N(t)$ indicate the numbers of potential smokers, light smokers, smokers, quit smokers and total smokers at time t, respectively. Here, b is the birth rate, μ is the natural death rate, γ is the recovery rate from smoking, $\beta_1(t)$ and $\beta_2(t)$ are transmission coefficients, d_1, d_2, d_3 and d_4 describes the death rates of potential, occasional, smokers and quit smokers concerning smoking disease, respectively. Furthermore, τ speaks to the rate at which a quitting smoker turns into a potential smoker once more.

Now we present an extension of the system (1) which involves a Caputo fractional derivative on this system for any arbitrary order α (where $0<\alpha\leq 1$). The main purpose for this extension is the smoking model of integer order (1) does not convey any data about the memory and learning component of the populace which influences the spread of sickness. It is well known that the integer derivative is a local operator but the fractional order operator is non-local. This implies the next state of the system of fractional order depends not only upon its current state but also upon all historical states. This is more sensible, and the outcomes inferred of the fractional systems are more broad nature. Then, the smoking model in the fractional order is given by

$$\begin{cases} {}^{C}D_t^{\alpha}P(t) = bN(t) - \beta_1(t)L(t)P(t) - (d_1+\mu)P(t) + \tau Q(t), \\ {}^{C}D_t^{\alpha}L(t) = \beta_1(t)L(t)P(t) - \beta_2(t)L(t)S(t) - (d_2+\mu)L(t), \\ {}^{C}D_t^{\alpha}S(t) = \beta_2(t)L(t)S(t) - (\gamma+d_3+\mu)S(t), \\ {}^{C}D_t^{\alpha}Q(t) = \gamma S(t) - (\tau+d_4+\mu)Q(t), \\ {}^{C}D_t^{\alpha}N(t) = (b-\mu)N(t) - (d_1P(t)+d_2L(t)+d_3S(t)+d_4Q(t)), \end{cases} \tag{3}$$

where ${}^{C}D_t^{\alpha}$ is the Caputo fractional derivative with respect to time of order α, with the use of initial conditions announced in Eq.(2).

There are numerous methods are used to find the approximation solutions for differential equations. However, for the fractional differential equations, there are only restricted methods, such as Laplace transform method, the homotopy analysis method, the Fourier transform method, the iteration method, the homotopy perturbation method, and the Adomian decomposition method. For the further detailed study, we indicate to see [15,16,17,18,19]. In most cases, the exact analytical solutions for the nonlinear problem are very scarce and will resort to various numerical methods, but these methods commonly requirement large computation work and have round-off error problems. For the sake of what has been said previously, we introduce in this work a simple and easy technique called GMLFM to find the analytical and approximation solutions of linear and nonlinear fractional differential equations, whereas the most scientific problems and phenomena are modeled by linear and nonlinear differential equations. It is necessary to mention that the motive or the main objective of this work is used of the GMLFM for solving a fractional-order smoking model.

This paper is organized as follows. In the next Section, we present some basic concepts of fractional calculus specifically pertinent to this work for understanding our main results are presented in this article, as these concepts are important to develop the main results. In Section 3, we define and formulate the generalized Mittag-Leffler function method and this proposed method is applied to solve a smoking model of fractional order. In Section 4, we introduce numerical simulations for the main results and illustrated it by graphics. Finally, in Section **??** presents some conclusions of this research.

2 Some preliminaries in fractional calculus

Here, we present an audit of certain definitions and primer actualities which are especially applicable for the aftereffects of this article. For extra detailed knowledge, we suggest seeing [20,21,22].

Definition 1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f(t)$, can be defined as

$$
{}_{t_0}I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau, \;\; t_0 \geq 0, \;\; t > t_0,
$$
$$
{}_{t_0}I_t^{0} f(t) = f(t),
$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2. The Caputo fractional derivative of a function $f(t)$, of order $\alpha > 0$ is defined as

$$
{}_{t_0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \;\; t_0 \geq 0, \;\; t > t_0,
$$

for $n-1 < \alpha \leq n$, $n \in \mathbb{N}$. *For* $0 < \alpha < 1$, *Caputo fractional derivative becomes*

$$
{}_{t_0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{\grave{f}(\tau)}{(t-\tau)^{\alpha}} d\tau, \;\; t_0 \geq 0, \;\; t > t_0.
$$

The Caputo fractional derivative has advantages when trying to represent real-world phenomena with fractional differential equations being the way that the fractional derivative of constants are zero.

Theorem 1. Considered $f(t)$ be a differentiable function in $[t_0, t]$, $\alpha > 0$. Then,

$$
{}_{t_0}^{C}D_t^{\alpha} \, {}_{t_0}I_t^{\alpha} f(t) = f(t),
$$

$$
{}_{t_0}I_t^{\alpha} \, {}_{t_0}^{C}D_t^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(t_0) \frac{(t-t_0)^k}{k!}.
$$

For more details of fractional calculus, a reader can look up in the mention references before

Definition 3. The Mittag-Leffler function of two-parameter defined by

$$
E_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(n\alpha+\beta)}, \qquad \alpha, \beta > 0, \tag{4}
$$

if $\beta = 1$, *this function is indicated by* $E_{\alpha}(\cdot)$, *and if* $\alpha = \beta = 1$ *this function represent the exponential function.*

3 Analyze the method and apply it to the smoking model

This section is dedicated to the proposed method used in the paper and is called GMLFM. This method suggests that $x_i(t)$, $i = 1, 2, 3, \ldots$ are decomposed by an infinite series of components:

$$
x_i(t) = E_{\alpha}(a_i t^{\alpha}) = \sum_{n=0}^{\infty} a_i^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \tag{5}
$$

by using Caputo fractional derivative we have

$$^{C}D^{\alpha}x_i(t)=\sum_{n=1}^{\infty}a_i^n\frac{t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)}. \tag{6}$$

The GMLFM has effectively demonstrated their proficiency as solutions of fractional order differential and integral equations and consequently have turned out to be significant components of the fractional calculus theory and applications [23,24], and the convergence of this method discussed in [25].

Now, we will explain how to solve system of nonlinear fractional differential equations (Smoking Dynamics System) through the imposition of the generalized Mittag-Leffler function $E_\alpha(\cdot)$. Let

$$\begin{aligned}
P(t) &= \textstyle\sum_{n=0}^{\infty}\frac{a^n t^{n\alpha}}{\Gamma(n\alpha+1)},\\
L(t) &= \textstyle\sum_{n=0}^{\infty}\frac{b^n t^{n\alpha}}{\Gamma(n\alpha+1)},\\
S(t) &= \textstyle\sum_{n=0}^{\infty}\frac{d^n t^{n\alpha}}{\Gamma(n\alpha+1)},\\
Q(t) &= \textstyle\sum_{n=0}^{\infty}\frac{a_*^n t^{n\alpha}}{\Gamma(n\alpha+1)},\\
N(t) &= \textstyle\sum_{n=0}^{\infty}\frac{b_*^n t^{n\alpha}}{\Gamma(n\alpha+1)}.
\end{aligned} \tag{7}$$

Applying Caputo fractional derivative for equations in (7) we obtain

$$\begin{aligned}
^{C}D_t^{\alpha}P(t) &= \textstyle\sum_{n=1}^{\infty}\frac{a^n t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)},\\
^{C}D_t^{\alpha}L(t) &= \textstyle\sum_{n=1}^{\infty}\frac{b^n t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)},\\
^{C}D_t^{\alpha}S(t) &= \textstyle\sum_{n=1}^{\infty}\frac{d^n t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)},\\
^{C}D_t^{\alpha}Q(t) &= \textstyle\sum_{n=1}^{\infty}\frac{a_*^n t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)},\\
^{C}D_t^{\alpha}N(t) &= \textstyle\sum_{n=1}^{\infty}\frac{b_*^n t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)}.
\end{aligned} \tag{8}$$

By replacing from Eqs. (7) and (8) in (3) we have

$$\begin{aligned}
0 &= \textstyle\sum_{n=0}^{\infty}\frac{a^{n+1}t^{n\alpha}}{\Gamma(n\alpha+1)}-b\sum_{n=0}^{\infty}\frac{b_*^n t^{n\alpha}}{\Gamma(n\alpha+1)}+\beta_1\sum_{n=0}^{\infty}c^n t^{n\alpha}\\
&\quad +(d_1+\mu)\textstyle\sum_{n=0}^{\infty}\frac{a^n t^{n\alpha}}{\Gamma(n\alpha+1)}-\tau\sum_{n=0}^{\infty}\frac{a_*^n t^{n\alpha}}{\Gamma(n\alpha+1)},\\
0 &= \textstyle\sum_{n=0}^{\infty}\frac{b^{n+1}t^{n\alpha}}{\Gamma(n\alpha+1)}-\beta_1\sum_{n=0}^{\infty}c^n t^{n\alpha}+\beta_2\sum_{n=0}^{\infty}c_*^n t^{n\alpha}\\
&\quad +(d_2+\mu)\textstyle\sum_{n=0}^{\infty}\frac{b^n t^{n\alpha}}{\Gamma(n\alpha+1)},\\
0 &= \textstyle\sum_{n=0}^{\infty}\frac{d^{n+1}t^{n\alpha}}{\Gamma(n\alpha+1)}-\beta_2\sum_{n=0}^{\infty}c_*^n t^{n\alpha}+(\gamma+d_3+\mu)\sum_{n=0}^{\infty}\frac{d^n t^{n\alpha}}{\Gamma(n\alpha+1)},\\
0 &= \textstyle\sum_{n=0}^{\infty}\frac{a_*^{n+1}t^{n\alpha}}{\Gamma(n\alpha+1)}-\gamma\sum_{n=0}^{\infty}\frac{d^n t^{n\alpha}}{\Gamma(n\alpha+1)}+(\tau+d_4+\mu)\sum_{n=0}^{\infty}\frac{a_*^n t^{n\alpha}}{\Gamma(n\alpha+1)},\\
0 &= \textstyle\sum_{n=0}^{\infty}\frac{b_*^{n+1}t^{n\alpha}}{\Gamma(n\alpha+1)}-(b-\mu)\sum_{n=0}^{\infty}\frac{b_*^n t^{n\alpha}}{\Gamma(n\alpha+1)}+(d_1\sum_{n=0}^{\infty}\frac{a^n t^{n\alpha}}{\Gamma(n\alpha+1)}\\
&\quad +d_2\textstyle\sum_{n=0}^{\infty}\frac{b^n t^{n\alpha}}{\Gamma(n\alpha+1)}+d_3\sum_{n=0}^{\infty}\frac{d^n t^{n\alpha}}{\Gamma(n\alpha+1)}+d_4\sum_{n=0}^{\infty}\frac{a_*^n t^{n\alpha}}{\Gamma(n\alpha+1)}),
\end{aligned} \tag{9}$$

where

$$c^n = \sum_{k=0}^{n} \frac{b^k a^{n-k}}{\Gamma(k\alpha+1)\Gamma((n-k)\alpha+1)}, \qquad c_*^n = \sum_{k=0}^{n} \frac{b^k d^{n-k}}{\Gamma(k\alpha+1)\Gamma((n-k)\alpha+1)}.$$

From system (9) we have

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty}(a^{n+1} - bb_*^n + \beta_1 c^n \Gamma(n\alpha+1) + (d_1+\mu)a^n - \tau a_*^n)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)},\\
0 &= \sum_{n=0}^{\infty}(b^{n+1} - \beta_1 c^n \Gamma(n\alpha+1) + \beta_2 c_*^n \Gamma(n\alpha+1) + (d_2+\mu)b^n)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)},\\
0 &= \sum_{n=0}^{\infty}(d^{n+1} - \beta_2 c_*^n \Gamma(n\alpha+1) + (\gamma+d_3+\mu)d^n)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)},\\
0 &= \sum_{n=0}^{\infty}(a_*^{n+1} - \gamma d^n + (\tau+d_4+\mu)a_*^n)\frac{t^{n\alpha}}{\Gamma(n\alpha+1)},\\
0 &= \sum_{n=0}^{\infty}(b_*^{n+1} - (b-\mu)b_*^n + (d_1a^n + d_2b^n + d_3d^n + d_4a_*^n))\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}.
\end{aligned} \tag{10}$$

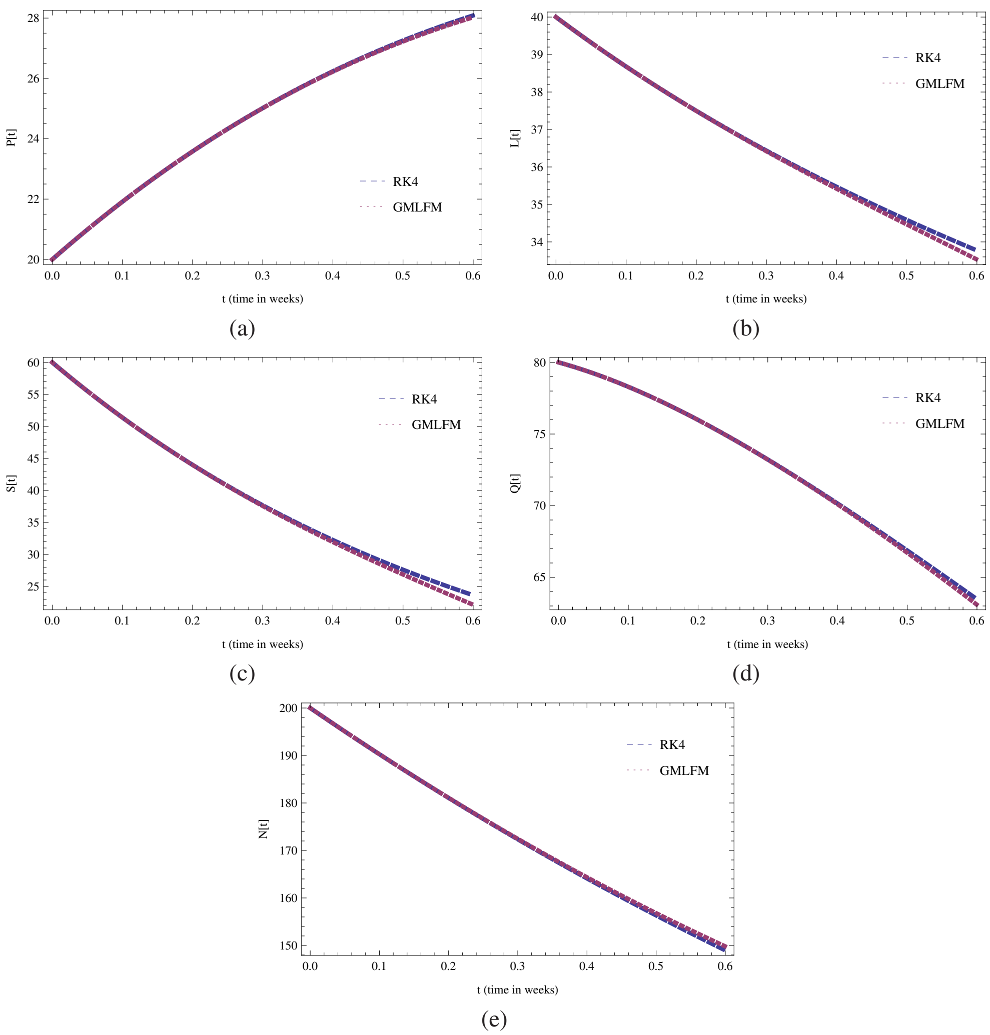

Fig. 1: Comparison between the results obtained by GMLFM and RK4 for $P(t)$, $L(t)$, $S(t)$, $Q(t)$ and $N(t)$ with $\alpha = 1$.

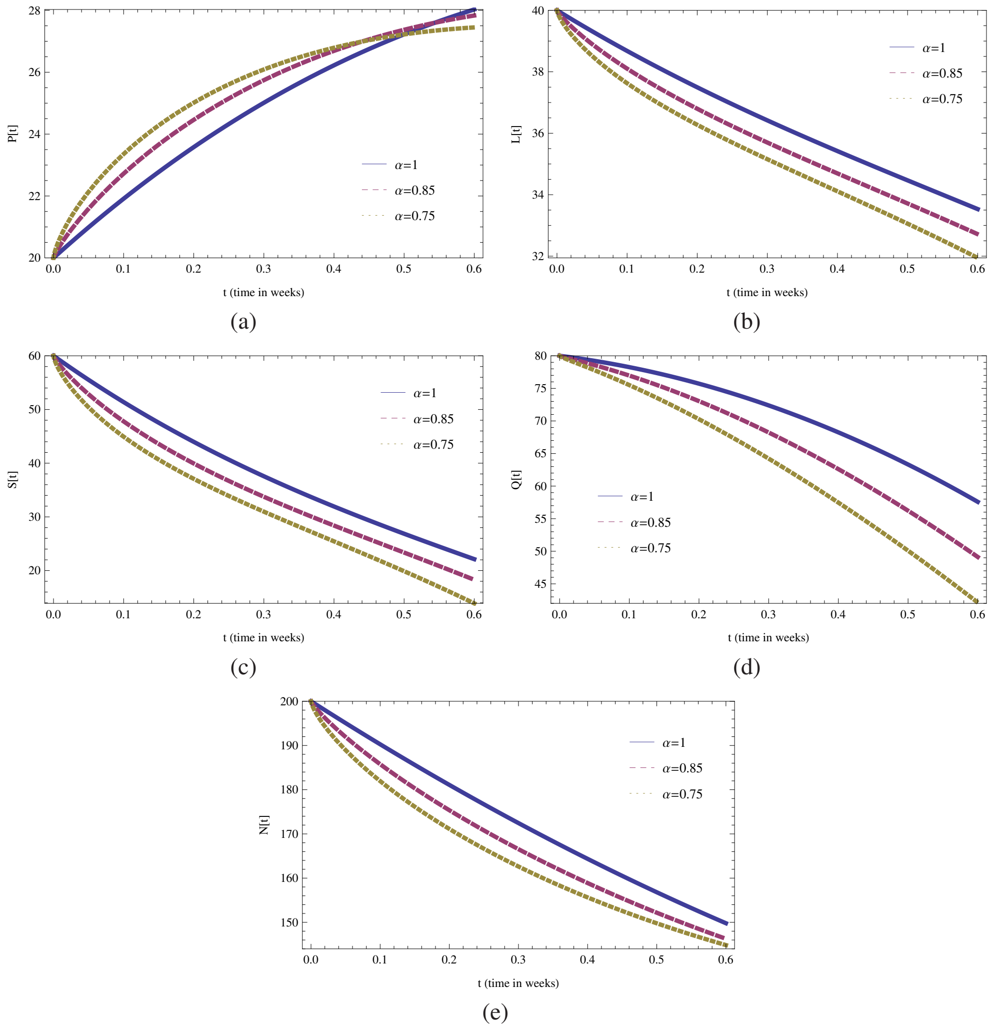

Fig. 2: The proximate solutions obtained by GMLFM for $P(t)$, $L(t)$, $S(t)$, $Q(t)$ and $N(t)$ with different values of α.

From system (10) we observe that impossible the variable $t^{n\alpha}$ is equal zero, then the coefficients are equal zero. So, we have

$$\begin{aligned}
a^{n+1} &= bb_*^n - \beta_1 c^n \Gamma(n\alpha+1) - (d_1+\mu)a^n + \tau a_*^n, \\
b^{n+1} &= \beta_1 c^n \Gamma(n\alpha+1) - \beta_2 c_*^n \Gamma(n\alpha+1) - (d_2+\mu)b^n, \\
d^{n+1} &= \beta_2 c_*^n \Gamma(n\alpha+1) - (\gamma+d_3+\mu)d^n, \\
a_*^{n+1} &= \gamma d^n - (\tau+d_4+\mu)a_*^n, \\
b_*^{n+1} &= (b-\mu)b_*^n - (d_1 a^n + d_2 b^n + d_3 d^n + d_4 a_*^n).
\end{aligned} \tag{11}$$

We start with the initial conditions

$$a^0 = m_1, \;\; b^0 = m_2, \;\; d^0 = m_3, \;\; a_*^0 = m_4, \;\; b_*^0 = m_5,$$

and, using the recurrence relation given in the system (11) we have

$$
\begin{aligned}
a^1 &= bm_5 - \beta_1 m_1 m_2 - (d_1+\mu)m_1 + \tau m_4,\\
b^1 &= \beta_1 m_1 m_2 - \beta_2 m_2 m_3 - (d_2+\mu)m_2,\\
d^1 &= \beta_2 m_2 m_3 - (\gamma + d_3 + \mu)m_3,\\
a^1_* &= \gamma m_3 - (\tau + d_4 + \mu)m_4,\\
b^1_* &= (b-\mu)m_5 - d_1 m_1 - d_2 m_2 - d_3 m_3 - d_4 m_4,
\end{aligned}
$$

$$
\begin{aligned}
a^2 = {} & b((b-\mu)m_5 - d_1m_1 - d_2m_2 - d_3m_3 - d_4m_4) - \beta_1 m_2(bm_5 - \beta_1 m_1 m_2\\
& -(d_1+\mu)m_1 + \tau m_4) - \beta_1 m_1(\beta_1 m_1 m_2 - \beta_2 m_2 m_3 - (d_2+\mu)m_2)\\
& -(d_1+\mu)(bm_5 - \beta_1 m_1 m_2 - (d_1+\mu)m_1 + \tau m_4) + \tau(\gamma m_3 - (\tau + d_4 + \mu)m_4),
\end{aligned}
$$

$$
\begin{aligned}
b^2 = {} & \beta_1 m_2(bm_5 - \beta_1 m_1 m_2 - (d_1+\mu)m_1 + \tau m_4) + \beta_1 m_1(\beta_1 m_1 m_2 - \beta_2 m_2 m_3\\
& -(d_2+\mu)m_2) - \beta_2 m_2(\beta_2 m_2 m_3 - (\gamma + d_3 + \mu)m_3) - \beta_2 m_3(\beta_1 m_1 m_2 - \beta_2 m_2 m_3\\
& -(d_2+\mu)m_2) - (d_2+\mu)(\beta_1 m_1 m_2 - \beta_2 m_2 m_3 - (d_2+\mu)m_2),
\end{aligned}
$$

$$
\begin{aligned}
d^2 = {} & \beta_2 m_2(\beta_2 m_2 m_3 - (\gamma + d_3 + \mu)m_3) + \beta_2 m_3(\beta_1 m_1 m_2 - \beta_2 m_2 m_3 - (d_2+\mu)m_2)\\
& -(\gamma + d_3 + \mu)(\beta_2 m_2 m_3 - (\gamma + d_3 + \mu)m_3),
\end{aligned}
$$

$$
a^2_* = \gamma(\beta_2 m_2 m_3 - (\gamma + d_3 + \mu)m_3) - (\tau + d_4 + \mu)(\gamma m_3 - (\tau + d_4 + \mu)m_4),
$$

$$
\begin{aligned}
b^2_* = {} & (b-\mu)((b-\mu)m_5 - d_1m_1 - d_2m_2 - d_3m_3 - d_4m_4) - d_1(bm_5 - \beta_1 m_1 m_2\\
& -(d_1+\mu)m_1 + \tau m_4) - d_2(\beta_1 m_1 m_2 - \beta_2 m_2 m_3 - (d_2+\mu)m_2) - d_3(\beta_2 m_2 m_3\\
& -(\gamma + d_3 + \mu)m_3) - d_4(\gamma m_3 - (\tau + d_4 + \mu)m_4),
\end{aligned}
$$

we can obtain the remaining terms similarly. Finally, by using the system (7) we obtain

$$
\begin{aligned}
P(t) &= a^0 + a^1\tfrac{t^\alpha}{\Gamma(\alpha+1)} + a^2\tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots = \textstyle\sum_{n=0}^{\infty} \tfrac{a^n t^{n\alpha}}{\Gamma(n\alpha+1)},\\
L(t) &= b^0 + b^1\tfrac{t^\alpha}{\Gamma(\alpha+1)} + b^2\tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots = \textstyle\sum_{n=0}^{\infty} \tfrac{b^n t^{n\alpha}}{\Gamma(n\alpha+1)},\\
S(t) &= d^0 + d^1\tfrac{t^\alpha}{\Gamma(\alpha+1)} + d^2\tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots = \textstyle\sum_{n=0}^{\infty} \tfrac{d^n t^{n\alpha}}{\Gamma(n\alpha+1)},\\
Q(t) &= a^0_* + a^1_*\tfrac{t^\alpha}{\Gamma(\alpha+1)} + a^2_*\tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots = \textstyle\sum_{n=0}^{\infty} \tfrac{a^n_* t^{n\alpha}}{\Gamma(n\alpha+1)},\\
N(t) &= b^0_* + b^1_*\tfrac{t^\alpha}{\Gamma(\alpha+1)} + b^2_*\tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots = \textstyle\sum_{n=0}^{\infty} \tfrac{b^n_* t^{n\alpha}}{\Gamma(n\alpha+1)}.
\end{aligned} \tag{12}
$$

4 Numerical results and discussions

In this section, we find numerical simulation of the considered problem (3), using values of the parameter: $m_1 = 20$, $m_2 = 40$, $m_3 = 60$, $m_4 = 80$, $m_5 = 200$, $d_1 = 0.33$, $d_2 = 0.44$, $d_3 = 0.55$, $d_4 = 0.66$, $\mu = 0.05$, $b = 0.1$, $\beta_1 = 0.01$, $\beta_2 = 0.001$, $\tau = 0.2$, $\gamma = 0.99$.

We presented in Figure 1 a comparison between solutions obtained by GMLFM and Runge-Kutta method for $P(t)$, $L(t)$, $S(t)$, $Q(t)$ and $N(t)$ using classical order $\alpha = 1$. The plain dotted line denotes the solutions of the system (3) by GMLFM while the dashed line denotes the solutions of the same system by using Runge-Kutta method when $\alpha = 1$. From the graphical results, it can be seen that the results obtained using the GMLFM match the results of the Runge-Kutta method very well, which infers that the introduced strategy can anticipate the conduct of these factors precisely for the region under consideration.

Figure (2) represents the fractional-order solutions (different values of α) which obtained by GMLFM for $P(t)$, $L(t)$, $S(t)$, $Q(t)$ and $N(t)$. From this representation in Fig.(2) , obviously, the solutions depend continuously on the fractional derivative α. It is obvious that the effectiveness of this methodology can be significantly improved by diminishing the step size and computing further terms or further components for each variable.

5 Conclusion

This article concerns the fractional-order nonlinear system in which described the model giving up smoking, and its approximate and analytical solutions are given by a GMLFM. The new generalization is based on the Caputo fractional derivative. The solution obtained via GMLFM closely consents to those gotten by Runge-Kutta method. The numerical simulation of the outcomes clarifies this strategy is an intense and proficient method for finding analytical solutions for wide classes of fractional nonlinear systems.

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