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Prediction of Exponentiated Family Distributions Observables under Type-II Hybrid Censored Data

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Abstract: The current study addresses Bayesian prediction intervals for exponentiated family distributions observables under Type-II hybrid censored data. It includes a set of exponentiated distributions such as exponentiated linear failure rate, exponentiated Weibull (EW), exponentiated modified Weibull, exponentiated Rayleigh (ER), exponentiated exponential (EE) distributions...etc. One and two-sample Bayesian predictive survival function under Type-II hybrid censored data are derived. Markov chain Monte Carlo (MCMC) sampling method utilized to generate samples from posterior distributions and Bayesian prediction intervals calculated. Numerical results obtained under two exponentiated distributions, EE(δ) distribution when δ is unknown and ER(δ,γ) distribution when parameters δ and γ are unknown.

Keywords: Bayesian prediction, Exponentiated exponential, Exponentiated Rayleigh, Gibbs sampling, MCMC, Hybrid censored.

1 Introduction

Type-I and Type-II censoring schemes are the most commonly applied censoring schemes. In Type-I censoring scheme, the number of observed failures is a random variable, while the experimental time is fixed. On the other hand, in Type-II censoring scheme the experimental time is a random variable, while number of observed failures is fixed. Hybrid censoring scheme is a mixture of Type-I and Type-II censoring schemes proposed firstly by Epstein[1] on 1954. The scheme can be specified as following:

let n items and its ordered failure times be $X_{1:n} \leq, \ldots, \leq X_{n:n}$. The experiment of Type-I hybrid censoring lasts till reaching a pre-determined time of T or failure of the pre-specified number $r < n$, out of n items. At the random time $T_1^* = \min(X_{r:n}, T)$, the experiment terminated. In Type-II hybrid censoring, the experiment termination time is $T_2^* = \max(X_{r:n}, T)$ providing at least r failures during the termination time. By taking $r = n$ and $T \to \infty$, Type-I and Type-II censoring are special cases of hybrid censoring scheme. Hybrid censoring scheme is a common censoring scheme utilized to investigate the statistical inference of various hybrid censoring scheme in reliability studies. Hybrid censored data arising from various parametric distributions have been analyzed by many authors, see, Chen and Bhattacharya [2], Ebrahimi [3], Gupta and Kundu [4], Childs et al. [5], Kundu [6], Kundu and Pradhan [7], Dube et al. [8], Balakrishnan and Shafay [9], Ganguly et al. [10], Singh et al. [11], Deya and Pradhan [12] and Mohie El-Din et al. [13].

The prediction problems of the life time models are essential as indicated by Dunsmore [14], AL-Hussaini ([15], [16]), Balakrishnan and Shafay [9] and Mohie El-Din et al. [13], among others. In this paper, we study the prediction problems for exponentiated family distributions observables under Type-II hybrid censored data.

The exponentiated family cumulative distribution function (cdf) is defined as

$$F_X(x;\underline{\theta}) = \left(1 - e^{-\lambda(x;\underline{\beta})}\right)^{\delta}, \tag{1}$$

Where $\delta > 0$ and $\lambda(x;\underline{\beta})$ is a non-negative, continuous, monotone increasing, differentiable function of x whereas $\lambda(x;\underline{\beta}) \to 0$ as $x \to 0^+$, $\lambda(x;\underline{\beta}) \to \infty$ as $x \to \infty$. $F_X(x;\underline{\theta})$ based on the unknown parameters $\underline{\theta} = (\delta, \underline{\beta})$. The probability

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density function (pdf) corresponding to $F_X(x;\underline{\theta})$ is

$$f_X(x;\underline{\theta}) = \delta\,\lambda'(x;\underline{\beta})\,e^{-\lambda(x;\underline{\beta})}\left(1-e^{-\lambda(x;\underline{\beta})}\right)^{\delta-1}, \tag{2}$$

and the survival function is

$$\bar{F_X}(x;\underline{\theta}) = 1-\left(1-e^{-\lambda(x;\underline{\beta})}\right)^{\delta}. \tag{3}$$

The exponentiated family distributions contains many exponentiated distributions such as exponentiated generalized linear exponential, exponentiated linear failure rate, exponentiated modified Weibull, exponentiated Gompertz, EW, EE, ER, exponentiated Burr Type XII, exponentiated Lomax, exponentiated Pareto, exponentiated Gamma distributions,...etc. Exponentiated family distributions have been applied by many authors. For instance, EW family utilized to analyze bathtub failure data by Mudholkar and Srivastava [17]. Gupta and Kundu [18] proposed various interesting properties of the EE distribution, it is clear that the EE distribution can be utilized effectively to analyze diverse lifetime data in place of gamma distribution or Weibull distribution. The EE has increasing or decreasing hazard function, like the gamma distribution. In many cases it might provide better data fit to a particular data set than the gamma or Weibull distribution.

The layout of the paper is as follows. In the following section, the prior and posterior distributions based on exponentiated family distributions are provided. One- and Two-Sample are presented, in addition to Bayesian predictive survival function discussed utilize MCMC under Type-II hybrid censored sample in section 3. In Section 4, we introduce the results for the EE(δ) distribution when δ is unknown and ER(δ,γ) distribution when both parameters δ and γ and are unknown. A real life data set is utilized to illustrate the findings for these examples in Section 5. Finally, we conclude the paper with concluding remarks in Section 6.

2 Posterior Distribution and Conditional Density Function

Suppose that $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ are the order statistics of random sample $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$. Based on Type-II hybrid censored sample, for fixed r and T the experiment is terminated at a random time $T^* = \max(X_{r:n}, T)$, where $1 \le r \le n$ and $0 < T < \infty$. In this case, we have two forms of observations as follows:
Case I. $X_{1:n} < \ldots < X_{r:n}$ when $X_{r:n} \ge T$.
Case II. $X_{1:n} < \ldots < X_{r:n} < X_{r+1:n} < \ldots < X_{k:n} < T < X_{k+1:n}$ if $r \le k < n$ and $X_{k:n} < T < X_{k+1:n}$.
Thence, the likelihood function for the above two cases can be written as

$$L(\underline{x}|\underline{\theta}) = \frac{n!}{(n-d)!}\prod_{i=1}^{d} f(x_i;\underline{\theta})\left[1-F(c;\underline{\theta})\right]^{n-d}, \tag{4}$$

where d and c are respectively given by

$$d = \begin{cases} r, & \text{for Case I} \\ k, & \text{for Case II} \end{cases}, \quad c = \begin{cases} X_{r:n}, & \text{for Case I} \\ T, & \text{for Case II.} \end{cases}$$

Consider a prior density in the formula (see, AL-Hussaini ([15], [16]))

$$\pi(\underline{\theta}) \equiv \begin{cases} \pi(\delta), & \text{when only } \delta \text{ is the unknown parameter of } F_x(x;\underline{\theta}), \\ \pi(\delta,\underline{\beta}), & \text{when all parameters of } F_x(x;\underline{\theta}) \text{ are unknown,} \end{cases} \tag{5}$$

where δ and $\underline{\beta}$ are independent.
From (4) and (5), the posterior density function, $\pi^*(\underline{\theta}|\underline{x})$, is

$$\pi^*(\underline{\theta}|\underline{x}) \propto \prod_{i=1}^{d} f(x_i)\left[1-F(c)\right]^{n-d}\pi(\underline{\theta}). \tag{6}$$

From [9] the conditional density function of $X_{s:n}$ is
Case I

$$f_1(x_s|\underline{\theta}) = \frac{(n-r)!}{(n-s)!\,(s-r-1)!}\,\frac{\left(F(x_s)-F(x_r)\right)^{s-r-1}\left(1-F(x_s)\right)^{n-s} f(x_s)}{\left(1-F(x_r)\right)^{n-r}}, \tag{7}$$

where $\underline{x} = (x_1, \ldots, x_r)$, $x_s > x_r$ and $r < s \le n$.
Equation (7) can be written as

$$f_1(x_s|\underline{\theta}) = \sum_{i=0}^{s-r-1}\sum_{j=0}^{n-s} \frac{(-1)^{i+j}\binom{s-r-1}{i}\binom{n-s}{j}(n-r)!}{(n-s)!\,(s-r-1)!}\,\frac{\Big(F(x_s)\Big)^{s-r-i+j-1}\Big(F(x_r)\Big)^{i} f(x_s)}{\Big(1-F(x_r)\Big)^{n-r}}. \tag{8}$$

Case II

$$\begin{aligned} f_2(x_s|\underline{\theta}) &= \frac{1}{P(r \le K \le s-1)} \sum_{k=r}^{s-1} f_2(x_s|\underline{x}, K=k)\,P(K=k) \\ &= \sum_{k=r}^{s-1} \frac{(n-k)!\,\phi_k(T)}{(s-k-1)!\,(n-s)!}\,\frac{\Big(F(x_s)-F(T)\Big)^{s-k-1}\Big(1-F(x_s)\Big)^{n-s} f(x_s)}{\Big(1-F(T)\Big)^{n-k}} \end{aligned} \tag{9}$$

where $\underline{x} = (x_1, \ldots, x_k)$, $x_k > T$ and $\phi_k(T) = \frac{P(K=k)}{\sum_{j=r}^{s-1} P(K=j)}$.
From (9), we get

$$f_2(x_s|\underline{\theta}) = \sum_{k=r}^{s-1}\sum_{i=0}^{s-k-1}\sum_{j=0}^{n-s} \frac{(-1)^{i+j}\binom{n}{k}\binom{s-k-1}{i}\binom{n-s}{j}(n-k)!}{(n-s)!\,(n-k-1)!}\,\frac{\Big(F(x_s)\Big)^{s-k-i+j-1}\Big(F(T)\Big)^{i+k} f(x_s)}{\sum_{j=r}^{s-1}\binom{n}{j}\Big(F(T)\Big)^{j}\Big(1-F(T)\Big)^{n-j}} \tag{10}$$

Let $Y_{1:m} \le Y_{2:m} \le \ldots \le Y_{m:m}$ be the order statistics from a future random sample of size m from the same population. Hence the marginal density function of the sth order statistics $Y_{s:m}$ is given by

$$\begin{aligned} g_{Y_{s:m}}(y_s|\underline{\theta}) &= \frac{m!}{(s-1)!\,(m-s)!}\Big(F(y_s)\Big)^{s-1}\Big(1-F(y_s)\Big)^{m-s} f(y_s) \\ &= \sum_{i=0}^{m-s} \frac{(-1)^i\binom{m-s}{i} m!}{(s-1)!\,(m-s)!}\Big(F(y_s)\Big)^{s+i-1} f(y_s), \end{aligned} \tag{11}$$

where $r \le s \le m$.

3 Bayesian prediction using MCMC technique

In this section, we use MCMC techniques to generate samples from posterior distributions, subsequently calculate the Bayesian prediction intervals.

3.1 One-sample Bayesian prediction

The predictive density function $h(x_s|\underline{x})$ of $X_{s:n}$ is

$$h(x_s|\underline{x}) = \int_{\underline{\theta}} f(x_s|\underline{\theta})\,\pi^*(\underline{\theta}|\underline{x})\,d\,\underline{\theta}, \tag{12}$$

where $f(x_s|\underline{\theta})$ is the conditional density function given by (8) or (10), $\pi^*(\underline{\theta}|\underline{x})$ is given by (6).
The predictive survival function of $X_{s:n}$ as $H(x_s|\underline{x})$ given by

$$H(x_s|\underline{\theta}) = \int_{\underline{\theta}} f^*(x_s|\underline{\theta})\,\pi^*(\underline{\theta}|\underline{x})\,d\,\underline{\theta}, \tag{13}$$

where

$$f^*(x_s|\underline{\theta}) = \int_{v}^{\infty} f(x_s|\underline{\theta})\,d\,x_s. \tag{14}$$

Upon substituting (8) and (10) in (14), we obtain
Case I

$$f_1^*(x_s|\underline{\theta}) = \sum_{i=0}^{s-r-1}\sum_{j=0}^{n-s} \frac{(-1)^{i+j}\binom{s-r-1}{i}\binom{n-s}{j}(n-r)!}{(n-s)!\,(s-r-1)!\,(s-r-i+j)} \frac{\left(1-(F(v))^{s-r-i+j}\right)\left(F(x_r)\right)^{i}}{\left(1-F(x_r)\right)^{n-r}}. \tag{15}$$

Case II

$$f_2^*(x_s|\underline{\theta}) = \sum_{k=r}^{s-1}\sum_{i=0}^{s-k-1}\sum_{j=0}^{n-s} \frac{(-1)^{i+j}\binom{n}{k}\binom{s-k-1}{i}\binom{n-s}{j}(n-k)!}{(n-s)!\,(n-k-1)!\,(s-k-i+j)} \frac{\left(1-\left(F(v)\right)^{s-k-i+j}\right)\left(F(T)\right)^{i+k}}{\sum_{j=r}^{s-1}\binom{n}{j}\left(F(T)\right)^{j}\left(1-F(T)\right)^{n-j}}. \tag{16}$$

3.2 Two-sample Bayesian Prediction

The predictive survival function of $Y_{s:m}$ under Type-II hybrid censoring scheme as follows:

$$G(y_s|\underline{\theta}) = \int_{\underline{\theta}} f^*_{Y_{s:m}}(y_s|\underline{\theta})\,\pi^*(\underline{\theta}|\underline{x})\,d\,\underline{\theta}, \tag{17}$$

where,

$$\begin{aligned} f^*_{Y_{s:m}}(y_s|\underline{\theta}) &= \int_v^{\infty} f_{Y_{s:m}}(y_s|\underline{\theta})\,d\,y_s \\ &= \sum_{i=0}^{m-s} \frac{(-1)^i\binom{m-s}{i}m!}{(s-1)!\,(m-s)!\,(s+i)}\left(1-(F(v))^{s+i}\right). \end{aligned} \tag{18}$$

Since (13) and (17) do not permit explicit solutions for the prediction bounds on $X_{s:n}$ and $Y_{s:m}$, we use MCMC samples $\{\underline{\theta_i} = (\delta_i, \underline{\beta_i}), i = M+1, M+2, ..., N\}$ a simulation consistent estimators of $H(x_s|\underline{\theta})$ and $G(y_s|\underline{\theta})$, see Chen et al.[19], we get

$$\hat{H}(x_s|\underline{\theta}) = \sum_{i=M+1}^{N} f^*(x_s|\underline{\theta_i})\,w_i, \tag{19}$$

and

$$\hat{G}(y_s|\underline{\theta}) = \sum_{i=M+1}^{N} f^*(y_s|\underline{\theta_i})\,w_i, \tag{20}$$

where $w_i = \frac{h(\underline{\theta_i}|\underline{x})}{\sum_{i=M+1}^{N} h(\underline{\theta_i}|\underline{x})}$, M is burn-in.
We noted that for all MCMC samples $\{\underline{\theta_i} = (\delta_i, \underline{\beta_i}), i = M+1, M+2, ..., N\}$ can be used to calculate $\hat{H}(x_s|\underline{\theta})$ and $\hat{G}(y_s|\underline{\theta})$. Moreover, we can obtain the two sided $100(1-\tau)\%$ prediction intervals (L,U) for $X_{s:n}$ and $Y_{s:m}$ through solving equations:

$$P[X_{s:n} > L|\underline{x}] = \hat{H}(L|\underline{\theta}) = 1-\frac{\tau}{2}, \qquad P[X_{s:n} > U|\underline{x}] = \hat{H}(U|\underline{\theta}) = \frac{\tau}{2}. \tag{21}$$

$$P[Y_{s:m} > L|\underline{x}] = \hat{G}(L|\underline{\theta}) = 1-\frac{\tau}{2}, \qquad P[Y_{s:m} > U|\underline{x}] = \hat{G}(U|\underline{\theta}) = \frac{\tau}{2}. \tag{22}$$

In this case the analytically solution is not possible, then we need numerical technique to compute (L,U) in (21) and (22).

4 Examples

This section computed the Bayesian prediction intervals for EE(δ) distribution when δ is unknown and ER(δ,γ) distribution when parameters δ and γ and are unknown.

4.1 Exponentiated Exponential Distribution

In this example, we take $\lambda(x;\underline{\beta}) = x$. The cdf and pdf of the EE(δ) distribution respectively given by

$$F_X(x;\delta) = \left(1-e^{-x}\right)^{\delta}, \qquad \delta > 0,\, x > 0. \tag{23}$$

$$f_X(x;\delta) = \delta\, e^{-x}\left(1-e^{-x}\right)^{\delta-1}. \tag{24}$$

We consider the conjugate gamma prior for δ is

$$\pi(\theta) \equiv \pi(\delta) = \frac{b^a}{\Gamma(a)}\,\delta^{a-1}\,e^{-b\,\delta}, \quad a,\ b,\ \delta > 0. \tag{25}$$

From (6), the posterior density function is

$$\pi^*(\delta|\underline{x}) \propto \delta^{a+d-1}\,e^{-\delta\left(b-\sum_{i=1}^{d}\ln(1-e^{-x_i})\right)}\left[1-\left(1-e^{-c}\right)^{\delta}\right]^{n-d}. \tag{26}$$

Hence, $\pi^*(\delta|\underline{x})$ of δ take the form

$$\pi^*(\delta|\underline{x}) \propto g(\delta|\underline{x})\,h(\delta|\underline{x}),$$

where, $g(\delta|\underline{x})$ is a gamma density with shape parameter $(d+a)$ and scale parameter $\left(b-\sum_{i=1}^{d}\ln(1-e^{-x_i})\right)$, and

$$h(\delta|\underline{x}) = \left[1-\left(1-e^{-c}\right)^{\delta}\right]^{n-d}.$$

Since $g(\delta|\underline{x})$ follows gamma, it is quite simple to generate from $g(\delta|\underline{x})$. Therefore the algorithm of Gibbs sampling procedures are as follows:
Algorithm 1:
(1): Start with an δ^0.
(2): Form $i=1$ to N, generate δ^0 form gamma distribution $g(\delta|\underline{x})$.
(3): In case of one-sample Bayesian prediction, we get the lower and upper limits through solving non-linear equations (27).

$$P[X_{s:n} > L|\underline{x}] = \frac{\sum_{i=M+1}^{N} f^*(L|\delta_i)\,h(\delta_i|\underline{x})}{\sum_{i=M+1}^{N} h(\delta_i|\underline{x})} = 1-\frac{\tau}{2},\ P[X_{s:n} > U|\underline{x}] = \frac{\sum_{i=M+1}^{N} f^*(U|\delta_i)\,h(\delta_i|\underline{x})}{\sum_{i=M+1}^{N} h(\delta_i|\underline{x})} = \frac{\tau}{2} \tag{27}$$

where,
Case I

$$f_1^*(v|\,\delta) = \sum_{i=0}^{s-r-1}\sum_{j=0}^{n-s}\frac{(-1)^{i+j}\binom{s-r-1}{i}\binom{n-s}{j}(n-r)!}{(n-s)!\,(s-r-1)!\,(s-r-i+j)}\,\frac{\left(1-\left(1-e^{-v}\right)^{\delta(s-r-i+j)}\right)\left(1-e^{-x_r}\right)^{i\,\delta}}{\left(1-\left(1-e^{-x_r}\right)^{\delta}\right)^{n-r}}.$$

Case II

$$f_2^*(v|\,\delta) = \sum_{k=r}^{s-1}\sum_{i=0}^{s-k-1}\sum_{j=0}^{n-s}\frac{(-1)^{i+j}\binom{n}{k}\binom{s-k-1}{i}\binom{n-s}{j}(n-k)!}{(n-s)!\,(n-k-1)!\,(s-k-i+j)}\,\frac{\left(1-\left(1-e^{-v}\right)^{\delta(s-k-i+j)}\right)\left(1-e^{-T}\right)^{\delta(i+k)}}{\sum_{j=r}^{s-1}\binom{n}{j}\left(1-e^{-T}\right)^{\delta j}\left(1-\left(1-e^{-T}\right)^{\delta}\right)^{n-j}}.$$

(4): In case of two-sample Bayesian prediction, we get lower and upper limits through solving non-linear equations (28).

$$P[Y_{s:m} > L|\underline{x}] = \frac{\sum_{i=M+1}^{N} f_{Y_{s:m}}^*(L|\delta_i)\,h(\delta_i|\underline{x})}{\sum_{i=M+1}^{N} h(\delta_i|\underline{x})} = 1-\frac{\tau}{2},\ P[Y_{s:m} > U|\underline{x}] = \frac{\sum_{i=M+1}^{N} f_{Y_{s:m}}^*(U|\delta_i)\,h(\delta_i|\underline{x})}{\sum_{i=M+1}^{N} h(\delta_i|\underline{x})} = \frac{\tau}{2}, \tag{28}$$

where,

$$f_{Y_{s:m}}^*(v|\,\delta) = \sum_{i=0}^{m-s}\frac{(-1)^i\binom{m-s}{i}m!}{(s-1)!\,(m-s)!\,(s+i)}\left(1-\left(1-e^{-v}\right)^{\delta(s+i)}\right).$$

4.2 Exponentiated Rayleigh Distribution

In this example, we take $\lambda(x;\underline{\beta}) = \gamma x^2,\ \gamma > 0$. cdf and pdf of the ER$(\delta,\gamma)$ distribution are given, respectively,

$$F_x(x;\delta,\gamma) = \left(1 - e^{-\gamma x^2}\right)^{\delta}, \qquad x > 0, \quad \delta,\gamma > 0. \tag{29}$$

$$f_x(x;\delta,\gamma) = 2\,\delta\,\gamma\,x\,e^{-\gamma x^2}\left(1 - e^{-\gamma x^2}\right)^{\delta-1}, \qquad x > 0, \quad \delta,\gamma > 0. \tag{30}$$

We assume δ and γ are independent, $\delta \sim gamma(a_1,b_1)$, $\gamma \sim gamma(a_2,b_2)$, thus

$$\pi(\delta) = \frac{b_1^{a_1}}{\Gamma(a_1)}\,\delta^{a_1-1}\,e^{-b_1\delta} \;\; and \;\; \pi(\gamma) = \frac{b_2^{a_2}}{\Gamma(a_2)}\,\gamma^{a_2-1}\,e^{-b_2\gamma},$$

where $a_1, b_1, a_2, b_2 > 0$.

$$\pi(\underline{\theta}) \equiv \pi(\delta,\gamma) = \pi(\delta)\,\pi(\gamma) = \frac{b_1^{a_1} b_2^{a_2}}{\Gamma(a_1)\,\Gamma(a_2)}\,\delta^{a_1-1}\,\gamma^{a_2-1}\,e^{-b_1\delta-b_2\gamma}.$$

From (6), $\pi^*(\delta,\gamma|\underline{x})$ is

$$\pi^*(\delta,\gamma|\underline{x}) \propto \delta^{a_1+d-1}\,\gamma^{a_2+d-1}\,e^{-\gamma(b_2+\sum_{i=1}^d x_i^2)}\,e^{-\delta\left(b_1-\sum_{i=1}^d \ln(1-e^{-\gamma x_i^2})\right)}\,e^{-\sum_{i=1}^d \ln(1-e^{-\gamma x_i^2})}\left[1-\left(1-e^{-\gamma c^2}\right)^{\delta}\right]^{n-d}. \tag{31}$$

Therefore, $\pi^*(\delta,\gamma|\underline{x})$ of δ and γ can be developed as

$$\pi^*(\delta,\gamma|\underline{x}) \propto g_1(\delta|\gamma,\underline{x})\,g_2(\gamma|\underline{x})\,h(\delta,\gamma,|\underline{x}),$$

where $g_1(\delta|\gamma,\underline{x})$ is a gamma density with shape parameter $(d+a_1)$ and scale parameter $\left(b_1 - \sum_{i=1}^d \ln(1-e^{-\gamma x_i^2})\right)$, and $g_2(\gamma|\underline{x})$ a proper density function is

$$g_2(\gamma|\underline{x}) = \frac{\gamma^{d+a_2-1}\,e^{-\gamma[b_2+\sum_{i=1}^d x_i^2]}}{\left[b_1 - \sum_{i=1}^d \ln(1-e^{-\gamma x_i^2})\right]^{d+a_1} e^{\sum_{i=1}^d \ln(1-e^{-\gamma x_i^2})}}.$$

Moreover,

$$h(\delta,\gamma,|\underline{x}) = \left[1-\left(1-e^{-\gamma c^2}\right)^{\delta}\right]^{n-d}.$$

In this case, we utilize Gibbs sampling procedure to draw MCMC samples with the help of importance sampling technique as proposed by [20]. Since $g_2(\gamma|\underline{x})$ is not known, but its plot shows that it is similar to normal distribution. So, we utilize the Metropolis-Hastings method to produce random numbers from these distributions. Moreover, since $g_1(\delta|\gamma,\underline{x})$ follows gamma, it is quite simple to produce random numbers from $g_1(\delta|\gamma,\underline{x})$. Hence, the algorithm of Gibbs sampling procedures for parameter δ with Metropolis-Hastings technique for parameter γ:

Algorithm 2:

(1): Start with an $(\delta^{(0)},\gamma^{(0)})$.

(2): Set $t = 1$.

(3): Generate $\delta^{(t)}$ from gamma distribution $g_1(\delta|\gamma,\underline{x})$.

(4): Generate $\gamma^{(t)}$ from $g_2(\gamma|\underline{x})$ using Metropolis-Hastings algorithm with the $N(\gamma^{(t-1)},\sigma)$ proposal distribution.

(5): Calculate $\delta^{(t)}$ and $\gamma^{(t)}$.

(6): Set $t = t+1$.

(7): Duplicate Step $3-6N$ times and obtain $(\delta_1,\gamma_1),(\delta_2,\gamma_2),\ldots,(\delta_N,\gamma_N)$.

(8): In case of one-sample Bayesian prediction, we compute the lower and upper limits through solving non-linear equations (32).

$$P[X_{s:n} > L|\underline{x}] = \frac{\sum_{i=M+1}^N f^*(L|\delta_i,\gamma_i)\,h(\delta_i,\gamma_i|\underline{x})}{\sum_{i=M+1}^N h(\delta_i,\gamma_i|\underline{x})} = 1-\frac{\tau}{2},\;\; P[X_{s:n} > U|\underline{x}] = \frac{\sum_{i=M+1}^N f^*(U|\delta_i,\gamma_i)\,h(\delta_i,\gamma_i|\underline{x})}{\sum_{i=M+1}^N h(\delta_i,\gamma_i|\underline{x})} = \frac{\tau}{2}, \tag{32}$$

where,
Case I

$$f_1^*(v|\,\delta,\gamma)=\sum_{i=0}^{s-r-1}\sum_{j=0}^{n-s}\frac{(-1)^{i+j}\binom{s-r-1}{i}\binom{n-s}{j}(n-r)!}{(n-s)!\,(s-r-1)!\,(s-r-i+j)}\,\frac{\left(1-\left(1-e^{-\gamma v^2}\right)^{\delta(s-r-i+j)}\right)\left(1-e^{-\gamma x_r^2}\right)^{i\,\delta}}{\left(1-\left(1-e^{-\gamma x_r^2}\right)^{\delta}\right)^{n-r}}.$$

Case II

$$f_2^*(v|\,\delta,\gamma)=\sum_{k=r}^{s-1}\sum_{i=0}^{s-k-1}\sum_{j=0}^{n-s}\frac{(-1)^{i+j}\binom{n}{k}\binom{s-k-1}{i}\binom{n-s}{j}(n-k)!}{(n-s)!\,(n-k-1)!\,(s-k-i+j)}\,\frac{\left(1-\left(1-e^{-\gamma v^2}\right)^{\delta(s-k-i+j)}\right)\left(1-e^{-\gamma T^2}\right)^{\delta(i+k)}}{\sum_{j=r}^{s-1}\binom{n}{j}\left(1-e^{-\gamma T^2}\right)^{\delta\,j}\left(1-\left(1-e^{-\gamma T^2}\right)^{\delta}\right)^{n-j}}.$$

(9): In case of two-sample Bayesian prediction, we get the lower and upper limits through solving non-linear equations (33).

$$P[Y_{s:m}>L|\underline{x}]=\frac{\sum_{i=M+1}^{N}f^*_{Y_{s:m}}(L|\delta_i,\gamma_i)\,h(\delta_i,\gamma_i|\underline{x})}{\sum_{i=M+1}^{N}h(\delta_i,\gamma_i|\underline{x})}=1-\frac{\tau}{2},\quad P[Y_{s:m}>U|\underline{x}]=\frac{\sum_{i=M+1}^{N}f^*_{Y_{s:m}}(U|\delta_i,\gamma_i)\,h(\delta_i,\gamma_i|\underline{x})}{\sum_{i=M+1}^{N}h(\delta_i,\gamma_i|\underline{x})}=\frac{\tau}{2},\quad (33)$$

where,

$$f^*_{Y_{s:m}}(v|\,\delta)=\sum_{i=0}^{m-s}\frac{(-1)^{i}\binom{m-s}{i}m!}{(s-1)!\,(m-s)!\,(s+i)}\left(1-\left(1-e^{-\gamma v^2}\right)^{\delta(s+i)}\right).$$

5 Numerical Results

The procedures developed in the previous sections can be explained through presenting a numerical study for EE(δ) distribution when δ is unknown and ER(δ,γ) distribution when both parameters δ and γ are unknown.

Example 5.1. To explain the prediction results of the EE(δ) distribution, when δ is unknown, we consider a real life data given by Bjerkedal [21]. The Kolmogorov-Smirnov (K-S) distance between empirical and fitted distribution function equals 0.224051 and corresponding p-value equals 0.00117451. We observed that EE(δ) distribution represents a good fit to these data, hence these data utilized to examine two different Type-II hybrid censoring schemes:
Scheme 1. Let $r=61$ and $T=170$, then $T>x_{61:72}$.
Scheme 2. Let $r=63$ and $T=170$, then $x_{63:72}>T$.
According to the two Schemes 1 and 2 , we utilize the findings introduced in Section 4 to construct 95% one-sample Bayesian prediction intervals for future order statistics $X_{s:72}$, where $s=64,\ldots,72$. In addition, we compute 95% two-sample Bayesian prediction intervals for future order statistics $Y_{s:20}$, where $s=1,\ldots,20$, from a future sample of size $m=20$. Tables 1 and 2 reported the lower and upper 95% one-sample Bayesian prediction bounds for $X_{s:72}$, where $s=64,\ldots,72$, for selecting $a=0,0.3,0.8$ and $b=0,2,5$. The lower and upper 95% two-sample Bayesian prediction bounds for $Y_{s:m}$, where $s=1,\ldots,20$, for selecting $a=0,0.5,1$ and $b=0,0.7,12$ are presented in Tables 3 and 4.

Example 5.2. To explain the prediction results of the ER(δ,γ) distribution, when both parameters δ and γ are unknown, let us consider the data given in Table 5. It indicates 39 liver cancers patients from El-Minia Cancer Center, Ministry of Health-Egypt, in (1999). The K-S space between empirical and fitted distribution function equals 0.22785 and corresponding p-value equals 0.02917. It is obvious that ER(δ,γ) distribution provides a good fit to the data. For computational ease, we have divided each data point by 100. We utilized these data to examine two different Type-II hybrid censoring schemes:
Scheme 1. Let $r=28$ and $T=70$, then $T>x_{28:39}$.
Scheme 2: Let $r=30$ and $T=70$, then $x_{30:39}>T$.
We can use the results presented in Section 4 to compute 95% one-sample Bayesian prediction intervals for future order statistics $X_{s:39}$, where $s=31,\ldots,39$. In addition, we compute 95% two-sample Bayesian prediction intervals for future order statistics $Y_{s:20}$, where $s=1,\ldots,20$, from a future sample of size $m=20$. Tables 6 and 7 reported the lower and upper 95% one-sample Bayesian prediction bounds for $X_{s:n}$, $s=31,\ldots,39$, for selecting (a_1,a_2,b_1,b_2), $(0.1,2,0.6,3)$, $(0.2,4,0.8,5)$. The lower and upper 95% two-sample Bayesian prediction bounds for $Y_{s:m}$, $s=1,\ldots,20$, for selecting (a_1,a_2,b_1,b_2), $(0.1,2,0.6,3)$, $(0.2,4,0.8,5)$, are presented in Tables 8 and 9.

Table 1: 95% One-sample Bayesian prediction bounds of $X_{s:72}, s = 64, \ldots, 72$ from the EE(δ) distribution in Case I

					$r = 63$	and	$T = 170$			
			$b = 0$			$b = 2$			$b = 5$	
a	s	$L_{X_{s:n}}$	$U_{X_{s:n}}$	Length	$L_{X_{s:n}}$	$U_{X_{s:n}}$	Length	$L_{X_{s:n}}$	$U_{X_{s:n}}$	Length
	64	175.294	217.427	42.1332	175.292	217.258	41.9663	175.290	216.983	41.6928
	65	177.979	242.927	64.9479	177.961	242.642	64.6810	177.941	242.307	64.3668
	66	183.106	269.164	86.0584	183.059	268.783	85.7272	183.006	268.405	85.3993
	67	190.318	298.505	108.186	190.245	298.157	107.912	190.138	297.641	107.503
0	68	199.724	333.316	133.592	199.610	332.916	133.307	199.456	332.380	132.924
	69	211.825	377.247	165.449	211.658	376.818	165.160	211.437	376.217	164.780
	70	227.732	437.938	210.206	227.529	437.501	209.971	227.232	436.844	209.612
	71	250.026	535.539	285.927	249.729	535.421	285.682	249.358	534.755	285.397
	72	286.716	766.505	479.789	286.338	765.954	479.616	285.900	765.320	479.419
	64	175.294	217.490	42.1490	175.292	217.236	41.9442	175.290	217.022	41.7319
	65	177.977	242.898	64.9208	177.963	242.664	64.7014	177.941	242.315	64.3734
	66	183.109	269.191	86.0814	183.066	268.860	85.7939	183.006	268.411	85.4045
	67	190.322	298.523	108.201	190.260	298.231	107.971	190.146	297.678	107.532
0.3	68	199.732	333.341	133.610	199.624	332.969	133.345	199.458	332.385	132.926
	69	211.840	377.312	165.473	211.680	376.878	165.198	211.462	376.285	164.823
	70	227.749	437.975	210.226	227.541	437.521	209.980	227.258	436.902	209.644
	71	250.043	535.986	285.943	249.780	535.515	285.735	249.425	534.878	285.453
	72	286.693	766.468	479.774	286.383	766.020	479.637	285.904	765.323	479.418
	64	175.294	217.461	42.1673	175.292	217.293	42.0004	175.290	217.046	41.7553
	65	177.983	242.994	65.0117	177.966	242.712	64.7462	177.945	242.375	64.4303
	66	183.118	269.251	86.1335	183.071	268.903	85.8319	183.018	268.497	85.4791
	67	190.347	298.645	108.298	190.261	298.229	107.968	190.156	297.728	107.571
0.8	68	199.760	333.447	133.687	199.644	333.042	133.397	199.467	332.415	132.948
	69	211.888	377.444	165.556	211.707	376.955	165.248	211.490	376.361	164.871
	70	227.813	438.116	210.303	227.588	437.626	210.039	227.309	437.014	209.705
	71	250.073	536.034	285.961	249.814	535.573	285.759	249.442	534.909	285.466
	72	286.816	766.650	479.834	286.417	766.069	479.616	285.961	765.407	479.445

6 Conclusions

1. The Bayesian prediction intervals for exponentiated family distributions utilizing Type-II hybrid censored is derived. Two examples are given: EE(δ) distribution when δ is unknown and ER$(\delta,\,\gamma)$ distribution when δ and γ are unknown. The result obtained in Section 3 can be applied to EW, exponentiated Burr Type XII distributions ,...etc.
2. It can be seen from Tables 1-4 and 6-9, the corresponding width of the Bayesian prediction bounds increase as s increases.
3. It has been noticed from Tables 1 and 2, in case of one-sample Bayesian prediction of the EE distribution, the lower as well as the upper bounds are comparatively insensitive to the specification of the hyper-parameters $(a,\,b)$.
4. It is evident from Tables 6 and 7 that, in the case of the ER distribution, lower and upper bounds are sensitive to the specification of the hyper-parameters $(a_1,\,a_2,\,b_1,\,b_2)$.
5. It has been noticed from Tables 3, 4 and 8, 9, in case of two-sample Bayesian prediction, lower and upper bounds are sensitive to the specification of the hyper-parameters $(a,\,b)$ and $(a_1,\,a_2,\,b_1,\,b_2)$.

Table 2: 95% One-sample Bayesian prediction bounds of $X_{s:72}, s = 64, \ldots, 72$ from the EE(δ) distribution in Case II

					$r = 61$	and	$T = 170$			
		$b = 0$			$b = 2$			$b = 5$		
a	s	$L_{X_{s:n}}$	$U_{X_{s:n}}$	Length	$L_{X_{s:n}}$	$U_{X_{s:n}}$	Length	$L_{X_{s:n}}$	$U_{X_{s:n}}$	Length
	64	171.822	239.699	67.8770	171.730	239.204	67.4738	171.594	238.423	66.3800
	65	174.251	260.322	86.0713	174.013	259.799	85.7864	173.580	258.704	85.1237
	66	179.121	284.019	104.898	178.534	283.264	104.730	177.646	282.048	104.402
	67	186.635	311.375	124.741	185.927	310.675	124.748	184.659	309.462	124.803
0	68	197.371	345.108	147.737	196.383	344.181	147.798	194.842	342.821	147.979
	69	210.650	387.801	177.151	209.607	386.826	177.219	208.076	385.511	177.436
	70	227.740	447.356	219.616	226.824	446.480	219.656	225.152	444.950	219.798
	71	251.373	544.436	293.064	250.363	543.441	293.078	248.682	541.824	293.141
	72	289.456	773.871	484.406	288.315	772.714	484.399	286.645	771.075	484.429
	64	171.833	239.754	67.9202	171.740	239.258	67.5186	171.605	238.502	66.8969
	65	174.332	260.515	86.1832	174.003	259.751	85.7482	173.637	258.870	85.2334
	66	179.110	283.971	104.860	178.654	283.432	104.778	177.722	282.155	104.433
	67	186.890	311.641	124.751	186.049	310.820	124.771	184.824	309.610	124.785
0.3	68	197.434	345.138	147.705	196.406	344.181	147.782	194.981	342.940	147.960
	69	210.737	387.874	177.137	209.799	387.043	177.244	208.333	385.581	177.477
	70	227.838	447.438	219.600	227.029	446.669	219.671	225.222	445.015	219.792
	71	251.427	544.482	293.054	250.363	543.421	293.058	248.739	541.853	293.114
	72	289.509	773.906	484.397	288.480	772.875	484.395	286.865	771.300	484.435
	64	171.850	239.851	68.0013	171.759	239.369	67.6096	171.619	238.559	66.9406
	65	174.354	260.572	86.2180	174.062	259.882	85.8198	173.687	258.985	85.2980
	66	179.289	284.252	104.963	178.666	283.406	104.740	177.795	282.245	104.450
	67	187.020	311.766	124.747	186.189	310.952	124.762	184.962	309.769	124.807
0.8	68	197.488	345.153	147.665	196.550	344.299	147.750	195.109	343.024	147.914
	69	210.998	388.158	177.160	209.895	387.089	177.194	208.398	385.750	177.352
	70	228.080	447.667	219.588	226.944	446.569	219.626	225.524	445.272	219.748
	71	251.673	544.718	293.045	250.595	543.641	293.047	249.053	542.180	293.127
	72	289.679	774.076	484.397	288.756	773.165	484.409	286.992	771.415	484.423

Table 3: 95% Two-sample Bayesian prediction bounds of $Y_{s:20}, s = 1, \ldots, 20$ from the EE(δ) distribution in Case I

				$r = 63$	and	$T = 170$			
(a, b)		$(0, 0)$			$(0.5, 7)$			$(1, 12)$	
s	$L_{X_{s:n}}$	$U_{X_{s:n}}$	Length	$L_{X_{s:n}}$	$U_{X_{s:n}}$	Length	$L_{X_{s:n}}$	$U_{X_{s:n}}$	Length
1	0.77184	35.6488	34.87696	0.40186	29.4732	29.07134	0.25869	25.95170	25.69301
2	4.00928	49.6047	45.59550	2.62063	42.4926	39.87200	1.91901	38.05560	36.13660
3	8.22637	61.9211	53.69470	5.82628	54.1920	48.36572	4.62178	49.30370	44.68190
4	12.8310	73.4937	60.66260	9.60859	64.8404	55.23180	7.86430	60.14170	52.27740
5	17.9060	85.4790	67.5731	13.7772	76.0993	62.32210	11.5634	70.33450	58.77110
6	23.1986	96.643	73.3443	18.3635	86.9843	68.6207	15.8072	81.4048	65.59760
7	28.8399	108.41	79.5704	23.4637	98.7961	75.3324	20.3948	92.6852	72.2905
8	34.7793	120.739	85.9599	28.8559	110.815	81.9595	25.4182	104.48	79.0617
9	41.2957	134.772	93.4762	34.6987	123.6	88.9013	30.9407	117.515	86.5743
10	48.3102	148.631	100.321	41.2532	138.072	96.8184	37.0407	131.152	94.1113
11	55.8574	164.402	108.544	48.1071	152.759	104.652	43.695	146.258	102.563
12	63.8974	181.541	117.644	55.866	169.754	113.888	50.9563	162.856	111.900
13	72.8263	200.118	127.291	64.198	188.527	124.329	59.4634	182.249	122.786
14	83.1246	223.13	140.006	74.0783	211.36	137.282	68.3716	203.133	134.762
15	94.3641	248.855	154.491	84.8968	236.983	152.086	79.4482	229.914	150.466
16	107.608	281.276	173.668	97.5308	268.802	171.271	91.6628	261.389	169.727
17	123.103	322.656	199.552	112.826	310.095	197.269	106.591	302.773	196.182
18	142.057	380.241	238.183	131.466	368.047	236.581	125.284	360.667	235.383
19	167.869	475.391	307.523	157.303	463.844	306.541	150.458	455.956	305.499
20	209.040	703.247	494.207	197.699	691.155	493.456	190.393	683.108	492.715

Table 4: 95% Two-sample Bayesian prediction bounds of $Y_{s:20}, s=1,\ldots,20$ from the EE(δ) distribution in Case II

				$r=63$	and	$T=170$			
(a,b)		$(0,0)$			$(0.5,7)$			$(1,12)$	
s	$L_{X_{s:n}}$	$U_{X_{s:n}}$	Length	$L_{X_{s:n}}$	$U_{X_{s:n}}$	Length	$L_{X_{s:n}}$	$U_{X_{s:n}}$	Length
1	0.79302	36.1067	35.31368	0.41251	29.9685	29.55599	0.26204	26.0077	25.74566
2	4.06531	50.6138	46.5485	2.65352	42.8279	40.1743	1.99429	38.8312	36.8369
3	8.26958	62.6093	54.3397	5.8488	53.8481	47.9993	4.61594	49.3700	44.7541
4	12.9341	73.6194	60.6852	9.66476	65.3284	55.6636	7.90223	59.9383	52.0361
5	17.9301	84.9416	67.0115	13.9061	77.5204	63.6143	11.6758	70.4746	58.7987
6	23.3825	97.1678	73.7853	18.5767	87.8353	69.2586	15.7969	81.2815	65.4846
7	28.9834	109.167	80.1836	23.5009	98.7146	75.2137	20.5653	93.2842	72.7190
8	34.9808	121.217	86.2358	28.9232	110.719	81.7958	25.6477	105.264	79.6160
9	41.3681	134.687	93.3193	34.9921	125.226	90.2335	30.9139	117.481	86.5673
10	48.3028	148.410	100.107	41.3257	138.196	96.8700	37.1095	131.280	94.1710
11	56.1538	165.025	108.871	48.2805	153.318	105.038	43.9472	146.912	102.965
12	63.9793	181.162	117.183	55.9008	169.786	113.885	51.2371	163.343	112.106
13	73.1566	200.494	127.338	64.4612	188.991	124.530	59.5119	182.224	122.712
14	82.8610	221.955	139.094	74.3209	211.340	137.019	68.7415	203.489	134.747
15	94.4406	248.649	154.208	85.2162	237.510	152.294	79.6852	230.185	150.499
16	107.769	281.576	173.807	97.9615	269.496	171.535	92.0608	262.018	169.958
17	123.256	322.574	199.318	112.564	309.528	196.964	106.865	302.930	196.065
18	142.505	380.874	238.369	132.081	368.797	236.715	125.403	360.828	235.425
19	168.355	475.975	307.521	157.557	463.965	306.408	150.614	456.300	305.686
20	209.314	703.557	494.242	197.698	690.976	493.278	190.423	683.138	492.716

Table 5: Survival times (in days) of liver cancers patients

10	14	14	14	14	14	15	17	18	20
20	20	20	20	23	23	24	26	30	30
31	40	49	51	52	60	61	67	71	74
75	87	96	105	107	107	107	116	150	

Table 6: 95% One-sample Bayesian prediction bounds of $X_{s:39}, s=31,\ldots,39$ from the ER(δ, γ) distribution in Case I

		$r=30$	and	$T=70$		
(a_1,b_1,a_2,b_2)		$(0.1,2,0.6,3)$			$(0.2,4,0.8,5)$	
s	$L_{X_{s:n}}$	$U_{X_{s:n}}$	Length	$L_{X_{s:n}}$	$U_{X_{s:n}}$	Length
31	74.1210	92.5085	18.3875	74.1465	96.1958	22.0493
32	75.2330	104.018	28.7845	75.4720	110.039	34.5675
33	77.1971	112.760	35.5625	77.9325	123.072	45.1396
34	79.8629	121.843	41.9800	81.3086	139.817	58.5087
35	83.4422	138.492	55.0499	85.5040	156.531	71.0272
36	88.0565	160.877	72.8209	90.9748	178.430	87.4554
37	93.1241	174.234	81.1095	97.5746	221.112	123.538
38	99.6728	193.681	94.0086	105.602	235.431	129.829
39	110.724	257.776	147.052	119.808	295.055	175.247

Table 7: 95% One-sample Bayesian prediction bounds of $X_{s:39}, s = 31, \ldots, 39$ from the ER(δ, γ) distribution in Case II

		$r = 28$	and	$T = 70$		
(a_1, b_1, a_2, b_2)		(0.1, 2, 0.6, 3)			(0.2, 4, 0.8, 5)	
s	$L_{X_{s:n}}$	$U_{X_{s:n}}$	Length	$L_{X_{s:n}}$	$U_{X_{s:n}}$	Length
31	70.4928	101.289	30.7962	70.7403	111.693	40.0953
32	71.0461	115.039	43.9925	71.6041	128.043	56.4386
33	71.9192	122.031	50.1119	72.6661	129.739	57.0729
34	73.1369	135.254	62.1175	75.9007	147.449	58.5087
35	76.2165	146.430	70.1239	79.7631	166.265	86.5019
36	80.9166	169.194	88.2770	86.8236	196.417	109.593
37	90.9685	206.007	115.039	94.2483	243.716	149.468
38	97.5879	211.852	114.264	103.448	249.167	145.718
39	108.174	261.025	152.850	117.471	298.560	181.089

Table 8: 95% One-sample Bayesian prediction bounds of $Y_{s:20}, s = 1, \ldots, 20$ from the ER(δ, γ) distribution in Case I

		$r = 30$	and	$T = 70$		
(a_1, b_1, a_2, b_2)		(0.1, 2, 0.6, 3)			(0.2, 4, 0.8, 5)	
s	$L_{X_{s:n}}$	$U_{X_{s:n}}$	Length	$L_{X_{s:n}}$	$U_{X_{s:n}}$	Length
1	0.03824	16.8492	16.8110	0.01602	14.6455	14.6295
2	0.52757	24.3935	23.8659	0.30029	22.5789	22.2786
3	1.45372	18.5058	17.0521	1.11478	29.5454	28.4306
4	2.67354	28.8629	26.1893	2.42722	37.2318	34.8046
5	5.15485	41.7161	36.5613	3.92650	44.0185	40.0920
6	7.66935	46.5578	38.8884	6.19380	49.3211	43.1273
7	10.6157	56.0661	45.4504	9.13160	64.2265	55.0949
8	12.9124	65.3622	52.4497	12.0233	72.1548	60.1315
9	16.5328	69.4780	52.9452	15.3964	71.0766	55.6802
10	20.5064	76.5500	56.0436	19.3209	81.3380	62.0171
11	24.5619	81.3755	56.8137	24.0498	92.0755	68.0257
12	28.7324	93.5942	64.8618	28.2890	98.5833	70.2943
13	33.3905	98.9126	65.5221	33.8132	112.162	78.3491
14	37.5301	101.216	63.6862	39.4961	121.781	82.2852
15	44.0727	119.586	75.5130	45.5419	131.563	86.0209
16	50.9219	131.271	80.3488	52.5118	149.640	97.1278
17	56.0859	142.284	86.1981	60.9624	185.198	124.235
18	65.4389	166.910	101.471	69.3278	184.977	151.649
19	73.8568	200.507	126.651	81.3575	225.582	144.224
20	89.4045	233.445	144.041	97.8258	288.318	190.492

Table 9: 95% One-sample Bayesian prediction bounds of $Y_{s:20}, s = 1, \dots, 20$ from the ER(δ, γ) distribution in Case II

		$r = 28$	and	$T = 70$		
(a_1, b_1, a_2, b_2)		$(0.1, 2, 0.6, 3)$			$(0.2, 4, 0.8, 5)$	
s	$L_{X_{s:n}}$	$U_{X_{s:n}}$	Length	$L_{X_{s:n}}$	$U_{X_{s:n}}$	Length
1	0.03447	15.7089	15.6744	0.01302	13.7507	13.7377
2	0.51945	23.1007	22.5812	0.24221	21.9969	21.7546
3	1.39082	32.6462	31.2554	1.00724	29.4567	28.4495
4	3.17143	36.1015	32.9301	2.25483	38.4734	36.2186
5	5.40420	46.2500	40.8458	3.87887	44.3140	40.4351
6	7.10627	48.0975	40.9912	5.83209	49.7065	43.8744
7	10.0213	56.8367	46.8154	8.64751	57.9587	55.0949
8	13.3055	63.3451	50.0396	11.9291	68.1493	56.2202
9	17.3587	69.6019	52.2432	15.1386	76.2789	61.1402
10	19.1977	74.0190	54.8213	19.6895	85.4603	65.7708
11	24.0745	83.9021	59.8275	23.4840	89.0166	65.5326
12	29.1537	93.6459	64.4923	28.9306	112.451	83.5203
13	33.2964	98.3151	65.0187	34.4942	120.436	85.9414
14	39.4523	108.258	68.8060	40.3178	129.787	89.4690
15	44.2093	118.523	74.3137	47.4760	153.077	105.601
16	51.5572	145.788	94.2304	54.7259	160.069	105.343
17	58.2895	148.468	90.1782	62.6288	179.496	116.868
18	68.8551	179.557	110.702	71.5247	200.879	129.354
19	77.4139	212.603	135.189	84.8466	239.624	154.778
20	91.4764	246.546	155.070	101.177	296.807	195.630

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