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Bayes Point Predictors of Exponential Distribution under Asymmetric Loss When Observations are Multiply Type II Censored

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Abstract: The crux of this paper is to obtain predictors of the future observation under multiply type II censored sample from exponential distribution. Bayes point predictors are obtained under asymmetric loss function (linear) as well as under symmetric loss function (squared error) using nature conjugate prior. Predictive risks are calculated under each loss. Predictors are compared for the smallest ordered future observation on the basis of predictive risk efficiencies for 1000 randomly generated sample using Monte Carlo simulation technique as well as for real informative data representing failure times for electric insulation.

Keywords: Point Predictor, Exponential distribution, Conjugate Prior, Symmetric Loss function, Asymmetric Loss function, Predictive risk, censored observation.

1 Introduction

The use of predictive inference got its appearance in recent past in which one wishes to infer about future sample on the basis of results obtained from the past sample of the same population. For example, a factory owner wishes to predict about lifetimes of certain type of machine tools to know about best inspection and replacement policy on the basis of recorded life time of machine tools of similar type. Such type of inference is known as predictive inference. A good deal of literature is available on the predictive inference for life time models using both classical and Bayesian approach (see for example, [6],[7], [9],[4], [1], [3]). Aitchen and Dunsmore [2] is a text exclusively devoted to this topic. Kaminsky and Nelson [5] described computational approach for obtaining interval and point prediction of ordered statistics.
The most frequent area of discussion under prediction is point prediction and interval prediction. When point prediction is under discussion, the consequence of being wrong must be viewed. Most of the above literature has assumed that loss due to consequence of being wrong is proportional to the square of error i.e. equal weightage has given for positive error or negative error. But this seems unjustified in the case if positive error is more serious than negative error or vice versa. In the example mentioned above if actual lifetime of machine exceeds inspection time the overhead scrapping loss is incurred for unused productive capacity. In contrary to it if inspection time exceeds the actual lifetime of machine there is a loss of production time. So under prediction and over prediction are not of equal importance in many practical situations, hence use of symmetric loss function is not justified.
The simplest asymmetric loss function for the prediction problems is the linear loss function suggested by [2] which associates unequal weights to under prediction and over prediction errors of equal magnitude. The loss function should be such that if we predict (y) correctly, the loss incurred must be zero, otherwise it should be proportional to the difference between predicted value (y^*) and the actual value (y). The constant of proportionality are chosen according to relative importance of under-prediction and over prediction. The asymmetric linear loss function can be given as

$$L(y^*,y)=\begin{cases}\xi\,(y^*-y) & if\ y\leq y^*\\ \eta\,(y-y^*) & if\ y>y^*\end{cases} \tag{1}$$

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where ξ is the loss per unit time for under prediction and η is the loss per unit time for over prediction. If η and ξ both are equal, then above loss function reduces to a symmetric loss function. Using this loss function [11] derived point predictors for exponential distribution whereas [10] obtained point predictors under asymmetric loss for Rayleigh distribution when data were doubly type II censored, but nothing has appeared in literature about point predictors when data is multiply type II censored.
To illustrate the use of linear loss function, we have considered the problem of point prediction for the future ordered observation from one parameter exponential distribution when the data available is multiply type II censored.

2 Proposed Procedure

Let us consider that N items $x_1, x_2, \ldots, x_N$ are put on test having exponential failure time distribution with probability density function (pdf)

$$f(x,\theta) = \frac{1}{\theta}\exp\left(-\frac{x}{\theta}\right) \quad ; x \geq 0, \;\; \theta > 0 \tag{2}$$

with cumulative distribution function (cdf)

$$F(x,\theta) = 1 - \exp\left(-\frac{x}{\theta}\right) \quad ; x \geq 0, \;\; \theta > 0 \tag{3}$$

be subjected to a life test. Due to unforeseen event experimenter could record only two groups of observations, i.e. $x_{r+1}, x_{r+2}, \ldots, x_{r+k}$ and $x_{r+k+l+1}, x_{r+k+l+2}, \ldots, x_{N-q}$, then this constitute a multiply type II censored observations. Its likelihood function can be written as

$$L(\underline{x},\theta) = \frac{N!}{r!l!q!}[F(x_{r+1})]^r [F(x_{r+k+l+1} - F(x_{r+k})]^l [1 - F(x_{N-q})]^q \prod_{i=r+1}^{r+k} f(x_i,\theta) \prod_{i=r+k+l+1}^{N-q} f(x_i,\theta)$$

$$L(\underline{x},\theta) = \frac{N!}{r!l!q!}\sum_{p=0}^{r}\sum_{g=0}^{l}\Omega_p\Omega_g\left(\frac{1}{\theta}\right)^A \exp\left(-\frac{S_{pg}+S}{\theta}\right) \tag{4}$$

where $A = N - r - l - q$, $\;\Omega_p = (-1)^p \binom{r}{p}$, $\;\Omega_g = (-1)^g \binom{l}{g}$,
$S_{pg} = p x_{r+1} + (l-g) x_{r+k} + g x_{r+k+l+1}$, $\;S = q x_{N-q} + \sum_{i=r+1}^{r+k} x_i + \sum_{i=r+k+l+1}^{N-q} x_i$.
Let the prior of θ be

$$g(\theta) = \frac{a^c}{\Gamma(c)}\frac{1}{\theta^{c+1}} exp\left[-\frac{a}{\theta}\right]; \;\; \theta > 0, \; a, c > 0 \tag{5}$$

Combining likelihood function (4) with the prior (5) via Bayes theorem, the posterior of θ will be

$$p(\theta|S) = \frac{L(\underline{x},\theta)\, g(\theta)}{\int_0^\infty L(\underline{x},\theta)\, g(\theta)\, d\theta}$$

On solving, we get

$$p(\theta|S) = D_1^{-1}(x)\sum_{p=0}^{r}\sum_{g=0}^{l}\Omega_p\Omega_g\left(\frac{1}{\theta}\right)^{A+c+1} \exp\left(-\frac{T_s}{\theta}\right) \tag{6}$$

where $T_s = S_{pg} + S + a$ and $D_1(x) = \Gamma(A+c)\sum_{p=0}^{r}\sum_{g=0}^{l}\Omega_p\Omega_g (T_s)^{-(A+c)}$.
Let $y_1, y_2, \ldots . y_m$ be an independent future sample of size m from (2), then the pdf of nth ordered future observation,where $1 \leq n \leq m$ is obtained from

$$f(y_{(n)}|\theta) = \frac{m!}{(n-1)!(m-n)!}\left[F(y_{(n)})\right]^{n-1} f(y_{(n)}) \left[1 - F(y_{(n)})\right]^{m-n}$$

where $F(.)$ is the cdf given in (3). On solving we obtain

$$f(y_{(n)}|\theta) = \beta^{-1}(n,M)\frac{1}{\theta}\sum_{i=0}^{n-1}\Omega_i exp\left[-\frac{(M+i)y_{(n)}}{\theta}\right] \tag{7}$$

where $\Omega_i = (-1)^i \binom{n-1}{i}$, $M = m - n + 1$.
Hence using (7), the predictive pdf of the nthordered future observation can be derived as

$$h\left(y_{(n)}|S\right) = \int_0^\infty f\left(y_{(n)}|\theta\right) p\left(\theta \mid S\right) d\theta$$

using (6) and (7), it becomes

$$h\left(y_{(n)}|S\right) = \beta^{-1}\left(n,M\right)\left(A+c\right)D^{-1}\left(x\right)\sum_{p=0}^{r}\sum_{g=0}^{l}\sum_{i=0}^{n-1}\Omega_p\Omega_g\Omega_i\left[T_s + \left(M+i\right)y_{(n)}\right]^{-(A+c+1)} \tag{8}$$

where $D\left(x\right) = \sum_{p=0}^{r}\sum_{g=0}^{l}\Omega_p\Omega_g\left(T_s\right)^{-(A+c)}$.
Further suppose that loss associated with the point predictor is linear, as given in (1), then the optimal value of point predictor may be obtained by differentiating the expected loss w.r.t. y^*. The expected loss can be written as

$$L(y^*) = E\left(L\left(y^*, y_{(n)}\right)\right) = \xi\int_0^{y^*}\left(y^* - y_{(n)}\right)h\left(y_{(n)}|S\right)dy_{(n)} + \eta\int_{y^*}^{\infty}\left(y_{(n)} - y^*\right)h\left(y_{(n)}|S\right)dy_{(n)} \tag{9}$$

Differentiating w.r.t. y^* and simplifying, we get

$$L'\left(y^*\right) = \left(\eta + \xi\right)\int_0^{y^*} h\left(y_{(n)}|S\right)dy_{(n)} - \eta \tag{10}$$

$$L''\left(y^*\right) = \left(\eta + \xi\right)h\left(y^* \mid S\right), \quad (>0)$$

which implies that the solution of (10) when equated to zero provides the optimal value of y^* for which expected loss is minimum. Hence point predictor, say $y_{(n)L^*}$, under linear loss is the solution of

$$\int_0^{y_{(n)L^*}} h\left(y_{(n)}|S\right)dy_{(n)} = \frac{\eta}{(\eta+\xi)} \tag{11}$$

On substituting value of $h\left(y_{(n)}|S\right)$ from (8) in (11) and simplifying, we have

$$\beta^{-1}\left(n,M\right)D^{-1}\left(x\right)\sum_{p=0}^{r}\sum_{g=0}^{l}\sum_{i=0}^{n-1}\Omega_p\Omega_g\Omega_i\left[\frac{\left(T_s\right)^{-(A+c)}}{(M+i)} - \frac{\left(T_s + (M+i)y_{(n)L}{}^*\right)^{-(A+c)}}{(M+i)}\right] = \frac{\eta}{(\eta+\xi)} \tag{12}$$

Above equation is solved for $y_{(n)L^*}$ by using Bisection method.
It is well known that point predictor under quadratic loss is the mean of predictive pdf. Thus for nth ordered future observation, the predictor is given by

$$y_{(n)Q^*} = E\left[y_{(n)}|S\right] = \int_0^\infty y_{(n)}.h\left(y_{(n)}|S\right)dy_{(n)} \tag{13}$$

On solving, we get

$$y_{(n)Q^*} = \beta^{-1}\left(n,M\right)D^{-1}\left(x\right)\sum_{p=0}^{r}\sum_{g=0}^{l}\sum_{i=0}^{n-1}\Omega_p\Omega_g\Omega_i\frac{\left(T_s\right)^{-(A+c)+1}}{\left\{(A+c)-1\right\}(M+i)^2} \tag{14}$$

Thus the point predictor $y_{(n)Q^*}$ is available in a nice closed form but its usage is justified only if under-prediction and over-prediction are of equal importance. Contrary to it if over-prediction and under-prediction are of unequal importance, the use of $y_{(n)Q^*}$ may not be appropriate and one might consider predictor under linear loss. Naturally, a question arises whether we lose enough due to the use of $y_{(n)Q^*}$ if the appropriate loss is linear. Similarly, it would be also worthwhile to investigate whether we lose enough due to the use of $y_{(n)L^*}$ instead of $y_{(n)Q^*}$ if over-prediction and under prediction are of equal importance. To get an answer to these queries, we propose to compare $y_{(n)Q^*}$ and $y_{(n)L^*}$under both linear and quadratic loss function. The comparison can be carried out on the basis of predictive risk which may be defined as the average loss incurred by the use of a particular predictor for a specified loss function. The predictor corresponding to which the predictive risk is minimum, may then be recommended for use. The predictive risk may be defined as

$$PR(y_{(n)}{}^*) = E\left[L\left\{y_{(n)}{}^*, y_{(n)}\right\}\right]$$

where $y_{(n)}{}^*$ is the predictor of $y_{(n)}$ and $L\left\{y_{(n)}{}^*, y_{(n)}\right\}$ denotes the specified loss. Naturally, the expectation E is to be taken over whole informative as well as future sample space. Thus

$$p\left(S|\theta\right) = S^{A-1} exp\left[-\frac{AS}{\theta}\right]\frac{1}{\Gamma(A)}\left(\frac{A}{\theta}\right)^{A}$$

$$PR\left(y_{(n)}{}^{*}\right) = \int_0^\infty\int_0^\infty L\left\{y_{(n)}{}^{*}, y_{(n)}\right\} h\left(y_{(n)}|S\right) p\left(S \mid \theta\right) dSdy_{(n)}$$

$$PR\left(y_{(n)}{}^{*}\right) = \frac{\beta^{-1}(n,M)(A+c)}{\Gamma(A)}\left(\frac{A}{\theta}\right)^{A}\int_0^\infty\int_0^\infty L\left\{y_{(n)}{}^{*}, y_{(n)}\right\} D^{-1}(x)\sum_{p=0}^{r}\sum_{g=0}^{l}\sum_{i=0}^{n-1}\Omega_p\Omega_g\Omega_i \times\left[T_s+(M+i)y_{(n)}\right]^{-(A+c+1)} S^{A-1} exp\left(-\frac{AS}{\theta}\right) dSdy_{(n)} \tag{15}$$

Assuming $L\left\{y_{(n)}{}^{*}, y_{(n)}\right\}$ to be linear, the predictive risks for $y_{(n)L^*}$ and $y_{(n)Q^*}$ can be obtained as

$$PR_L\left(y_{(n)L^*}\right) = \frac{\beta^{-1}(n,M)(A+c)}{\Gamma(A)}\left(\frac{A}{\theta}\right)^{A}\int_0^\infty\left[\xi\int_0^{y_{(n)L^*}}\left(y_{(n)L^*}-y_{(n)}\right)+\eta\int_{y_{(n)L^*}}^{\infty}\left(y_{(n)}-y_{(n)L^*}\right)\right] \times S^{A-1} exp\left(-\frac{AS}{\theta}\right) D^{-1}(x)\sum_{p=0}^{r}\sum_{g=0}^{l}\sum_{i=0}^{n-1}\Omega_p\Omega_g\Omega_i\left[T_s+(M+i)y_{(n)}\right]^{-(A+c+1)} dSdy_{(n)} \tag{16}$$

and

$$PR_L\left(y_{(n)Q^*}\right) = \frac{\beta^{-1}(n,M)(A+c)}{\Gamma(A)}\left(\frac{A}{\theta}\right)^{A}\int_0^\infty\left[\xi\int_0^{y_{(n)Q^*}}\left(y_{(n)Q^*}-y_{(n)}\right)+\eta\int_{y_{(n)Q^*}}^{\infty}\left(y_{(n)}-y_{(n)Q^*}\right)\right] \times S^{A-1} exp\left(-\frac{AS}{\theta}\right) D^{-1}(x)\sum_{p=0}^{r}\sum_{g=0}^{l}\sum_{i=0}^{n-1}\Omega_p\Omega_g\Omega_i\left[T_s+(M+i)y_{(n)}\right]^{-(A+c+1)} dSdy_{(n)} \tag{17}$$

Similarly, the predictive risks of the predictors $y_{(n)L^*}$ and $y_{(n)Q^*}$ under quadratic loss are

$$PR_Q\left(y_{(n)L^*}\right) = \frac{\beta^{-1}(n,M)(A+c)}{\Gamma(A)}\left(\frac{A}{\theta}\right)^{A}\int_0^\infty\int_0^\infty\left(y_{(n)L^*}-y_{(n)}\right)^2 \times S^{A-1} exp\left(-\frac{AS}{\theta}\right) D^{-1}(x)\sum_{p=0}^{r}\sum_{g=0}^{l}\sum_{i=0}^{n-1}\Omega_p\Omega_g\Omega_i\left[T_s+(M+i)y_{(n)}\right]^{-(A+c+1)} dSdy_{(n)} \tag{18}$$

and

$$PR_Q\left(y_{(n)Q^*}\right) = \frac{\beta^{-1}(n,M)(A+c)}{\Gamma(A)}\left(\frac{A}{\theta}\right)^{A}\int_0^\infty\int_0^\infty\left(y_{(n)Q^*}-y_{(n)}\right)^2 \times S^{A-1} exp\left(-\frac{AS}{\theta}\right) D^{-1}(x)\sum_{p=0}^{r}\sum_{g=0}^{l}\sum_{i=0}^{n-1}\Omega_p\Omega_g\Omega_i\left[T_s+(M+i)y_{(n)}\right]^{-(A+c+1)} dSdy_{(n)} \tag{19}$$

respectively.

3 Comparison of Predictors for the smallest observation

In this section, comparison of the predictors for the smallest observation from a future sample has been made. The predictors and their corresponding risks, for this particular case, may be obtained by putting n=1 in (12), (14), (16), (17), (18) and (19). The predictor under linear loss is obtained as

$$D^{-1}(x)\sum_{p=0}^{r}\sum_{g=0}^{l}\Omega_p\Omega_g\left[T_s{}^{-(A+c)}-\left(T_s+my_{(1)L}{}^{*}\right)^{-(A+c)}\right] = \frac{\eta}{(\eta+\xi)} \tag{20}$$

Similarly, the predictor under quadratic loss comes out to be

$$y_{(1)Q^*} = \frac{D^{-1}(x)}{m} \sum_{p=0}^{r} \sum_{g=0}^{l} \Omega_p \Omega_g \frac{T_s{}^{-(A+c)+1}}{(A+c-1)} \tag{21}$$

The predictive risks of the predictors $y_{(1)L^*}$ and $y_{(1)Q^*}$ under linear loss are

$$\begin{aligned} PR_L\left(y_{(1)L^*}\right) &= \frac{1}{\Gamma(A)}\left(\frac{A}{\theta}\right)^A \int_0^\infty D^{-1}(x) \sum_{p=0}^{r}\sum_{g=0}^{l} \Omega_p\Omega_g S^{A-1} exp\left(-\frac{AS}{\theta}\right) \\ &\times \left[\xi\left\{y_{(1)L}{}^* T_s{}^{-(A+c)} - \frac{T_s{}^{-(A+c)+1}}{m(A+c-1)}\right\} + (\eta+\xi)\frac{\left(T_s + my_{(1)L}{}^*\right)^{-(A+c)+1}}{m(A+c-1)}\right] ds \end{aligned} \tag{22}$$

and

$$\begin{aligned} PR_L\left(y_{(1)Q^*}\right) &= \frac{1}{\Gamma(A)}\left(\frac{A}{\theta}\right)^A \int_0^\infty D^{-1}(x) \sum_{p=0}^{r}\sum_{g=0}^{l} \Omega_p\Omega_g S^{A-1} exp\left(-\frac{AS}{\theta}\right) \\ &\times \left[\xi\left\{y_{(1)Q}{}^* T_s{}^{-(A+c)} - \frac{T_s{}^{-(A+c)+1}}{m(A+c-1)}\right\} + (\eta+\xi)\frac{\left(T_s + my_{(1)Q}{}^*\right)^{-(A+c)+1}}{m(A+c-1)}\right] ds \end{aligned} \tag{23}$$

respectively.
In the same way, the predictive risks of the predictors $y_{(1)L^*}$ and $y_{(1)Q^*}$ under quadratic loss function are

$$\begin{aligned} PR_Q\left(y_{(1)L^*}\right) &= \frac{1}{\Gamma(A)}\left(\frac{A}{\theta}\right)^A \int_0^\infty D^{-1}(x) \sum_{p=0}^{r}\sum_{g=0}^{l} \Omega_p\Omega_g S^{A-1} exp\left(-\frac{AS}{\theta}\right) \\ &\times \left[y_{(1)L}{}^{*2}\, T_s{}^{-(A+c)} - \frac{2y_{(1)L}{}^* T_s{}^{-(A+c)+1}}{m(A+c-1)} + \frac{2T_s{}^{-(A+c)+2}}{m^2(A+c-1)(A+c-2)}\right] ds \end{aligned} \tag{24}$$

and

$$\begin{aligned} PR_Q\left(y_{(1)Q^*}\right) &= \frac{1}{\Gamma(A)}\left(\frac{A}{\theta}\right)^A \int_0^\infty D^{-1}(x) \sum_{p=0}^{r}\sum_{g=0}^{l} \Omega_p\Omega_g S^{A-1} exp\left(-\frac{AS}{\theta}\right) \\ &\times \left[y_{(1)Q}{}^{*2} T_s{}^{-(A+c)} - \frac{2y_{(1)Q}{}^* T_s{}^{-(A+c)+1}}{m(A+c-1)} + \frac{2T_s{}^{-(A+c)+2}}{m^2(A+c-1)(A+c-2)}\right] ds \end{aligned} \tag{25}$$

respectively.
It may be noted here that as the predictors and predictive risks are not in closed form, therefore can be evaluated using 15-point Gauss-Laguerre quadrature formula.
Now the PRE_{LIN} of $y_{(1)L^*}$ w.r.t. $y_{(1)Q^*}$ may be defined as

$$PRE_{LIN} = \frac{PR_L(y_{(1)Q^*})}{PR_L(y_{(1)L^*})} \tag{26}$$

Similarly, the PRE_{QRD} of $y_{(1)L^*}$ w.r.t. $y_{(1)Q^*}$ may be defined as

$$PRE_{QRD} = \frac{PR_Q(y_{(1)Q^*})}{PR_Q(y_{(1)L^*})} \tag{27}$$

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4 Discussion

In this section, we have obtained numerical results for predictor under linear loss and predictor under quadratic loss along with the predictive risks. Results have been obtained for simulated data as well as for real data set.

4.1 Simulation study

In the present section we compare the point predictor under linear loss for the smallest order future observation with the point predictor obtained under quadratic loss, on the basis of their predictive risk efficiency. For the comparison purpose a Monte Carlo study of 1000 randomly generated samples from exponential distribution was conducted for different values of θ. We considered a number of values for the different constants involved in (26) and (27), but the results have been reported only for some of the considered values, because of a number of reasons. For example, we considered three different values of θ, namely 0.5, 1.0 and 2.0 and it was found that although risks differs by varying θ, the risk efficiencies remains mostly unchanged so $\theta = 2.0$ has been reported. Similarly, three different sample sizes namely 6,10 and 25 were taken for both informative and future samples but N=m=20 is only reported because no significant change observed in the results with variation in sample sizes.

A number of values 2.0, 4.0, 6.0 were assigned to the hyperparameter a, but less variation in predictive risk efficiencies was noticed, therefore we have fixed hyperparameter a at 2.0 everywhere. As the variation in hyperparameter c was found significant, five different values, namely 0.50, 1.0, 2.0, 4.0 and 6.0 were considered for hyperparameter c. Appropriate values were assigned to r, l and q so as to cover different censoring schemes, i,e. multiply, doubly, mid, left and right. Number of observed lifetimes kept fixed i.e. at 6 for these censoring schemes except multiply. For multiply censoring schemes results were reported for different numbers of observed lifetimes. A number of values were assigned to linear loss parameter (η, ξ) so as to keep the ratio η/ξ fixed at 0.25, 0.50, 1.0, 1.5 and 2.0. The results are summarized in tables 1-10.

Tables 1-5 shows the relative efficiencies of $y_{((1)L^*)}$ w.r.t. $y_{((1)Q^*)}$ under linear loss. It is deduced from the tables that, in most of the cases $y_{((1)L^*)}$ performs better than that of $y_{((1)Q^*)}$. It is observed that PRE_{LIN} decreases as the ratio η/ξ increases. Hence for $\eta/\xi \leq 1.5$ predictor under linear loss performs better than predictor under quadratic loss. Though PRE_{LIN} is less than unity for $\eta/\xi > 1.5$ but seems close to unity, so it can be inferred that predictor under linear loss performs equally as good as predictor under quadratic loss. It may be observed from tables that as we increase the value of hyperparameter c, PRE_{LIN} decreases almost everywhere except in the case of mid censoring scheme where it increases. For multiply censoring similar trend in the results is noticed where number of observed life times are more but with less number of observed lifetimes somewhere trend is reversed.
Table 6-10 summaries relative efficiencies of $y_{((1)L^*)}$ w.r.t $y_{((1)Q^*)}$ under squared error loss function. As expected the PRE_{QRD} is observed to be more than unity everywhere. PRE_{QRD} decreases with the increase in the ratio η/ξ. For $\eta/\xi \leq 1.5$, it is found that predictor under linear loss performs better than predictor under quadratic loss but for $\eta/\xi > 1.5$ it seems that one can use predictor under linear loss over predictor under quadratic loss without any significant loss even if quadratic loss seems to be more appropriate. It may be noted that PRE_{QRD} decreases with increase in hyperparameter c almost everywhere.

4.2 Real data study

The following data represent failure times (in minutes) for electric insulation in an experiment in which insulation was subjected to a continuously voltage stress (Lawless[8], p.138)

$$12.3, 21.8, 24.4, 28.6, 43.2, 46.9, -, 75.3, 95.5, 98.1, 138.6, -$$

Since the experimenter failed to record the failure time of 7th unit hence 7th observation is censored. Similarly the experimenter could not wait till the last observation gets failed, hence he stopped recording after 11th failure, due to this 12th observation get censored. Therefore, we have following multiply type II censoring parameters

$N = 12, r = 0, k = 6, l = 1, q = 1.$

Predictive risk under quadratic loss were obtained using Bayes estimator of parameter under multiply type II censoring whereas quantile estimator under multiply type II censoring is used to obtain predictive risk under linear loss. For above data, the predictive risk efficiencies are calculated for different ratio of η/ξ and since no significant variation is seen of changing hyperparameters in simulation study, we have fixed hyperparameters as a=1300.0, c=27 is summarized in table 11. From table 11, it is to be noted that as η/ξ increases PRE_{LIN} decreases. It is evident that PRE_{LIN} is greater than unity everywhere which implies that predictor under linear loss performs better than predictor under quadratic loss even if actual loss is symmetric. From table 11 it is seen that PRE_{QRD} first increases with increase in / up to 1.0 then it decreases. It means PRE_{QRD} attains its maximum at $\eta/\xi = 1.0$ and decreases on either side of it. PRE_{QRD} is maximum for symmetric loss that is at $\eta/\xi = 1.0$. PRE_{QRD} is slightly greater than unity for η/ξ less than equal to 2, but even though it is smaller than PRE_{LIN}. It seems that predictor under linear loss can be used safely over predictor under quadratic loss everywhere.

5 Conclusion

As per the discussion of previous section it may be concluded that with more number of observed life time under multiply type II censoring one can safely use the predictor obtained under linear loss because it is either more efficient (in case when asymmetric loss is actual loss) or almost equally efficient (in case when quadratic loss is actual loss) compared to the usual predictor obtained under quadratic loss. It needs to be pointed out here that the use of quadratic loss is advisable if one is quite sure about its sustainability. However, in all other cases one may safely use the proposed linear loss as it provides both symmetric and asymmetric loss functions.

Table 1: Predictive risk efficiencies of $y_{(}(1)L^{*}$) w.r.t $y_{(}(1)Q^{*}$) under linear loss for $\eta/\xi = 0.25$, $a = 2.0$

Censoring Scheme	r k l q	c=0.5	c=1.0	c=2.0	c=4.0	c=6.0
Multiply	2 6 2 2	7.79063	7.78322	7.76926	7.74680	7.72854
	3 6 3 2	7.41671	7.41340	7.40736	7.39848	7.39265
	3 6 3 4	7.12186	7.11859	7.11387	7.10741	7.10449
	4 6 4 4	6.04679	6.04945	6.05563	6.06998	6.08578
	3 3 3 8	6.83862	6.83238	6.82276	6.80977	6.80291
	3 3 5 8	5.66046	5.66266	5.66871	5.68433	5.70345
Right	0 6 0 6	7.81920	7.80480	7.77927	7.73778	7.70587
Left	6 6 0 0	7.70134	7.69118	7.67256	7.63885	7.60944
Doubly	3 6 0 3	7.73734	7.72666	7.70627	7.67201	7.64438
Mid	0 6 6 0	7.23222	7.23405	7.23803	7.24691	7.25660

Table 2: Predictive risk efficiencies of $y_{(}(1)L^{*})$ w.r.t $y_{(}(1)Q^{*})$ under linear loss for $\eta/\xi = 0.5$, $a = 2.0$

Censoring Scheme	r k l q	c=0.5	c=1.0	c=2.0	c=4.0	c=6.0
Multiply	2 6 2 2	3.32367	3.32090	3.31603	3.30887	3.30333
	3 6 3 2	3.17528	3.17349	3.17057	3.16651	3.16388
	3 6 3 4	3.06346	3.06167	3.05861	3.05435	3.05173
	4 6 4 4	2.73732	2.73701	2.73669	2.73754	2.73938
	3 3 3 8	2.92315	2.91999	2.91468	2.90715	2.90218
	3 3 5 8	2.57619	2.57507	2.57354	2.57260	2.57351
Right	0 6 0 6	3.36408	3.35919	3.35045	3.33705	3.32731
Left	6 6 0 0	3.37042	3.36797	3.36336	3.35609	3.35125
Doubly	3 6 0 3	3.38141	3.37762	3.37119	3.36112	3.35363
Mid	0 6 6 0	3.12307	3.12285	3.12276	3.12367	3.12591

Table 3: Predictive risk efficiencies of $y_{(}(1)L^{*})$ w.r.t $y_{(}(1)Q^{*})$ under linear loss for $\eta/\xi = 1$, $a = 2.0$

Censoring Scheme	r k l q	c=0.5	c=1.0	c=2.0	c=4.0	c=6.0
Multiply	2 6 2 2	1.64685	1.64532	1.64260	1.63780	1.63360
	3 6 3 2	1.61790	1.61666	1.61449	1.61070	1.60733
	3 6 3 4	1.59738	1.59605	1.59352	1.58936	1.58597
	4 6 4 4	1.51543	1.51464	1.51317	1.51099	1.50928
	3 3 3 8	1.57240	1.57015	1.56628	1.55981	1.55505
	3 3 5 8	1.47619	1.47483	1.47249	1.46894	1.46652
Right	0 6 0 6	1.66373	1.66117	1.65652	1.64859	1.64243
Left	6 6 0 0	1.64833	1.64704	1.64448	1.64002	1.63619
Doubly	3 6 0 3	1.65684	1.65498	1.65154	1.64569	1.64070
Mid	0 6 6 0	1.60212	1.60140	1.60017	1.59797	1.59617

Table 4: Predictive risk efficiencies of $y_((1)L^*)$ w.r.t $y_((1)Q^*)$ under linear loss for $\eta/\xi = 1.5$, $a = 2.0$

Censoring Scheme	r k l q	c=0.5	c=1.0	c=2.0	c=4.0	c=6.0
Multiply	2 6 2 2	1.15378	1.15271	1.15078	1.14736	1.14449
	3 6 3 2	1.14793	1.14697	1.14516	1.14207	1.13950
	3 6 3 4	1.14628	1.14515	1.14307	1.13951	1.13660
	4 6 4 4	1.12704	1.12617	1.12454	1.12179	1.11960
	3 3 3 8	1.14869	1.14680	1.14349	1.13812	1.13400
	3 3 5 8	1.12391	1.12247	1.11997	1.11600	1.11294
Right	0 6 0 6	1.16515	1.16342	1.16019	1.15486	1.15060
Left	6 6 0 0	1.15113	1.15026	1.14857	1.14564	1.14315
Doubly	3 6 0 3	1.15733	1.15604	1.15383	1.14993	1.14667
Mid	0 6 6 0	1.14229	1.14158	1.14026	1.13792	1.13593

Table 5: Predictive risk efficiencies of $y_((1)L^*)$ w.r.t $y_((1)Q^*)$ under linear loss for $\eta/\xi = 2.0$, $a = 2.0$.

Censoring Scheme	r k l q	c=0.5	c=1.0	c=2.0	c=4.0	c=6.0
Multiply	2 6 2 2	0.92194	0.92111	0.91958	0.91695	0.91476
	3 6 3 2	0.92498	0.92412	0.92260	0.91997	0.91779
	3 6 3 4	0.93169	0.93066	0.92882	0.92566	0.92308
	4 6 4 4	0.94085	0.93993	0.93821	0.93526	0.93280
	3 3 3 8	0.94675	0.94506	0.94208	0.93724	0.93349
	3 3 5 8	0.95613	0.95469	0.95209	0.94788	0.94453
Right	0 6 0 6	0.93082	0.92944	0.92695	0.92290	0.91964
Left	6 6 0 0	0.91819	0.91753	0.91628	0.91416	0.91234
Doubly	3 6 0 3	0.92313	0.92218	0.92048	0.91760	0.91522
Mid	0 6 6 0	0.92317	0.92246	0.92117	0.91892	0.91695

Table 6: Predictive risk efficiencies of $y_((1)L^*)$ w.r.t $y_((1)Q^*)$under Quadratic loss for$\eta/\xi = 0.25$, $a = 2.0$

Censoring Scheme	r k l q	c=0.5	c=1.0	c=2.0	c=4.0	c=6.0
Multiply	2 6 2 2	13.48227	13.53450	13.63262	13.81049	13.96707
	3 6 3 2	7.40479	7.44742	7.52944	7.68229	7.82271
	3 6 3 4	5.47138	5.50736	5.57608	5.70321	5.81908
	4 6 4 4	2.69043	2.70949	2.74656	2.81722	2.88405
	3 3 3 8	4.25656	4.29457	4.36542	4.49152	4.60220
	3 3 5 8	2.12220	2.14185	2.17978	2.25176	2.31977
Right	0 6 0 6	18.64233	18.70007	18.80303	18.96922	19.10026
Left	6 6 0 0	21.14014	21.25642	21.48313	21.90586	22.29466
Doubly	3 6 0 3	21.42228	21.48066	21.58621	21.77445	21.94114
Mid	0 6 6 0	6.55624	6.61722	6.73733	6.96971	7.19170

Table 7: Predictive risk efficiencies of $y_((1)L^*)$ w.r.t $y_((1)Q^*)$ under Quadratic loss for $\eta/\xi = 0.50$, $a = 2.0$

Censoring Scheme	r k l q	c=0.5	c=1.0	c=2.0	c=4.0	c=6.0
Multiply	2 6 2 2	5.88108	5.88977	5.90592	5.93503	5.96053
	3 6 3 2	4.34306	4.35583	4.38000	4.42391	4.46326
	3 6 3 4	3.53645	3.55002	3.57551	3.62108	3.66127
	4 6 4 4	2.24169	2.25204	2.27188	2.30848	2.34205
	3 3 3 8	2.79035	2.80646	2.83581	2.88559	2.92706
	3 3 5 8	1.79885	1.80990	1.83068	1.86816	1.90173
Right	0 6 0 6	6.27519	6.28356	6.29806	6.32120	6.34004
Left	6 6 0 0	7.13074	7.14041	7.15894	7.19477	7.23088
Doubly	3 6 0 3	7.04031	7.04437	7.05204	7.06628	7.08017
Mid	0 6 6 0	4.06447	4.08256	4.11767	4.18433	4.24684

Table 8: Predictive risk efficiencies of $y_((1)L^*)$ w.r.t $y_((1)Q^*)$ under Quadratic loss for $\eta/\xi = 1.0$, a=2.0

Censoring Scheme	r k l q	c=0.5	c=1.0	c=2.0	c=4.0	c=6.0
Multiply	2 6 2 2	2.50191	2.49851	2.49251	2.48172	2.47213
	3 6 3 2	2.35292	2.35145	2.34898	2.34452	2.34024
	3 6 3 4	2.23731	2.23683	2.23567	2.23386	2.23232
	4 6 4 4	1.89168	1.89383	1.89783	1.90531	1.91186
	3 3 3 8	2.06984	2.07047	2.07160	2.07270	2.07368
	3 3 5 8	1.69796	1.70169	1.70857	1.72034	1.73045
Right	0 6 0 6	2.55429	2.54834	2.53745	2.51854	2.50368
Left	6 6 0 0	2.54570	2.54248	2.53608	2.52489	2.51526
Doubly	3 6 0 3	2.56606	2.56122	2.55225	2.53693	2.52378
Mid	0 6 6 0	2.29457	2.29475	2.29535	2.29639	2.29741

Table 9: Predictive risk efficiencies of $y_((1)L^*)$ w.r.t $y_((1)Q^*)$ under Quadratic loss for $\eta/\xi = 1.5$, $a = 2.0$

Censoring Scheme	r k l q	c=0.5	c=1.0	c=2.0	c=4.0	c=6.0
Multiply	2 6 2 2	1.31942	1.31704	1.31273	1.30510	1.29871
	3 6 3 2	1.30909	1.30700	1.30303	1.29624	1.29058
	3 6 3 4	1.30614	1.30379	1.29938	1.29184	1.28561
	4 6 4 4	1.26207	1.26054	1.25763	1.25265	1.24861
	3 3 3 8	1.30755	1.30395	1.29760	1.28712	1.27896
	3 3 5 8	1.24482	1.24277	1.23920	1.23333	1.22864
Right	0 6 0 6	1.34448	1.34061	1.33338	1.32146	1.31198
Left	6 6 0 0	1.31216	1.31021	1.30644	1.29990	1.29437
Doubly	3 6 0 3	1.32618	1.32331	1.31835	1.30965	1.30237
Mid	0 6 6 0	1.29732	1.29576	1.29286	1.28770	1.28327

Table 10: Predictive risk efficiencies of $y_((1)L^*)$ w.r.t $y_((1)Q^*)$ under Quadratic loss for $\eta/\xi = 2.0$, $a = 2.0$

Censoring Scheme	r k l q	c=0.5	c=1.0	c=2.0	c=4.0	c=6.0
Multiply	2 6 2 2	0.85105	0.84956	0.84682	0.84214	0.83827
	3 6 3 2	0.85311	0.85155	0.84875	0.84397	0.84004
	3 6 3 4	0.86312	0.86116	0.85767	0.85171	0.84689
	4 6 4 4	0.87479	0.87289	0.86936	0.86336	0.85839
	3 3 3 8	0.89006	0.88662	0.88060	0.87090	0.86343
	3 3 5 8	0.90474	0.90154	0.89582	0.88656	0.87927
Right	0 6 0 6	0.86744	0.86493	0.86043	0.85311	0.84727
Left	6 6 0 0	0.84608	0.84493	0.84274	0.83903	0.83587
Doubly	3 6 0 3	0.85467	0.85298	0.84995	0.84484	0.84066
Mid	0 6 6 0	0.84875	0.84747	0.84516	0.84115	0.83767

Table 11: Predictive risk efficiencies under linear loss and under quadratic loss for real dataset

η/ξ	PRE_{LIN}	PRE_{QRD}
0.25	4.00657	1.00896
0.5	2.32859	1.24591
0.75	1.77982	1.40089
1	1.51332	1.44961
1.25	1.35966	1.40865
1.5	1.26237	1.31426
1.75	1.19724	1.19887
2	1.15215	1.08268
2.5	1.09795	0.88045
3.5	1.06079	0.61335
4	1.05973	0.52818

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