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Soliton Solutions for Fractional Choquard Equations

Quanqing Li[1], *Jian Zhang*[2], *Wenbo Wang*[3] *and Jianjun Nie*[4,*]

[1] Department of Mathematics, Honghe University, Mengzi, Yunnan 661100, P.R.China
[2] School of Mathematics and Statistics, Hunan University of Technology and Business, Changsha, Hunan 410205, P.R.China
[3] Department of Mathematics and Statistics, Yunnan University, Kunming, Yunnan 650091, P.R.China
[4] School of Mathematics and Physics, North China Electric Power University, Beijing, Beijing, 102206, P. R. China

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Abstract: In this paper, we establish the existence of soliton type solutions for a class of fractional Choquard equations. Our main technique is based on constrained minimization arguments.

Keywords: Fractional Choquard equations, Minimization arguments, Soliton solutions.

1 Introduction, motivation and preliminaries

In this paper, we address the existence of soliton type solutions for a class of fractional Choquard equations of the following form

$$(-\Delta)^s u+V(x)u+\kappa[(-\Delta)^s u^2]u=\lambda[|x|^{-\mu}*|u|^p]|u|^{p-2}u,\ u>0,\ x\in\mathbb{R}^N, \tag{1.1}$$

where $0<s<1$, $(-\Delta)^s$ denotes the fractional Laplacian of order s, $N>2s$, $0<\mu<N$, $\frac{2N-\mu}{N}\leq p<\frac{2N-\mu}{N-2s}$ and κ is a real constant.

Solutions of (1.1) are related to the existence of solitary wave solutions for the following equations

$$i\psi_t=(-\Delta)^s\psi+[V(x)+E]\psi-\lambda[|x|^{-\mu}*|\psi|^p]|\psi|^{p-2}\psi+\kappa[(-\Delta)^s\rho(|\psi|^2)]\rho'(|\psi|^2)\psi, \tag{1.2}$$

where $\psi:\mathbb{R}^N\times\mathbb{R}\to\mathbb{C}$, κ is a real constant and ρ is a real function. Here, we handle the existence of solutions for Eq. (1.2) with $\rho(t)=t$. If we set $\psi(x,t)=e^{-iEt}u(x)$ in (1.2), where $E\in\mathbb{R}$ and $u>0$ is a real function, then (1.2) turns into (1.1).

For $\kappa=0$, as anticipated, problem (1.1) has nonlocal characteristics in the nonlinearity as well as in the (fractional) diffusion. When $s=1$; $p=2$; $N=3$ and $\mu=\lambda=1$, (1.1) boils down to the so-called Choquard equation

$$-\Delta u+V(x)u=[|x|^{-1}*|u|^2]u,\ x\in\mathbb{R}^3. \tag{1.3}$$

This equation goes back to the description of the quantum theory of a polaron at rest by Peak in 1954 [1] and the modeling of an electron trapped in its own hole in 1976 in the work of Choquard, as a certain approximation to Hartree-Fock theory of one-component plasma [2]. In some particular cases, this equation is also known as the Schrödinger-Newton equation, which was introduced by Penrose in his discussion on the selfgravitational collapse of a quantum mechanical wave function [3]. The first investigations for existence and symmetry of the solutions to (1.3) go back to the works of Lieb [2] and Lions [4]. Since then, many efforts have been made to study the existence of nontrivial solutions for nonlinear Choquard equations. For related results, we refer the readers to [5,6] for the existence of sign-changing solutions, [7,8] for the existence and concentration behavior of the semiclassical solutions, [9] for the existence of multi-bump solutions and [10] for the critical nonlocal part with respect to the Hardy-Littlewood-Sobolev inequality.

For fractional Laplacian with local type nonlinearities, a great attention has been devoted in recent years, such as [11, 12,13] and their references therein. In [15], the authors investigated the existence of positive solutions for a class of

* Corresponding author e-mail: niejjun@126.com

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critical fractional Schrödinger equations with potential vanishing at infinity by the variational method. Moreover, when the potential and nonlinearity are asymptotically periodic in x, we obtain the existence of ground state solutions by the Nehari method ([14]). See [16,17,18,19] and the references cited therein for recent results of soliton solutions about fractional equations. Only few papers, including [20], addressed the existence of solutions for problem (1.1) with $\kappa = 1$ and local type nonlinearities. Quantum systems [21,22] can be manipulated via fractional Schrödinger equations.

For fractional Laplacian with nonlocal Hartree-type nonlinearities, the problem has also attracted much interest, which arises in various branches of mathematical physics, such as the mean field limit of weakly interacting molecules, the quantum mechanical theory, physics of multiple-particle systems, etc., see [23]. In the case $s = \frac{1}{2}$, problem (1.1) has been used to model the dynamics of pseudo-relativistic boson stars. For example, Frank and Lenzmann in [24] proved analyticity and radial symmetry of ground state solutions $u > 0$ for the L^2- critical boson star equation

$$\sqrt{-\Delta}u + u = [|x|^{-1} * |u|^2]u,\ x \in \mathbb{R}^3.$$

The square root of the Laplacian also appears in the semi-relativistic Schrödinger-Poisson-Slater systems ([25]). In [26], it is shown that the dynamical evolution of boson stars is described by the nonlinear evolution equation

$$i\partial_t \psi = \sqrt{-\Delta + m^2}\psi - [|x|^{-1} * |\psi|^2]\psi\ (m \geq 0),$$

for a field $\psi :\ [0,T) \times \mathbb{R}^3 \to \mathbb{C}$. More results about the fractional Choquard equation can be found in [27] and their references therein. The case $\kappa = 0$ has been extensively studied in recent years. However, to our best knowledge, there is no result for (1.1) with $\kappa \neq 0$. For simplicity, set $\kappa = 1$. Furthermore, we also assume that the following condition holds.

(V) $\lim\limits_{|x| \to \infty} V(x) = +\infty$.

For any $0 < s < 1$, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{N+2s}{2}}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N)\},$$

endowed with the natural norm

$$\|u\|_{H^s(\mathbb{R}^N)} = (\int_{\mathbb{R}^N} u^2 dx + \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy)^{\frac{1}{2}},$$

where the term

$$[u]_{H^s(\mathbb{R}^N)} := (\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy)^{\frac{1}{2}}$$

is the so-called Gagliardo semi-norm of u. Let $\mathscr{S}$ be the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^N$. For any $u \in \mathscr{S}$ and $s \in (0,1)$, $(-\Delta)^s$ is defined as

$$(-\Delta)^s u(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy = C_{N,s} \lim_{\varepsilon \to 0^+} \int_{\mathscr{C}B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dy, \tag{1.4}$$

where $\mathscr{C}B_\varepsilon(x) = \mathbb{R}^N \setminus B_\varepsilon(x)$. The symbol P.V. stands for the Cauchy principal value and $C_{N,s}$ is a dimensional constant that depends on N and s and is given by

$$C_{N,s} = (\int_{\mathbb{R}^N} \frac{1 - cos\zeta_1}{|\zeta|^{N+2s}} d\zeta)^{-1}.$$

Indeed, the fractional Laplacian $(-\Delta)^s$ can be viewed as a pseudo-differential operator of symbol $|\xi|^{2s}$, as stated in the following, check relevance to the previous part Proposition 3.3 in [29].

Let $s \in (0,1)$ and $(-\Delta)^s : \mathscr{S} \to L^2(\mathbb{R}^N)$ be the fractional Laplacian operator defined by (1.4). Then, for any $u \in \mathscr{S}$,

$$(-\Delta)^s u(x) = \mathscr{F}^{-1}(|\xi|^{2s}(\mathscr{F}u))(x),\ \forall \xi \in \mathbb{R}^N.$$

Here, $\mathscr{F}u := \hat{u} := (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} u(x) e^{-ix\cdot\xi} dx$ denotes the Fourier transform of u. Then we can see that an alternative definition of the fractional Sobolev space $H^s(\mathbb{R}^N)$ via the Fourier transform, as follows:

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (1 + |\xi|^{2s}) |\hat{u}(\xi)|^2 d\xi < +\infty\}.$$

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Propositions 3.4 and 3.6 in [29] imply that

$$2C_{N,s}^{-1}\int_{\mathbb{R}^N}|\xi|^{2s}|\hat{u}(\xi)|^2d\xi=2C_{N,s}^{-1}\|(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^N)}^2=[u]_{H^s(\mathbb{R}^N)}^2.$$

Set

$$H=\{u\in H^s(\mathbb{R}^N):u^2\in H^s(\mathbb{R}^N)\}.$$

Using a standard argument, we know that a weak solution of problem (1.1) is a critical point of the following functional

$$\begin{aligned}J(u)=&\frac{1}{2}\int_{\mathbb{R}^N}|\xi|^{2s}|\hat{u}(\xi)|^2d\xi+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2dx+\frac{1}{4}\int_{\mathbb{R}^N}|\xi|^{2s}|\widehat{u^2}(\xi)|^2d\xi\\&-\frac{\lambda}{2p}\int_{\mathbb{R}^N}[|x|^{-\mu}*|u|^p]|u|^pdx\\=&\frac{1}{2}\int_{\mathbb{R}^N}|\xi|^{2s}|\hat{u}(\xi)|^2d\xi+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2dx+\frac{1}{4}\int_{\mathbb{R}^N}|\xi|^{2s}|\hat{u}(\xi)*\hat{u}(\xi)|^2d\xi\\&-\frac{\lambda}{2p}\int_{\mathbb{R}^N}[|x|^{-\mu}*|u|^p]|u|^pdx,\end{aligned}$$

where $\widehat{u^2}(\xi)=\hat{u}(\xi)*\hat{u}(\xi)$. Here, $(f*g)(x)=\int_{\mathbb{R}^N}f(x-y)g(y)dy$. Moreover,

$$\begin{aligned}\langle J'(u),\varphi\rangle=&\int_{\mathbb{R}^N}(-\Delta)^su\varphi dx+\int_{\mathbb{R}^N}V(x)u\varphi dx+\int_{\mathbb{R}^N}[(-\Delta)^su^2]u\varphi dx\\&-\lambda\int_{\mathbb{R}^N}[|x|^{-\mu}*|u|^p]|u|^{p-2}u\varphi dx\\=&\int_{\mathbb{R}^N}|\xi|^{2s}\hat{u}(\xi)\bar{\hat{\varphi}}(\xi)d\xi+\int_{\mathbb{R}^N}V(x)u\varphi dx+\int_{\mathbb{R}^N}|\xi|^{2s}[\hat{u}(\xi)*\hat{u}(\xi)]\cdot[\bar{\hat{u}}(\xi)*\bar{\hat{\varphi}}(\xi)]d\xi\\&-\lambda\int_{\mathbb{R}^N}[|x|^{-\mu}*|u|^p]|u|^{p-2}u\varphi dx\\=&\frac{1}{2}C_{N,s}\int_{\mathbb{R}^{2N}}\frac{[u(x)-u(y)][\varphi(x)-\varphi(y)]}{|x-y|^{N+2s}}dxdy+\int_{\mathbb{R}^N}V(x)u\varphi dx\\&+\frac{1}{2}C_{N,s}\int_{\mathbb{R}^{2N}}\frac{[u^2(x)-u^2(y)][u(x)\varphi(x)-u(y)\varphi(y)]}{|x-y|^{N+2s}}dxdy\\&-\lambda\int_{\mathbb{R}^N}[|x|^{-\mu}*|u|^p]|u|^{p-2}u\varphi dx\end{aligned}$$

for all $u,\ \varphi\in H$.

It seems quite clear that H is the working space for studying the problem (1.1). However, H is not a linear space. Consequently, the classical critical point theory and usual min-max techniques cannot be directly applied to the energy functional J, so we encounter the difficulties caused by the lack of an appropriate working space. Motivated by [30], we use a constrained minimization argument to give a solution of Eq. (1.1).

2 Main result

Set

$$X:=\{u\in H^s(\mathbb{R}^N):\int_{\mathbb{R}^N}V(x)u^2dx<+\infty\}$$

with the norm

$$\|u\|_X^2=\int_{\mathbb{R}^N}|\xi|^{2s}|\hat{u}(\xi)|^2d\xi+\int_{\mathbb{R}^N}V(x)u^2dx.$$

For any $a>0$, we define

$$m_a:=\inf_{u\in M_a}E(u),$$

where

$$M_a:=\{u\in H:\int_{\mathbb{R}^N}[|x|^{-\mu}*|u|^p]|u|^pdx=a\}$$

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and

$$E(u)=\int_{\mathbb{R}^N}|\xi|^{2s}|\hat{u}(\xi)|^2d\xi+\int_{\mathbb{R}^N}V(x)u^2dx+\int_{\mathbb{R}^N}|\xi|^{2s}|\hat{u}(\xi)*\hat{u}(\xi)|^2d\xi.$$

Because we are concerned with the nonlocal problems, we would like to recall the well-known Hardy-Littlewood-Sobolev inequality, which will be frequently used throughout the paper.

Proposition 2.1. [31] (Hardy-Littlewood-Sobolev inequality) Let $r,\ t>1$ and $0<\mu<N$ with $\frac{1}{r}+\frac{\mu}{N}+\frac{1}{t}=2$. Let $g\in L^r(\mathbb{R}^N)$ and $h\in L^t(\mathbb{R}^N)$. Then, there exists a sharp constant $C_{r,N,\mu,t}$ independent of g and h such that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{g(x)h(y)}{|x-y|^{\mu}}dxdy\le C_{r,N,\mu,t}\|g\|_{L^r(\mathbb{R}^N)}\|h\|_{L^t(\mathbb{R}^N)}.$$

Remark 2.2. In general, set $F(u)=|u|^q$ for some $q>0$. By Hardy-Littlewood-Sobolev inequality, $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(u(x))F(u(y))}{|x-y|^{\mu}}dxdy$ is well defined if $F(u)\in L^t(\mathbb{R}^N)$ for $t>1$ defined by $\frac{2}{t}+\frac{\mu}{N}=2$. Thus, for $u\in H^s(\mathbb{R}^N)$, there must hold

$$\frac{2N-\mu}{N}\le q\le\frac{2N-\mu}{N-2s}.$$

Our main result is the following:

Theorem 2.3 Assume that (V) holds. Then, there exists $\lambda_n\to+\infty$ such that Eq. (1.1) with $\lambda=\lambda_n$ has a solution.

Proof. We divide the proof into two steps.

Step 1: We claim that for each $a>0$, m_a is achieved at some $u_a\in M_a$, which is a weak solution of equation (1.1) with $\lambda=\lambda_a$ satisfying $\lambda_a=\frac{m_a}{a^{p+1}}$.

Indeed, we fix $a>0$. Let $\{u_n\}\subset M_a$ be a minimizing sequence for m_a. We may assume $u_n\ge 0$ for all n. Then, by the proof of Lemma 3.4 in [32], we know that the embedding $X\hookrightarrow L^q(\mathbb{R}^N)$ is compact for $2\le q<2_s^*$. Hence, up to a subsequence, we have $u_n\rightharpoonup u_a$ in X, $u_n\to u_a$ in $L^q(\mathbb{R}^N)$ for $2\le q<2_s^*$ and $u_n(x)\to u_a(x)\ge 0$ for almost all $x\in\mathbb{R}^N$. Note that

$$\begin{aligned}\int_{\mathbb{R}^N}|\xi|^{2s}|\hat{u}_n(\xi)*\hat{u}_n(\xi)|^2d\xi=&\int_{\mathbb{R}^N}|\xi|^{2s}|\widehat{u_n^2}(\xi)|^2d\xi\\ =&\frac{1}{2}C_{N,s}\int_{\mathbb{R}^{2N}}\frac{|u_n^2(x)-u_n^2(y)|^2}{|x-y|^{N+2s}}dxdy.\end{aligned}$$

By Fatou Lemma, we have

$$\int_{\mathbb{R}^{2N}}\frac{|u_a^2(x)-u_a^2(y)|^2}{|x-y|^{N+2s}}dxdy\le\liminf_{n\to\infty}\int_{\mathbb{R}^{2N}}\frac{|u_n^2(x)-u_n^2(y)|^2}{|x-y|^{N+2s}}dxdy.$$

Consequently,

$$\int_{\mathbb{R}^N}|\xi|^{2s}|\hat{u}_a(\xi)*\hat{u}_a(\xi)|^2d\xi\le\liminf_{n\to\infty}\int_{\mathbb{R}^N}|\xi|^{2s}|\hat{u}_n(\xi)*\hat{u}_n(\xi)|^2d\xi.$$

Hence, by weakly lower semi-continuity of the norm, one has

$$\begin{aligned}E(u_a)\le&\liminf_{n\to\infty}[\int_{\mathbb{R}^N}|\xi|^{2s}|\hat{u}_n(\xi)|^2d\xi+\int_{\mathbb{R}^N}V(x)u_n^2dx]+\liminf_{n\to\infty}\int_{\mathbb{R}^N}|\xi|^{2s}|\hat{u}_n(\xi)*\hat{u}_n(\xi)|^2d\xi\\ \le&\liminf_{n\to\infty}E(u_n).\end{aligned}$$

Since $\frac{2N-\mu}{N}\le p<\frac{2N-\mu}{N-2s}$, the Hardy-Littlewood-Sobolev inequality implies that

$$\begin{aligned}&|\int_{\mathbb{R}^N}[|x|^{-\mu}*|u_n|^p]|u_n|^pdx-\int_{\mathbb{R}^N}[|x|^{-\mu}*|u_a|^p]|u_a|^pdx|\\ \le&C(\|u_n\|^p_{\frac{2Np}{2N-\mu}}+\|u_a\|^p_{\frac{2Np}{2N-\mu}})\||u_n|^p-|u_a|^p\|_{\frac{2N}{2N-\mu}}\\ \to&0\end{aligned}$$

as $n\to\infty$. It follows that $\int_{\mathbb{R}^N}[|x|^{-\mu}*|u_a|^p]|u_a|^pdx=a$, so $m_a\le E(u_a)\le\liminf_{n\to\infty}E(u_n)=m_a$, i.e., $E(u_a)=m_a$. Hence, m_a is achieved at some $u_a\in M_a$. We can conclude that u_a is a weak solution of

$$(-\Delta)^su+V(x)u+[(-\Delta)^su^2]u=\lambda_a[|x|^{-\mu}*|u|^p]|u|^{p-2}u. \tag{2.1}$$

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Multiplying the Eq. (2.1) by u_a and integrating over $\mathbb{R}^N$, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}|\xi|^{2s}|\hat{u}_a(\xi)|^2d\xi+\int_{\mathbb{R}^N}V(x)u_a^2dx+\int_{\mathbb{R}^N}|\xi|^{2s}|\hat{u}_a(\xi)*\hat{u}_a(\xi)|^2d\xi\\ =&\lambda_a\int_{\mathbb{R}^N}[|x|^{-\mu}*|u_a|^p]|u_a|^pdx=\lambda_a\cdot a,\end{aligned}$$

which means that $\lambda_a\cdot a=E(u_a)=m_a$, i.e., $\lambda_a=\frac{m_a}{a}$.

Step 2: We prove that $\lambda_a\to+\infty$ as $a\to 0$.

Indeed, suppose the conclusion is false, then there exists $a_n\to 0$ such that $\lambda_n:=\lambda_{a_n}\le C_1$. Set $u_n:=u_{a_n}$. By $\int_{\mathbb{R}^N}[|x|^{-\mu}*|u_n|^p]|u_n|^pdx=a_n\to 0$ as $n\to\infty$, one has

$$\|u_n\|_X^2+\int_{\mathbb{R}^N}|\xi|^{2s}|\hat{u}_n(\xi)*\hat{u}_n(\xi)|^2d\xi=\lambda_n\int_{\mathbb{R}^N}[|x|^{-\mu}*|u_n|^p]|u_n|^pdx\le C_1a_n\to 0.$$

On the other hand, the Hardy-Littlewood-Sobolev inequality together with Sobolev embedding theorem implies that

$$\int_{\mathbb{R}^N}[|x|^{-\mu}*|u_n|^p]|u_n|^pdx\le C_2\|u_n\|^{2p}_{\frac{2Np}{2N-\mu}}\le C_3\|u_n\|_X^{2p}.$$

Consequently,

$$\begin{aligned}&\|u_n\|_X^2+\int_{\mathbb{R}^N}|\xi|^{2s}|\hat{u}_n(\xi)*\hat{u}_n(\xi)|^2d\xi\\ =&\lambda_n\int_{\mathbb{R}^N}[|x|^{-\mu}*|u_n|^p]|u_n|^pdx\\ \le&C_1C_3[\|u_n\|_X^2+\int_{\mathbb{R}^N}|\xi|^{2s}|\hat{u}_n(\xi)*\hat{u}_n(\xi)|^2d\xi]^p,\end{aligned}$$

which implies that $\|u_n\|_X^2+\int_{\mathbb{R}^N}|\xi|^{2s}|\hat{u}_n(\xi)*\hat{u}_n(\xi)|^2d\xi\ge C_4>0$ since $p\ge\frac{2N-\mu}{N}>1$, a contradiction. By Steps 1 and 2, we complete the proof of Theorem 2.3.

3 Conclusion

The existence of soliton type solutions for the fractional Choquard equation (1.1) was established using a constrained minimization argument. To the best of our knowledge, there is no paper considering the fractional Choquard equation (1.1) with $\kappa\neq 0$.

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