Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/040405

Numerical Treatment for Solving the Fractional Two-Group Influenza Model

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Received: 2 Feb. 2018, Revised: 17 Apr. 2018, Accepted: 21 Apr. 2018
Published online: 1 Oct. 2018

Abstract: In this article, a general model for Influenza of two groups is presented as a fractional order model. The fractional derivatives for this model which consist of eight differential equations are defined in the sense of Caputo definition. To obtain an efficient numerical method, the fraction order derivatives are approximated by the shifted Jacobi polynomials. The proposed scheme reduces the solution of the main problem to the solution of a system of nonlinear algebraic equations. Comparative studies between the proposed method and both the fourth-order Runge-Kutta method and the generalized Euler method are done.

Keywords: Influenza mathematical model, Jacobi polynomials, spectral method, Caputo derivative, generalized Euler method.

1 Introduction

Influenza is a common respiratory disease caused by the influenza virus. This disease spreads easily from person to person through coughing, sneezing and hands touching your eyes, mouth or nose. Several papers considered modeling Influenza such as ([1]-[8]). On other hand, mathematical models provide important and efficient tool to describe several problems in natural sciences disciplines such as biology, physics, meteorology science and many other fields ([9]-[13]).

Through the past three decades, there has been an interest in the study of the fractional differential equations (FDEs) ([14]-[20]). The applications of fractional calculus are used in various problems in science and biology ([12],[13]), magnetic plasma [21], physics [22] and other phenomena.

In the mean time, the Jacobi polynomials have been used as basis functions of the spectral collocation method for approximating the solution of different types of differential equations, see for example [23] and [24], [25].

In this paper, a novel Influenza model in two groups of fraction order derivatives is presented and modified parameters are introduced to account for the fractional order. The main aim of this work is to study the approximate solutions for Influenza model in two groups of fraction order derivatives using the shifted Jacobi polynomials. Shifted Legendre polynomials and shifted Chebyshev polynomials as special cases from Jacobi polynomials, are introduced to solve the the proposed system and compare the numerical results obtained by the proposed method with those numerical solutions using fourth-order Runge-Kutta (RK4) and the generalized Euler method (GEM).

This paper is organized as follows: In section 2, the basic and necessary definitions of the shifted Jacobi Polynomials and the fractional order derivatives with their properties are presented. In section 3, fractional order derivatives for Influenza in two-group model and the basic reproduction number are introduced; in addition, the stability of equilibrium points is studied. In section 4, procedure solution of the fractional Influenza in two-group model is presented. Numerical simulations are given in section 5. In section 6, the conclusions are given.

2 Elementary Definitions

In the following, some definitions and mathematical tools of the fractional calculus theory which are used in this paper are introduced.

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Definition 21[18] *The Caputo fractional derivative operator* ${}_{a}^{c}D_{t}^{\alpha}$ *of order* α *is defined in the following form*

$$ {}_{a}^{c}D_{t}^{\alpha}f(x)=\frac{1}{\Gamma(m-\alpha)}\int_{0}^{x}\frac{f^{m}(t)}{(x-t)^{\alpha-m+1}}dt, $$

where $\alpha>0,\ \ x>0$, and $m-1<\alpha\leq m,\ \ m\in\mathbb{N}$,
similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation ${}_{a}^{c}D_{t}^{\alpha}(\lambda f(x)+\mu g(x))=\lambda{}_{a}^{c}D_{t}^{\alpha}f(x)+\mu{}_{a}^{c}D_{t}^{\alpha}g(x)$, where λ and μ are constant. For the Caputo's derivative we have $D^{\alpha}C=0,\ \ C$ is a constant and

$$ {}_{a}^{c}D_{t}^{\alpha}x^{n}=\begin{cases}0, & n\in\mathbb{N}_{0},\quad n<\lceil\alpha\rceil\,,\\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}x^{n-\alpha}, & n\in\mathbb{N}_{0},\quad n\geq\lceil\alpha\rceil.\end{cases} \tag{1} $$

Where, $\lceil\alpha\rceil$ to denote the smallest integer greater than or equal to α , $\mathbb{N}_{0}=0,1,...$ and it's called the ceiling function. For more details on fractional derivatives definitions and theirs properties see [18] and [19].

2.1 Jacobi polynomials

The well-known Jacobi polynomials are defined on the interval $[-1,1]$ and can be generated with the aid of the following recurrence formula:

$$ \begin{aligned} P_{i}^{(\mu,\beta)}(t)=&\frac{(\mu+\beta+2i-1)\{\mu^{2}-\beta^{2}+t(\mu+\beta+2i)(\mu+\beta+2i-2)\}}{2i(\mu+\beta+i)(\mu+\beta+2i-2)}P_{i-1}^{(\mu,\beta)}(t)\\ &-\frac{(\mu+i-1)(\beta+i-1)(\mu+\beta+2i)}{i(\mu+\beta+i)(\mu+\beta+2i-2)}P_{i-2}^{(\mu,\beta)}(t), \end{aligned} $$

$i=2,3,...,$ where,

$$ P_{0}^{(\mu,\beta)}(t)=1,\qquad P_{1}^{(\mu,\beta)}(t)=\frac{\mu+\beta+2}{2}t+\frac{\mu-\beta}{2}. $$

In order to use these polynomials on the interval $x\in[0,L]$ we define the so-called shifted Jacobi polynomials by introducing the change of variable $t=\frac{2x}{L}-1$. Let the shifted Jacobi polynomials $P_{i}^{(\mu,\beta)}(\frac{2x}{L}-1)$ be denoted by $P_{L,i}^{(\mu,\beta)}(x)$. Then $P_{L,i}^{(\mu,\beta)}(x)$ can be generated from:

$$ \begin{aligned} P_{L,i}^{(\mu,\beta)}(x)=&\frac{(\mu+\beta+2i-1)\{\mu^{2}-\beta^{2}+(\frac{2x}{L}-1)(\mu+\beta+2i)(\mu+\beta+2i-2)\}}{2i(\mu+\beta+i)(\mu+\beta+2i-2)}P_{i-1}^{(\mu,\beta)}(x)\\ &-\frac{(\mu+i-1)(\beta+i-1)(\mu+\beta+2i)}{i(\mu+\beta+i)(\mu+\beta+2i-2)}P_{i-2}^{(\mu,\beta)}(x), \end{aligned} $$

$i=2,3,...,$ where,

$$ P_{L,0}^{(\mu,\beta)}(x)=1,\qquad P_{L,1}^{(\mu,\beta)}(x)=\frac{\mu+\beta+2}{2}(\frac{2x}{L}-1)+\frac{\mu-\beta}{2}. $$

The analytic form of the shifted Jacobi polynomials $P_{L,i}^{(\mu,\beta)}(x)$ of degree i is given by

$$ P_{L,i}^{(\mu,\beta)}(x)=\sum_{k=0}^{i}(-1)^{i-k}\frac{\Gamma(\beta+i+1)\Gamma(\mu+\beta+i+k+1)}{\Gamma(\beta+k+1)\Gamma(\mu+\beta+i+1)(i-k)!k!L^{k}}x^{k}, \tag{2} $$

where,

$$ P_{L,i}^{(\mu,\beta)}(0)=(-1)^{i}\frac{\Gamma(\beta+i+1)}{\Gamma(\beta+1)i!},\qquad P_{L,i}^{(\mu,\beta)}(L)=\frac{\Gamma(\mu+i+1)}{\Gamma(\mu+1)i!}. $$

The orthogonality condition of shifted Jacobi polynomials is

$$\int_0^L P_{L,j}^{(\mu,\beta)}(x)P_{L,k}^{(\mu,\beta)}(x)w_L^{(\mu,\beta)}(x)dx = h_k, \tag{3}$$

where $w_L^{(\mu,\beta)}(x)$ is the weighted function, which is defined as follows:

$$w_L^{(\mu,\beta)}(x) = x^{\beta}(L-x)^{\mu},$$

$$h_k = \begin{cases} \frac{L^{\mu+\beta+1}\Gamma(k+\mu+1)\Gamma(k+\beta+1)}{(2k+\mu+\beta+1)\Gamma(k+\beta+\mu+1)k!}, & j = k\,, \\ 0, & j \neq k. \end{cases}$$

Let $v(x)$ be an analytic function, then it may be expressed in terms of shifted Jacobi polynomials as follows [26]:

$$v(x) \cong \sum_{j=0}^{m} c_j P_{L,j}^{(\mu,\beta)}(x), \tag{4}$$

where the coefficients c_j are given by:

$$c_j = \frac{1}{h_k} = \int_0^L P_{L,j}^{(\mu,\beta)}(x)v(x)w_L^{(\mu,\beta)}(x)dx, \quad j = 0,1,...,N. \tag{5}$$

2.2 *Fractional Jacobi spectral collocation method*

Let us consider $P_{L,k}^{(\mu,\beta)}(x)$ of degree i is given by (2). Using Eqs. (1) and by noting that the Caputo fractional derivative operator is a linear operator, we can claim from (2):

$$\begin{aligned} {}_a^cD_t^{\alpha}(P_{L,j}^{(\mu,\beta)}(x)) &= \sum_{k=0}^{j}(-1)^{j-k}\frac{\Gamma(\beta+j+1)\Gamma(\mu+\beta+j+k+1)}{\Gamma(\beta+k+1)\Gamma(\mu+\beta+j+1)(j-k)!k!L^k}\,{}_a^cD_t^{\alpha}(x^k), \\ &= \sum_{k=\lceil\alpha\rceil}^{j}(-1)^{j-k}\frac{\Gamma(\beta+j+1)\Gamma(\mu+\beta+j+k+1)}{\Gamma(\beta+k+1)\Gamma(\mu+\beta+j+1)(j-k)!\Gamma(k-\alpha+1)L^k}x^{k-\alpha}, \\ j &= \lceil\alpha\rceil, \lceil\alpha\rceil+1, \cdots. \end{aligned} \tag{6}$$

Now, the spectral collocation method is applied by setting the residual of Eqs. (6) equal zero at the $m+1-\lceil\alpha\rceil$ collocation points:

$${}_a^cD_t^{\alpha}(v(x)) = \sum_{j=0}^{m} c_j c_a D_t^{\alpha}(P_{L,j}^{(\mu,\beta)}(x_p)) = \sum_{j=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{j} a_j \Theta_{j,k}^{\alpha} x_p^{k-\alpha}, \quad p = 0,1,2,\cdots,m-\lceil\alpha\rceil, \tag{7}$$

where,

$$\begin{aligned} \Theta_{j,k}^{\alpha} &= (-1)^{j-k}\frac{\Gamma(\beta+j+1)\Gamma(\mu+\beta+j+k+1)}{\Gamma(\beta+k+1)\Gamma(\mu+\beta+j+1)(j-k)!\Gamma(k-\alpha+1)L^k}, \\ j &= \lceil\alpha\rceil, \lceil\alpha\rceil+1, \cdots, m. \end{aligned} \tag{8}$$

3 Mathematical Model

In this section, the fractional Influenza model of two groups with modified parameters is presented, it is more general model than the model given in [27]. The new parameters of the proposed model are θ^{α}, β_1^{α}, ξ^{α}, ρ^{α} as described in Table 2, this modified parameters are introduced to account for the fractional order [28]. The population in this model is divided into two groups. The first group is the individuals belonging to economically higher strata and the second group is the individuals belonging to economically lower strata [27]. All interpretation and meaning of the variables for the proposed model are given in Table 1.

The new system is described by fractional order derivatives as follows:

$$
\begin{aligned}
{}_a^cD_t^\alpha S_1 =&\theta^\alpha - \beta_1^\alpha S_1 I_1 - \beta_1^\alpha S_1 I_2 - (\xi^\alpha + \theta^\alpha) S_1, && (9)\\
{}_a^cD_t^\alpha V_1 =&\xi^\alpha S_1 - (1-\rho^\alpha)\beta_1^\alpha V_1 I_1 - (1-\rho^\alpha)\beta_1^\alpha V_1 I_2 - \theta^\alpha V_1, && (10)\\
{}_a^cD_t^\alpha I_1 =&\beta_1^\alpha S_1 (I_1 + I_2) + (1-\rho^\alpha)\beta_1^\alpha V_1 (I_1 + I_2) - (\theta^\alpha + \eta^\alpha) I_1, && (11)\\
{}_a^cD_t^\alpha R_1 =&\eta^\alpha I_1 - \theta^\alpha R_1, && (12)\\
{}_a^cD_t^\alpha S_2 =&\theta^\alpha - \beta_1^\alpha S_2 I_2 - (\xi^\alpha + \mu^\alpha) S_1, && (13)\\
{}_a^cD_t^\alpha V_2 =&\xi^\alpha S_2 - (1-\rho^\alpha)\beta_1^\alpha V_2 I_2 - \theta^\alpha V_2, && (14)\\
{}_a^cD_t^\alpha I_2 =&\beta_1^\alpha I_1 S_2 + (1-\rho^\alpha)\beta_1^\alpha V_2 I_2 - (\theta^\alpha + \eta^\alpha) I_2, && (15)\\
{}_a^cD_t^\alpha R_2 =&\eta^\alpha I_2 - \theta^\alpha R_2. && (16)
\end{aligned}
$$

with the following initial conditions

$$S_i(0) = s_{i0}, I_i(0) = i_{i0}, R_i(0) = r_{i0}, C_i(0) = c_{i0},$$

$$i = 1, 2.$$

Table 1: Definitions of the variable states of system (9) − (16).

Variable	Definition
$S_1(t)$	The proportion of susceptible at time t, of higher strata group.
$V_1(t)$	The proportion of vaccinated at time t, of higher strata group.
$I_1(t)$	The proportion of infectious at time t, of higher strata group.
$R_1(t)$	The proportion of removed individuals at time t, of higher strata group.
$S_2(t)$	The proportion of susceptible of lower strata group.
$V_2(t)$	The proportion of vaccinated of lower strata group.
$I_2(t)$	The proportion of infectious of lower strata group.
$R_2(t)$	The proportion of removed individuals of lower strata group.

Table 2: All parameters and their values of the system (9) − (16).

parameter	Definition	Value
θ^α	birth and death rates.	$(\frac{1.5\times10^{-3}}{day})^\alpha$
η^α	Removal rate due to hospitalization and isolation.	$(\frac{1.0\times10^{-2}}{day})^\alpha$
β_1^α	Transmission rate.	$(\frac{1.5\times10^{-1}}{day})^\alpha$
ξ^α	Vaccination rate.	$(\frac{0.08}{day})^\alpha$
ρ^α	Vaccination efficacy.	$(\frac{0.99}{day})^\alpha, (\frac{0.94}{day})^\alpha, \ldots$

One of the main assumptions of this model is that:

$$\Omega = \{(S_i, V_i, R_i, I_i) : S_i, V_i, R_i, I_i \geq 0; S_i + V_i + R_i + I_i = 1; i = 1, 2\}$$

is positively invariant for the model $(9)-(16)$. Using $R_i = 1 - S_i - I_i - Vi;\;\; i = 1,2$ in Ω equations (12) and (16) can be removed from the model. Therefore we have to study only following six equations:

$$
\begin{aligned}
{}^c_aD^\alpha_t S_1 &= \theta^\alpha - \beta_1^\alpha S_1 I_1 - \beta_1^\alpha S_1 I_2 - (\xi^\alpha + \theta^\alpha) S_1, && (17)\\
{}^c_aD^\alpha_t V_1 &= \xi^\alpha S_1 - (1-\rho^\alpha)\beta_1^\alpha V_1 I_1 - (1-\rho^\alpha)\beta_1^\alpha V_1 I_2 - \theta^\alpha V_1, && (18)\\
{}^c_aD^\alpha_t I_1 &= \beta_1^\alpha S_1 (I_1 + I_2) + (1-\rho^\alpha)\beta_1^\alpha V_1 (I_1 + I_2) - (\theta^\alpha + \eta^\alpha) I_1, && (19)\\
{}^c_aD^\alpha_t S_2 &= \theta^\alpha - \beta_1^\alpha S_2 I_2 - (\xi^\alpha + \theta^\alpha) S_2, && (20)\\
{}^c_aD^\alpha_t V_2 &= \xi^\alpha S_2 - (1-\rho^\alpha)\beta_1^\alpha V_2 I_2 - \theta^\alpha V_2, && (21)\\
{}^c_aD^\alpha_t I_2 &= \beta_1^\alpha I_1 S_2 + (1-\rho^\alpha)\beta_1^\alpha V_2 I_2 - (\theta^\alpha + \eta^\alpha) I_2. && (22)
\end{aligned}
$$

3.1 Stability of equilibrium points

Let $\alpha \in (0,1]$ and consider the system $(17)-(22)$.
${}^c_aD^\alpha_t S_1(t) = f_1,\;\; {}^c_aD^\alpha_t V_1(t) = f_2,\;\; {}^c_aD^\alpha_t I_1(t) = f_3,\;\; {}^c_aD^\alpha_t S_2(t) = f_4,\;\; {}^c_aD^\alpha_t V_2(t) = f_5,\;\; {}^c_aD^\alpha_t I_2(t) = f_6.$

$$f_i = f_i(S_1, V_1, I_1, S_2, V_2, I_2)\;\; i = 1,2,...6.$$

With the initial values $(S_1(0), V_1(0), I_1(0), S_2(0), V_2(0), I_2(0))$. To evaluate the equilibrium point let ${}^c_aD^\alpha_t S_1 = {}^c_a D^\alpha_t V_1 = {}^c_a D^\alpha_t I_1 = {}^c_a D^\alpha_t S_2 = {}^c_a D^\alpha_t V_2 = {}^c_a D^\alpha_t I_2 = 0$

$$\Rightarrow f_i(S_1^{eq}, V_1^{eq}, I_1^{eq}, S_2^{eq}, V_2^{eq}, I_2^{eq}) = 0,\;\; i = 1,2,3,...,6.$$

Then we can get the equilibrium points $(S_1^{eq}, V_1^{eq}, I_1^{eq}, S_2^{eq}, V_2^{eq}, I_2^{eq})$. To evaluate the asymptotic stability let

$$
\begin{aligned}
S_1(t) &= S_1^{eq} + \varepsilon_1(t), && (23)\\
V_1(t) &= V_1^{eq} + \varepsilon_2(t), && (24)\\
I_1(t) &= I_1^{eq} + \varepsilon_3(t), && (25)\\
S_2(t) &= S_2^{eq} + \varepsilon_4(t), && (26)\\
V_2(t) &= V_2^{eq} + \varepsilon_5(t), && (27)\\
I_2(t) &= I_2^{eq} + \varepsilon_6(t). && (28)
\end{aligned}
$$

So the equilibrium point $(S^{eq}, L_s^{eq}, L_m^{eq}, L_x^{eq}, I_s^{eq}, I_m^{eq}, I_x^{eq}, R^{eq})$ is locally asymptotically stable if all eignvalues of Jacobian evaluated at the equilibrium point satisfies Matignon's conditions given by

$$|arg\lambda_i| > \frac{\alpha\pi}{2},\quad where\;\; i = 1,2,...,6\quad ([29],[30]).$$

3.2 Control reproduction number

Control reproductive number R_c for the model $(17)-(22)$ is defined as follows:

$$R_c^\alpha = \frac{\beta_1^\alpha[\theta^\alpha + (1-\rho^\alpha)\xi^\alpha]}{(\eta^\alpha + \theta^\alpha)(\xi^\alpha + \theta^\alpha)}. \tag{29}$$

For more details see [27]. If there is no vaccination or the vaccination is totally ineffective, i.e., $\xi^\alpha = 0$ or $\rho^\alpha = 0$. Then the control reproductive number become basic reproductive number R_0^α and

$$R_0^\alpha = \frac{\beta_1^\alpha}{\theta^\alpha + \eta^\alpha}. \tag{30}$$

It is clear that, $R_c^\alpha \leq R_0^\alpha$.

3.3 Stability of the disease free equilibrium point

To evaluate the equilibrium points:
Let $D^\alpha_t S_1(t) = {}^c_a D^\alpha_t V_1(t) = {}^c_a D^\alpha_t I_1(t) = {}^c_a D^\alpha_t S_2(t) = {}^c_a D^\alpha_t V_2(t) {}^c_a D^\alpha_t I_2(t) = 0 \Rightarrow f_i(S_1^{eq}, V_1^{eq}, I_1^{eq}, S_2^{eq}, V_2^{eq}, I_2^{eq}) = 0 \qquad i = 1,2,...,6$. Now, if $I_1 = I_2 = 0$, then DFE $(\frac{\mu^\alpha}{\mu^\alpha+\xi^\alpha}, \frac{\xi^\alpha}{\mu^\alpha+\xi^\alpha}, 0, \frac{\mu^\alpha}{\mu^\alpha+\xi^\alpha}, \frac{\xi^\alpha}{\mu^\alpha+\xi^\alpha}, 0)$ i. e., the model $(17)-(22)$ has exactly one equilibrium point $E_0 = (\frac{\mu^\alpha}{\mu^\alpha+\xi^\alpha}, \frac{\xi^\alpha}{\mu^\alpha+\xi^\alpha}, 0, \frac{\mu^\alpha}{\mu^\alpha+\xi^\alpha}, \frac{\xi^\alpha}{\mu^\alpha+\xi^\alpha}, 0)$ in Ω, with no disease in the population. We calculate the Jacobian matrix of the system $(17)-(22)$ at DFE point as following:

$$J(E_0)=\begin{pmatrix} -(\xi^\alpha+\theta^\alpha) & 0 & \frac{-\beta_1^\alpha\theta^\alpha}{\xi^\alpha+\theta^\alpha} & 0 & 0 & \frac{-\beta_1^\alpha\theta^\alpha}{\xi^\alpha+\theta^\alpha} \\ \xi^\alpha & -\theta^\alpha & -\frac{(1-\rho^\alpha)\beta_1^\alpha\xi^\alpha}{\xi^\alpha+\theta^\alpha} & 0 & 0 & -\frac{(1-\rho^\alpha)\beta_1^\alpha\xi^\alpha}{\xi^\alpha+\theta^\alpha} \\ 0 & 0 & \frac{\beta_1^\alpha[\theta^\alpha+(1-\rho^\alpha)\xi^\alpha]}{\xi^\alpha+\theta^\alpha}-\theta^\alpha-\eta^\alpha & 0 & 0 & \frac{\beta_1^\alpha[\theta^\alpha+(1-\rho^\alpha)\xi^\alpha]}{\xi^\alpha+\theta^\alpha} \\ 0 & 0 & 0 & -(\xi^\alpha+\theta^\alpha) & 0 & \frac{-\beta_1^\alpha\theta^\alpha}{\xi^\alpha+\theta^\alpha} \\ 0 & 0 & 0 & \xi^\alpha & -\theta^\alpha & -\frac{(1-\rho^\alpha)\beta_1^\alpha\xi^\alpha}{\xi^\alpha+\theta^\alpha} \\ 0 & 0 & 0 & 0 & 0 & \frac{\beta_1^\alpha[\theta^\alpha+(1-\rho^\alpha)\xi^\alpha]}{\xi^\alpha+\theta^\alpha}-\theta^\alpha-\eta^\alpha \end{pmatrix}$$

The characteristic equation of E_0 is as follows:

$$(\lambda+\theta^\alpha)^2(\lambda+\xi^\alpha+\theta^\alpha)^2)[\lambda-\frac{\beta_1^\alpha\{\theta^\alpha+(1-\rho^\alpha)\xi^\alpha\}}{\xi^\alpha+\theta^\alpha}+\theta^\alpha+\eta^\alpha]^2=0$$

or,

$$(\lambda+\theta^\alpha)^2(\lambda+\xi^\alpha+\theta^\alpha)^2[\lambda-(\theta^\alpha+\eta^\alpha)(R_c^\alpha-1)]^2=0.$$

Thus four roots are negative and remaining two roots are negative if $R_c^\alpha<1$ and positive if $R_c^\alpha>1$. Hence by Hurwitz's criteria DFE is locally asymptotically stable if all eigenvalues of the Jacobian matrix satisfies Matignon's conditions given by [31]:

$$|arg\lambda_i|>\frac{\alpha\pi}{2}, \quad where \;\; i=1,2,....,6.$$

Theorem 31*The proposed model* $(17)-(22)$ *has a unique solution* $(S_1(t),V_1(t),I_1(t),S_2(t),V_2(t),I_2(t))$ *and the solution remains in* R_+^6.

*Proof.*The existence and the uniqueness of the solution of system $(17)-(22)$ in $(0,\infty)$ can be obtained from [6]. We need to show that the domain R_+^6 is positively invariant.
Since ${}_a^cD_t^\alpha S_1(t)|_{s_1=0}=\theta^\alpha\geq 0$, ${}_a^cD_t^\alpha V_1(t)|_{v_1=0}=\xi^\alpha S_1\geq 0$, ${}_a^cD_t^\alpha I_1(t)|_{i_1=0}=\beta_1^\alpha S_1I_2+(1-\rho^\alpha)\beta_1V_1I_2\geq 0$, ${}_a^cD_t^\alpha S_2(t)|_{s_2=0}=\theta^\alpha\geq 0$, ${}_a^cD_t^\alpha V_2(t)|_{v_2=0}=\xi^\alpha S_2\geq 0$, ${}_a^cD_t^\alpha I_2(t)|_{i_2=0}=\beta_1^\alpha S_2I_1\geq 0$ and in view of the generalized mean-value theorem (GMVT), on each hyperplane bounding the non-negative octant, the vector field point into R_+^6.

4 Procedure Solution of the Fractional Influenza in Two-Group Model

Consider the system given in Eqs.$(17)-(22)$. In order to use Jacobi polynomials of m-degree, first we approximate $S_1(t)$, $V_1(t)$, $I_1(t)$,$S_2(t)$,$V_2(t)$ and $I_2(t)$ as follows:

$$S_1(t)=\sum_{i=0}^m a_iP_{L,i}^{(\mu,\beta)}(t), \quad V_1(t)=\sum_{i=0}^m b_iP_{L,i}^{(\mu,\beta)}(t), \tag{31}$$

$$I_1(t)=\sum_{i=0}^m c_iP_{L,i}^{(\mu,\beta)}(t), \quad S_2(t)=\sum_{i=0}^m d_iP_{L,i}^{(\mu,\beta)}(t), \tag{32}$$

$$V_2(t)=\sum_{i=0}^m e_iP_{L,i}^{(\mu,\beta)}(t), \quad I_2(t)=\sum_{i=0}^m f_iP_{L,i}^{(\mu,\beta)}(t), \tag{33}$$

now, we collocate the solution at $(m+1-\lceil\alpha\rceil)$ points $t_p \;\; (p=0,1,..,m+1-\lceil\alpha\rceil)$ as follows

$$\begin{aligned}\sum_{i=\lceil\alpha\rceil}^m\sum_{k=\lceil\alpha\rceil}^i a_i\Theta_{i,k}^\alpha t_p^{k-\alpha} =&\theta^\alpha-\beta_1^\alpha\sum_{i=0}^m a_iP_{L,i}^{(\mu,\beta)}(t_p)\sum_{i=0}^m c_iP_{L,i}^{(\mu,\beta)}(t_p)-\beta_1^\alpha\sum_{i=0}^m a_iP_{L,i}^{(\mu,\beta)}(t_p)\sum_{i=0}^m f_iP_{L,i}^{(\mu,\beta)}(t_p)\\ &-(\xi^\alpha+\theta^\alpha)\sum_{i=0}^m a_iP_{L,i}^{(\mu,\beta)}(t_P),\end{aligned} \tag{34}$$

$$\begin{aligned}\sum_{i=\lceil\alpha\rceil}^m\sum_{k=\lceil\alpha\rceil}^i b_i\Theta_{i,k}^\alpha t_p^{k-\alpha} =&\xi^\alpha\sum_{i=0}^m a_iP_{L,i}^{(\mu,\beta)}(t_p)-(1-\rho^\alpha)\beta_1^\alpha\sum_{i=0}^m b_iP_{L,i}^{(\mu,\beta)}(t_p)\sum_{i=0}^m c_iP_{L,i}^{(\mu,\beta)}(t_p)\\ &-(1-\rho^\alpha)\beta_1^\alpha\sum_{i=0}^m b_iP_{L,i}^{(\mu,\beta)}(t_p)\sum_{i=0}^m f_iP_{L,i}^{(\mu,\beta)}(t_p)-\theta^\alpha\sum_{i=0}^m b_iP_{L,i}^{(\mu,\beta)}(t_p),\end{aligned} \tag{35}$$

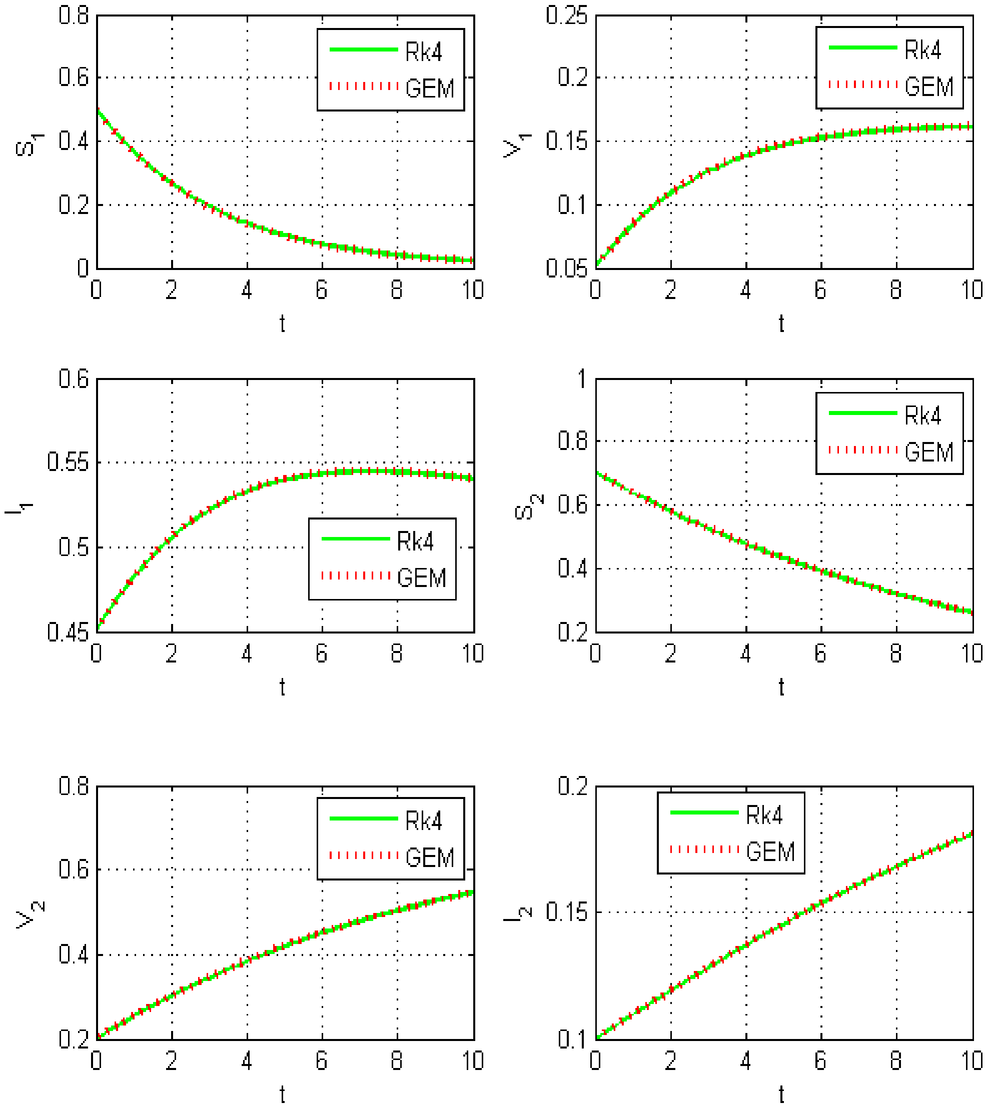

Fig. 1: The solutions graph using GEM and RK4 with $\alpha = 1$, $\rho = 0.94$, $R_c^\alpha = 1.0082$.

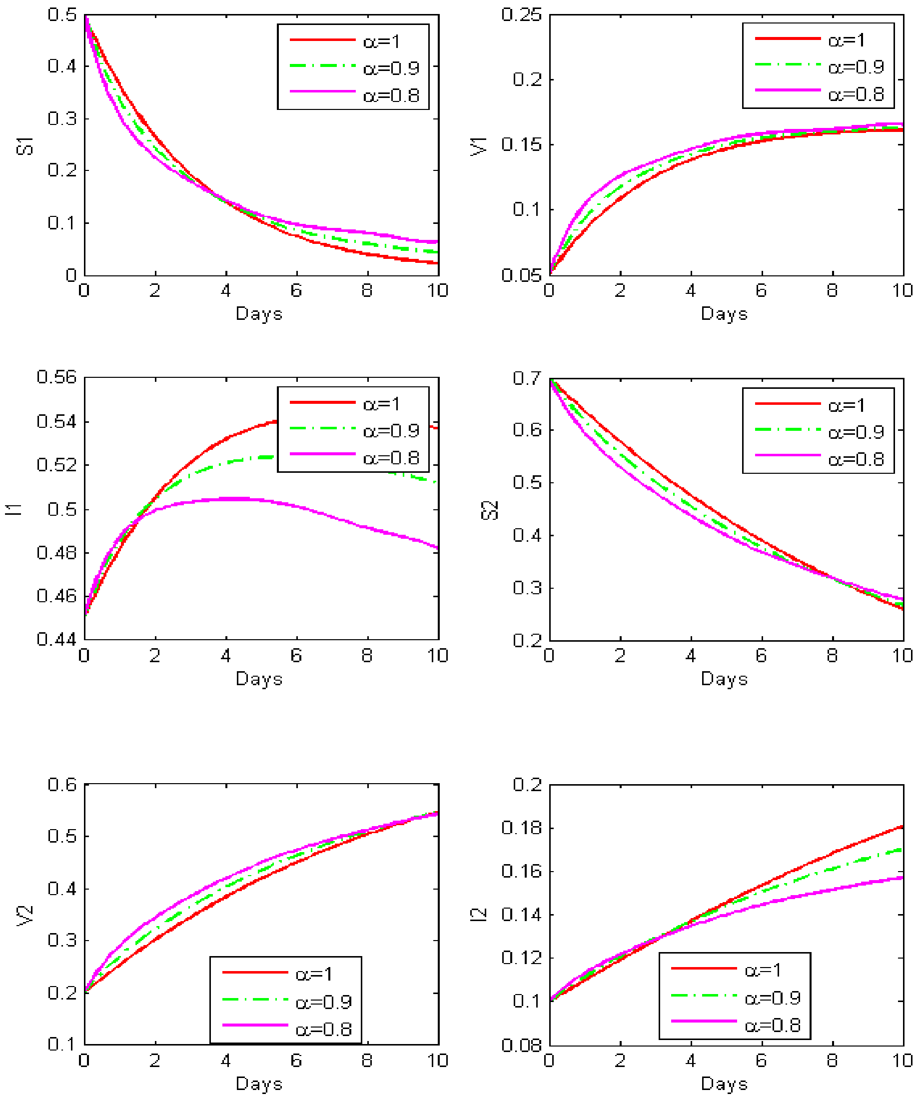

Fig. 2: The solutions graph using the shifted Jacobi spectral method with different α and, $\rho^{\alpha} = (0.94)^{\alpha}$, $\mu = 0$, $\beta = 0$.

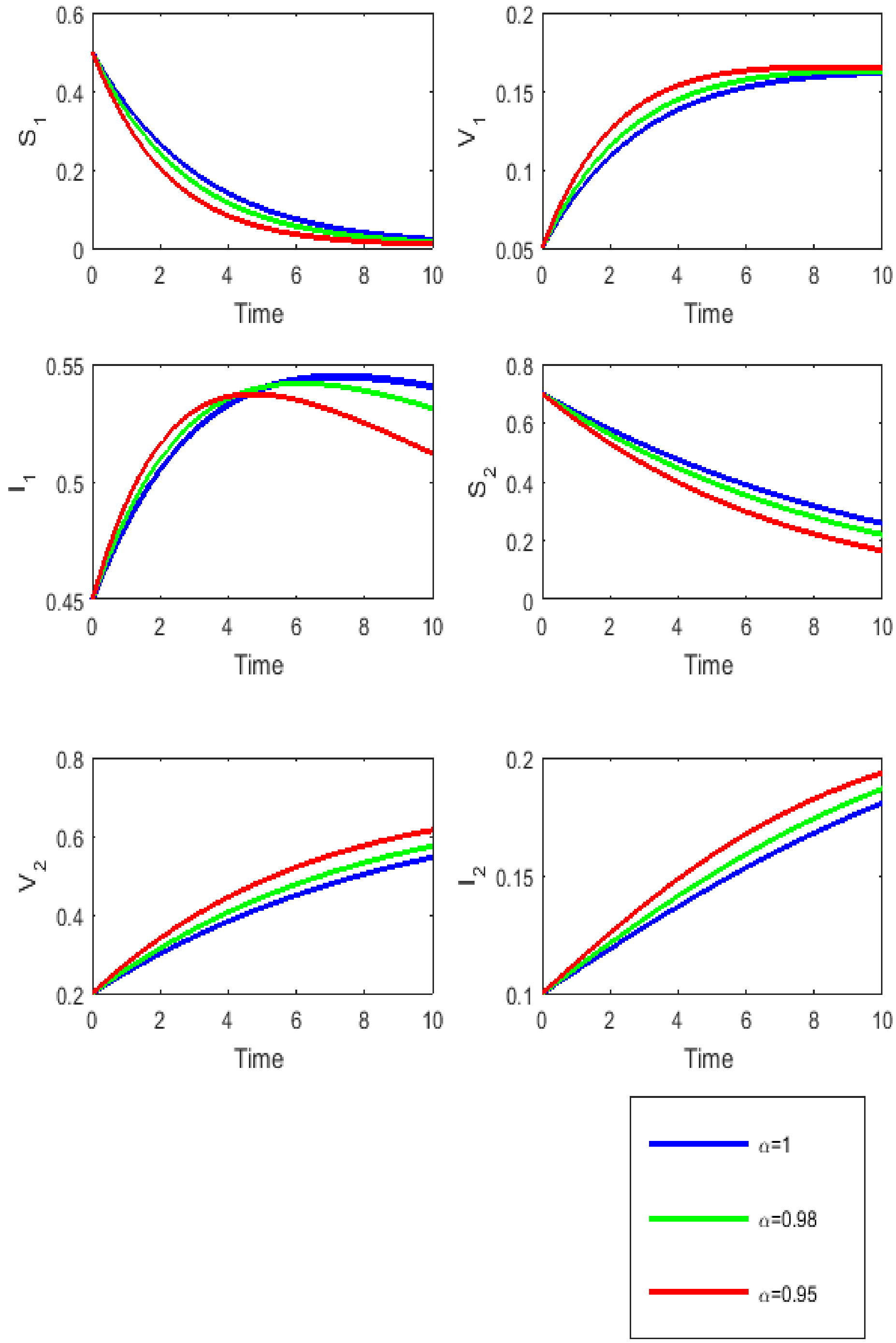

Fig. 3: The solutions graph with different α by using GEM, $\rho^{\alpha} = (0.94)^{\alpha}$.

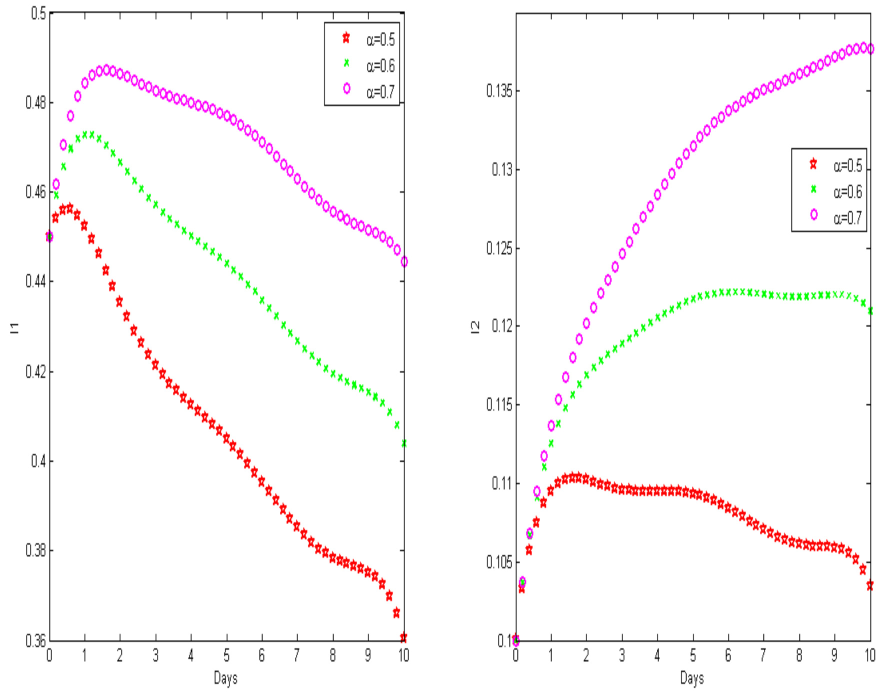

Fig. 4: The solutions graph with different α by using the shifted Jacobi spectral method, $\rho^{\alpha}=(0.99)^{\alpha}$, $\mu=0.5$, $\beta=0.5$.

$$\begin{aligned}\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} c_i\Theta_{i,k}^{\alpha}t_p^{k-\alpha} =&\beta_1^{\alpha}\sum_{i=0}^{m}a_iP_{L,i}^{(\mu,\beta)}(t_p)(\sum_{i=0}^{m}c_iP_{L,i}^{(\mu,\beta)}(t_p)+\sum_{i=0}^{m}f_iP_{L,i}^{(\mu,\beta)}(t_p)\\ &+(1-\rho^{\alpha})\beta_1^{\alpha}\sum_{i=0}^{m}b_iP_{L,i}^{(\mu,\beta)}(t_p)\sum_{i=0}^{m}c_iP_{L,i}^{(\mu,\beta)}(t_p)+\sum_{i=0}^{m}f_iP_{L,i}^{(\mu,\beta)}(t_p)\\ &-(\theta^{\alpha}+\xi^{\alpha})\sum_{i=0}^{m}c_iP_{L,i}^{(\mu,\beta)}(t_p),\end{aligned} \tag{36}$$

$$\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} d_i\Theta_{i,k}^{\alpha}t_p^{k-\alpha} =\theta^{\alpha}-\beta_1^{\alpha}\sum_{i=0}^{m}d_iP_{L,i}^{(\mu,\beta)}(t_p)\sum_{i=0}^{m}f_iP_{L,i}^{(\mu,\beta)}(t_p)-(\xi^{\alpha}+\theta^{\alpha})\sum_{i=0}^{m}d_iP^{(\mu,\beta)}(t_p), \tag{37}$$

$$\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} e_i\Theta_{i,k}^{\alpha}t_p^{k-\alpha} =\xi^{\alpha}\sum_{i=0}^{m}d_iP^{(\mu,\beta)}(t_p)-(1-\rho^{\alpha})\beta_1^{\alpha}\sum_{i=0}^{m}e_iP_{L,i}^{(\mu,\beta)}(t_p)\sum_{i=0}^{m}f_iP_{L,i}^{(\mu,\beta)}(t_p)-\theta^{\alpha}\sum_{i=0}^{m}e_iP_{L,i}^{(\mu,\beta)}(t_p), \tag{38}$$

$$\begin{aligned}\sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} f_i\Theta_{i,k}^{\alpha}t_p^{k-\alpha} =&\beta_1^{\alpha}\sum_{i=0}^{m}d_iP_{L,i}^{(\mu,\beta)}(t_p)\sum_{i=0}^{m}f_iP_{L,i}^{(\mu,\beta)}(t_p)+(1-\rho^{\alpha})\beta_1^{\alpha}\sum_{i=0}^{m}e_iP_{L,i}^{(\mu,\beta)}(t_p)\sum_{i=0}^{m}f_iP_{L,i}^{(\mu,\beta)}(t_p)\\ &-(\theta^{\alpha}+\eta^{\alpha})\sum_{i=0}^{m}f_iP_{L,i}^{(\mu,\beta)}(t_p).\end{aligned} \tag{39}$$

For suitable collocation points we use roots of Jacobi spectral collocation method $P_{L,i}^{(\mu,\beta)}(t)$. By substituting the initial conditions in Eqs. (31) − (33), we can obtain six equations as follows:

$$\sum_{i=0}^{m}(-1)^i\frac{\Gamma(\beta+i+1)}{\Gamma(\beta+1)i!}a_i=S_{10},\quad \sum_{i=0}^{m}(-1)^i\frac{\Gamma(\beta+i+1)}{\Gamma(\beta+1)i!}b_i=V_{10},\quad \sum_{i=0}^{m}(-1)^i\frac{\Gamma(\beta+i+1)}{\Gamma(\beta+1)i!}c_i=I_{10}, \tag{40}$$

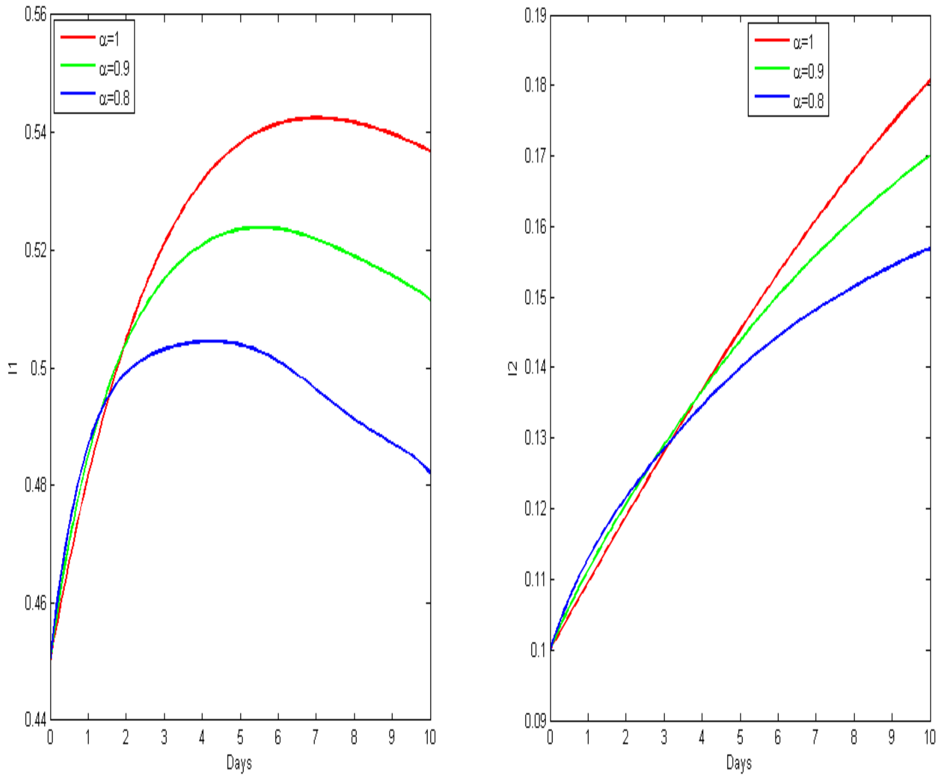

Fig. 5: The solutions graph using the shifted Chebyshev polynomials with different α and $\rho^{\alpha} = (0.94)^{\alpha}$.

$$\sum_{i=0}^{m} (-1)^i \frac{\Gamma(\beta+i+1)}{\Gamma(\beta+1)i!} d_i = S_{20}, \quad \sum_{i=0}^{m} (-1)^i \frac{\Gamma(\beta+i+1)}{\Gamma(\beta+1)i!} e_i = V_{20}, \quad \sum_{i=0}^{m} (-1)^i \frac{\Gamma(\beta+i+1)}{\Gamma(\beta+1)i!} f_i = I_{20}. \tag{41}$$

Equations $(34) - (39)$, together with the equations $(40) - (41)$, give $6(m+1)$ of algebraic equations. This system is solved by the Newton's iteration method for the unknowns a_i, b_i, c_i, d_i, e_i and f_i, $\quad i = 0, 1, ..., m$.

5 Numerical Experiment

In the following, the shifted Jacobi polynomials are used to obtain approximate solutions for the system $(17) - (22)$ with the initial condition $S_1(0) = 0.5, V_1(0) = 0.05, I_1(0) = 0.45, S_2(0) = 0.7, V_2(0) = 0.2, I_2(0) = 0.1$ and the parameters given in Table 2. Let $m = 6$ and $T = 10$, Fig. 1 shows the behavior of the approximate solutions using GEM and RK4 with $\alpha = 1$, $\rho^{\alpha} = 0.94$ and $R_c^{\alpha} = 1.0082$. Using the same data given in Fig. 1, consider $\mu = 0$, $\beta = 0$ in Eq. (7), Fig. 2 shows the behavior of the approximate solutions with different values of α using the shifted Jacobi spectral method. We have noted that, when $\alpha = 1$, the approximate solutions using the shifted Jacobi spectral method are in excellent agreement with the solution by using RK4 and GEM. Fig. 3 shows the behavior of the approximate solutions with different α by using GEM. Fig. 4 shows the behavior of the approximate solutions with different α by using the shifted Jacobi spectral method when $\mu = 0.5$, $\beta = 0.5$. The obtained results using the shifted Jacobi spectral method with different values of μ, β and $\alpha = 1$ are listed in Table 3. We noted that, the results almost equal in all cases of μ, β. The results using the shifted Jacobi spectral method when $\mu = 0.5$, $\beta = 0.5$ and different values of α are listed in Table 4. Moreover, we can generate the shifted Chebyshev polynomials of first kind from the Jacobi polynomials according to the following relationship $T_{L,i}(x) = \frac{i!\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+i)} P_{L,i}^{(-\frac{1}{2},-\frac{1}{2})}(x)$. Fig. 5 shows the behavior of the approximate solutions using shifted Chebyshev spectral method with different α and $\rho^{\alpha} = (0.94)^{\alpha}$. We have noted from Fig. 5 and Table 5 that the obtained results are the same when $\mu = 0$, $\beta = 0$. Fig. 6 shows the behavior of the approximate solutions with different ρ^{α} by using the shifted Jacobi spectral method $\mu = 0$ and $\beta = 0$. We have noted that whenever the

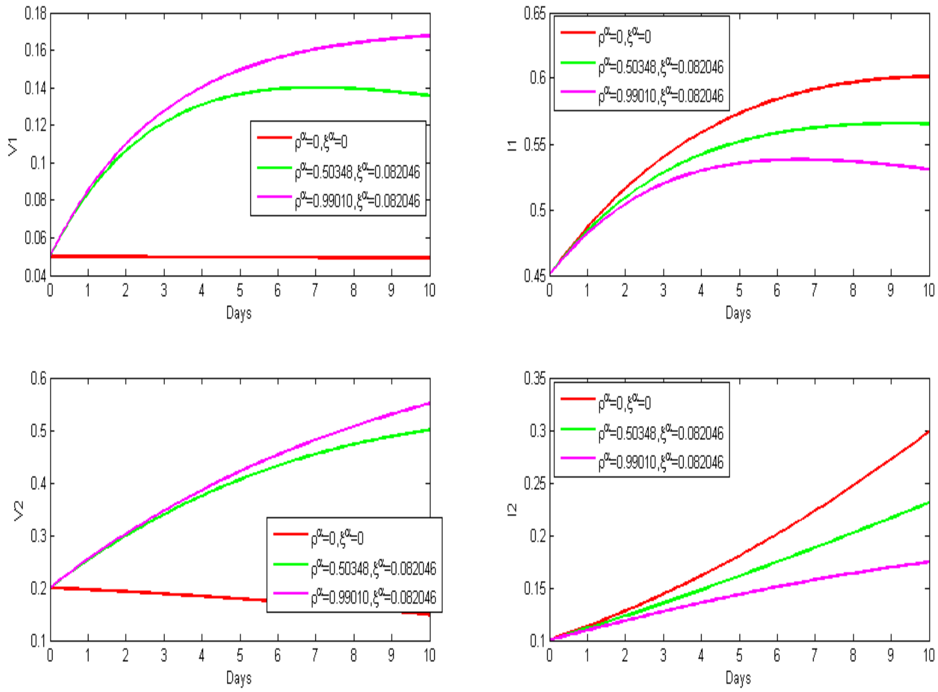

Fig. 6: The solutions graph with different ρ^{α} by using the shifted Jacobi spectral method, $\mu = 0$, $\beta = 0$ and $\alpha = 0.99$.

values of ρ^{α} increase the values of I_1, I_2 decrease. Fig. 7 illustrates the relation between the variables when $\alpha = 0.90$, $\rho^{\alpha} = (0.50)^{0.90}$ by using the shifted Jacobi spectral method and $\mu = 1$, $\beta = 1$.

Table 3: Numerical solutions by the shifted Jacobi spectral method when, $\alpha = 1$, $T = 10$.

ρ^{α}	R_c^{α}	μ	β	V_1	I_1	V_2	I_2
0.99	0.3689	0.5	0.5	1.6763×10^{-1}	5.3289×10^{-1}	5.5071×10^{-1}	1.7537×10^{-1}
		-0.5	-0.5	1.6763×10^{-1}	5.3289×10^{-1}	5.5071×10^{-1}	1.7537×10^{-1}
		1	1	1.6763×10^{-1}	5.3289×10^{-1}	5.5071×10^{-1}	1.7537×10^{-1}
		-0.5	0.5	1.6763×10^{-1}	5.3289×10^{-1}	5.5071×10^{-1}	1.7537×10^{-1}
		0	0	1.6764×10^{-1}	5.3288×10^{-1}	5.5071×10^{-1}	1.7537×10^{-1}
0.90	1.5204	0.5	0.5	1.5681×10^{-1}	5.3984×10^{-1}	5.4214×10^{-1}	1.8499×10^{-1}
		-0.5	0.5	1.5681×10^{-1}	5.3984×10^{-1}	5.4214×10^{-1}	1.8499×10^{-1}
		0	0	1.5681×10^{-1}	5.3983×10^{-1}	5.4214×10^{-1}	1.8499×10^{-1}

Table 4: Numerical solutions by the shifted Jacobi spectral method with different values of α, R_c^{α} and $T = 10$, $\mu = 0.5$, $\beta = 0.5$.

α	R_c^{α}	V_1	I_1	V_2	I_2
0.98	1.4267	1.5707×10^{-1}	5.3509×10^{-1}	5.4219×10^{-1}	1.8300×10^{-1}
0.90	1.1153	1.5852×10^{-1}	5.1442×10^{-1}	5.4144×10^{-1}	1.7400×10^{-1}
0.80	0.8410	1.6112×10^{-1}	4.8424×10^{-1}	5.3839×10^{-1}	1.6033×10^{-1}
0.50	0.4622	1.6720×10^{-1}	3.6319×10^{-1}	5.0630×10^{-1}	1.0677×10^{-1}
0.30	0.3692	1.5385×10^{-1}	2.9292×10^{-1}	4.2715×10^{-1}	7.4409×10^{-2}

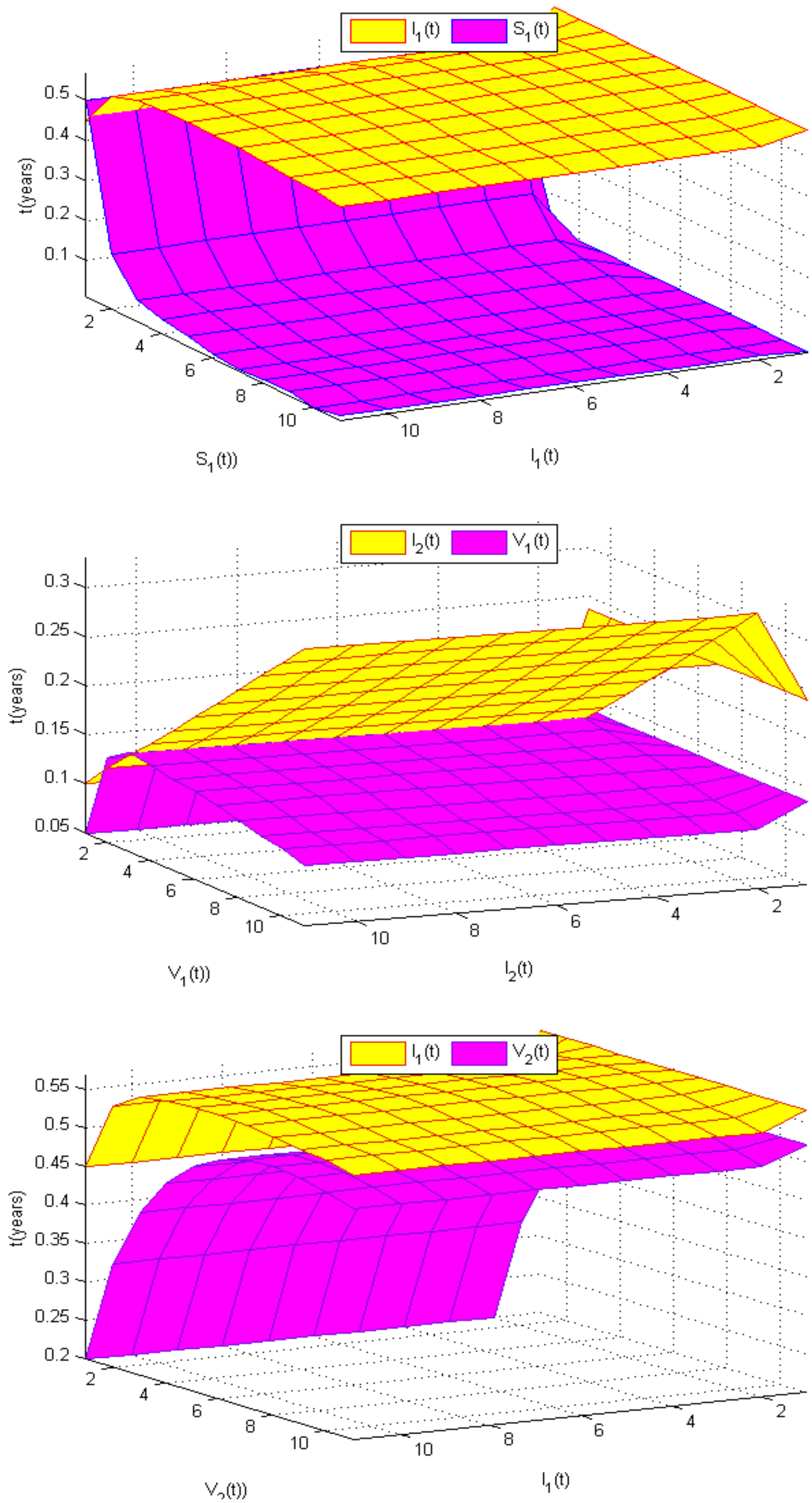

Fig. 7: Relation between the state variables when $\alpha = 0.90$, $\rho^{\alpha} = (0.50348)^{\alpha}$ by using the shifted Jacobi spectral method and $\mu = 1$, $\beta = 1$.

Table 5: Numerical solutions by using the shifted Chebyshev polynomials and different values of α, R_c^α and $T = 10$.

α	R_c^α	V_1	I_1	V_2	I_2
1	0.3689	1.6764×10^{-1}	5.3288×10^{-1}	5.5071×10^{-1}	1.7537×10^{-1}
1	1.5204	1.5681×10^{-1}	5.3983×10^{-1}	5.4214×10^{-1}	1.8499×10^{-1}
0.98	1.4267	1.5709×10^{-1}	5.3507×10^{-1}	5.4225×10^{-1}	1.8302×10^{-1}
0.90	1.1153	1.5865×10^{-1}	5.1443×10^{-1}	5.4172×10^{-1}	1.7406×10^{-1}
0.80	0.8410	1.6163×10^{-1}	4.8446×10^{-1}	5.3907×10^{-1}	1.6042×10^{-1}

6 Conclusions

In this paper, the shifted Jacobi polynomials and their properties together with the collocation method are used to solve two-group Influenza model of the fractional order derivatives. The fractional derivative is considered in the Caputo sense. Two polynomials from Jacobi polynomials; the shifted Legendre polynomials and shifted Chebyshev polynomials as special cases from the shifted Jacobi polynomial are introduced to solve the proposed model. The obtained results by proposed method are compared with the results obtianed by RK4 and GEM methods. It's found that the results obtained by the method suggested here are in excellent agreement with the results obtianed by RK4 and GEM, in integer-order case. Some figures are given to demonstrate how the fractional model is a generalization of the integer-order model. All computed results are obtained using Matlab programming.

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