Applied Mathematics & Information Sciences **2**(2) (2008), 135–141
– An International Journal

Variation of Parameters Method for Solving Fifth-Order Boundary Value Problems

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Received January 11, 2008; Revised February 15, 2008; Accepted March 25, 2008

In this paper, we consider the variation of parameters method for solving fifth-order boundary value problems. The proposed technique is quite efficient and is practically well suited for solving these problems. The suggested iterative scheme finds the solution without any perturbation, discritization, linearization or restrictive assumptions and is free from the identification of Lagrange multipliers. Several examples are given to verify the reliability and efficiency of the proposed method and its comparison with other methods.

Keywords: Variation of parameters method, boundary value problems, nonlinear equations, error estimates.

2000 Mathematics Subject Classification: 65N99.

1 Introduction

In this paper, we consider the general fifth-order boundary value problem

$$y^{(v)}(x) = f(x, y) \tag{1.1}$$

with boundary conditions

$$y(a) = A_1, \quad y'(a) = A_2, \quad y''(x) = A_3, \quad y(b) = B_1, \quad y'(b) = B_2,$$

where $y(x)$ and $f(x, y)$ are real and as many times differentiable as required for $x \in [a, b]$ and $A_i,\ i = 1, 2, 3$ and $B_i,\ i = 1, 2$ are real finite constants. This type of boundary value problems arises in the mathematical modeling of the viscoelastic flows and other branches of mathematical, physical and engineering sciences, see [1-3, 6, 12, 17] and the references therein. Several numerical methods including spectral Galerkin and collocation, decomposition, sixth order B-spline have been developed for solving fifth-order boundary value problems, see [1-3, 6, 12, 17] and the references therein. Noor and Mohyud-Din [4-14] used homotopy perturbation, variational iteration and variational iteration decomposition methods for solving fifth-order and other higher order boundary value problems. Inspired and motivated by the ongoing research in this area, we use the variation of parameters method to the fifth-order boundary value problems. It is worth mentioning that the developed method is free from perturbation, discritization, restrictive assumptions, identification of Lagrange multipliers and calculation of Adomian's polynomials. The proposed method provides the solution in a rapid convergent series with easily computable components, see [15, 16] and the references therein. The results are very encouraging and reveal the complete reliability of the suggested algorithm. Several examples are given to verify the efficiency and accuracy of the proposed scheme.

2 Variation of Parameters Method

To illustrate the basic concept of the technique, we consider the general fifth-order boundary value problem

$$y^{(v)}(x) = f(x, y) \tag{2.1}$$

with boundary conditions

$$y(a) = A_1, \quad y'(a) = A_2, \quad y''(a) = A_3, \quad y(b) = A_4, \quad y'(b) = A_5.$$

The variation of parameters method gives the general solution of (2.1) as

$$y = \sum_{i=1}^{n} A_i y^{(i)}(x) + \sum_{i=1}^{n} y_i(x) \int_0^x f(s, y_i) g(s) ds.$$

Using the boundary conditions, we have the following iterative method for finding the approximate solution y_{n+1} as

$$y_{n+1}(x) = h(x) + \int_a^b f(s, y_n) g(x, s) ds + \int_0^x f(s, y_n) g(x, s) ds,$$

where $h(x)$ is determined using the boundary conditions.

3 Numerical Application

In this section, we apply the method of variation of parameters developed in section 2 for solving the fifth-order boundary value problems. Numerical results are very encouraging. For the sake of comparison, we take the same examples as used in [3, 6, 12, 17].

Example 3.1. Consider the following nonlinear fifth-order boundary value problem of the form

$$y^{(v)}(x) = e^{-x}y^2(x), \quad 0 < x < 1 \tag{3.1}$$

with boundary conditions

$$y(0) = y'(0) = y''(0) = 1, \quad y(1) = y'(1) = e. \tag{3.2}$$

The exact solution for this problem is $y(x) = e^x$. The method of variation of parameters gives the solution of nonlinear boundary value problem (3.1) and (3.2) as

$$\begin{aligned} y_{n+1}(x) = {} & A_1 + A_2x + A_3\frac{x^2}{2!} + A_4\frac{x^3}{3!} + A_5\frac{x^4}{4!} \\ & + \int_0^x e^{-x}y_n^2(s)\left(\frac{1}{24}s^4 - \frac{x}{6}s^3 + \frac{x^2}{4}s^2 - \frac{x^3}{6}s + \frac{x^4}{24}\right)ds. \end{aligned}$$

Using the boundary conditions (3.2), we have

$$\begin{aligned} A_1 &= 1, \\ A_2 &= 1, \\ A_3 &= \frac{1}{2}, \\ A_4 &= -8 + 3e - \frac{1}{6}\int_0^1 e^{-s}y_n^2(s)\left(s^4 - 3s^3 + 3s^2 - s\right)ds, \\ A_5 &= \frac{11}{2} - 2e + \frac{1}{24}\int_0^1 e^{-s}y_n^2(s)\left(3s^4 - 8s^3 + 6s^2 - 1\right)ds. \end{aligned}$$

Consequently, the following approximants are obtained

$$\begin{aligned} y_0(x) = {} & 1 + x + \frac{1}{2}x^2 + 0.154845484x^3 + 0.63436344x^4 \\ y_1(x) = {} & 1 + x + 0.5000000000x^2 + 0.1666632966x^3 + 0.0416712806x^4 \\ & + 0.008333333304x^5 + 0.0013888888880x^6 + 0.0001984126956x^7 \\ & + 0.00000212337631x^8 + O(x^9) \\ y_2(x) = {} & 1 + x + 0.5000000000x^2 + 0.1666632641x^3 + 0.04166666917x^4 \\ & + 0.008333333304x^5 + 0.0013888888880x^6 + 0.0001984126956x^7 \\ & + 2.4800584888xio^{-5}x^8 + O(x^9). \\ & \vdots \end{aligned}$$

Table 5.1 exhibits the errors obtained by using the B-spline method, the Adomian's decomposition method (ADM), homotopy perturbation method (HPM), variational iteration method (VIM) and variation of parameters (VOP). It is obvious that evaluation of more components of $y(x)$ will reasonably improve the accuracy of solution.

Table 3.1: Error Estimates; Error = Exact solution - Series solution

x	Exact	VIM	HPM	VOP	ADM	B-spline
0.0	1.000000000000000	0.00000	0.00000	0.00000	0.00000	0.00000
0.1	1.105170918074275	1.0E-09	1.0E-09	1.3E-12	1.0E-09	-7.0E-04
0.2	1.221402758149936	2.0E-09	2.0E-09	1.0E-11	2.0E-09	-7.2E-04
0.3	1.349858807543966	1.0E-08	1.0E-08	3.2E-11	1.0E-08	4.1E-04
0.4	1.491824697571211	2.0E-08	2.0E-08	7.0E-11	2.0E-08	4.6E-04
0.5	1.648721270574672	3.1E-08	3.1E-08	1.2E-10	3.1E-08	4.7E-04
0.6	1.822118800193176	3.7E-08	3.7E-08	1.9E-10	3.7E-08	4.8E-04
0.7	2.013752707187650	4.1E-08	4.1E-08	2.8E-10	4.1E-08	3.9E-04
0.8	2.225540928115266	3.1E-08	3.1E-08	3.7E-10	3.1E-08	3.1E-04
0.9	2.459603110682982	1.4E-08	1.4E-08	4.7E-10	1.4E-08	1.6E-04
1.0	2.718281827894025	0.00000	0.00000	5.6E-10	0.00000	0.00000

Example 3.2. Consider the following nonlinear fifth-order boundary value problem

$$y^{(v)}(x) = y(x) - 15e^x - 10xe^x, \quad 0 < x < 1 \tag{3.3}$$

with boundary conditions

$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y(1) = 0, \quad y'(1) = -e. \tag{3.4}$$

The exact solution for this problem is $y(x) = x(1-x)e^x$. The method of variation of parameters gives the solution of nonlinear boundary value problem (3.3) and (3.4) as

$$\begin{aligned} y_{n+1}(x) &= A_1 + A_2 x + A_3 \frac{x^2}{2!} + A_4 \frac{x^3}{3!} + A_5 \frac{x^4}{4!} \\ &\quad + \int_0^x (y_n - 15e^s - 10se^s) \left(\frac{1}{24}s^4 - \frac{x}{6}s^3 + \frac{x^2}{4}s^2 - \frac{x^3}{6}s + \frac{x^4}{24} \right) ds. \end{aligned}$$

Using the boundary conditions, we have

$$\begin{aligned} A_1 &= 0, \\ A_2 &= 1, \\ A_3 &= 0, \\ A_4 &= e - 3 + \int_0^1 (y(s) - 15e^s - 10se^s) \left(-\frac{1}{6}s^4 + \frac{1}{2}s^3 - \frac{1}{2}s^2 + \frac{1}{6}s \right) ds, \\ A_5 &= 2 - e + \int_0^1 (y(s) - 15e^s - 10se^s) \left(\frac{1}{8}s^4 - \frac{1}{3}s^3 + \frac{1}{4}s^2 - \frac{1}{24} \right) ds. \end{aligned}$$

Consequently, we have the following approximants

$$y_0(x) = x - 0.281718172x^3 - 0.718281828x^4$$

$$\begin{aligned} y_1(x) &= x - 0.4999676963x^3 - 0.3333774763x^4 - 0.12499999999x^5 \\ &- 0.03333333350x^6 - 0.006944444231x^7 - 0.001157993683x^8 \\ &- 0.001990706960x^9 - 0.1791225780x10^{-5}x^{10} + O(x^{11}) \\ y_2(x) &= x - 0.4999999948x^3 - 0.33333333405x^4 - 0.1249999999x^5 \\ &- 0.03333333350x^6 - 0.006944444231x^7 - 0.001190471495x^8 \\ &- 0.0001736140287x^9 - 2.204584519x10^{-5}x^{10} + O(x^{11}). \\ &\vdots \end{aligned}$$

Table 3.2: Error Estimates; Error = Exact solution - Series solution

x	Exact	VIM	HPM	VOP	ADM	B-spline
0.0	0.000000000000000	0.00000	0.00000	0.00000	0.00000	0.00000
0.1	.099465382626452	-3.0E-11	-3.0E-11	3.5E-13	-3.0E-11	-8.0E-03
0.2	.195424441303144	-2.0E-10	-2.0E-10	2.4E-12	-2.0E-10	-1.2E-03
0.3	.283470349583825	-4.0E-10	-4.0E-10	7.1E-12	-4.0E-10	-5.0E-03
0.4	.358037927419948	-8.0E-10	-8.0E-10	1.3E-11	-8.0E-10	3.0E-03
0.5	.412180317653458	-1.2E-09	-1.2E-09	2.1E-11	-1.2E-09	8.0E-03
0.6	.437308512065821	-2.0E-09	-2.0E-09	2.7E-11	-2.0E-09	6.0E-03
0.7	.422888068537963	-2.2E-09	-2.2E-09	3.0E-11	-2.2E-09	-0.0000
0.8	.356086548529305	-1.9E-09	-1.9E-09	2.9E-11	-1.9E-09	9.0E-03
0.9	.221364279978394	-1.4E-09	-1.4E-09	2.5E-11	-1.4E-09	-9.0E-03
1.0	-.000000000024272	0.00000	0.00000	2.4E-11	0.00000	0.00000

Table 5.2 exhibits the errors obtained by using the B-spline method, the Adomian's decomposition method (ADM), homotopy perturbation method (HPM), variational iteration method (VIM) and variation of parameters (VOP). It is obvious that evaluation of more components of $y(x)$ will reasonably improve the accuracy of solution.

4 Conclusion

In this paper, we have used the method of variation of parameters to find the solution of boundary value problems for fifth-order. The method is applied in a direct way without using linearization, transformation, discritization, perturbation or restrictive assumptions. It may be concluded that the proposed technique is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. It is worth mentioning that the method is capable of reducing the volume of the computational work as compare to the classical methods while still maintaining the high accuracy of the numerical result, the size reduction amounts to the improvement of performance of approach. The fact that the developed algorithm solves nonlinear problems without using the Adomian's polynomials and the identification of Lagrange multipliers are the clear advantages of this technique over the variational iteration method and the decomposition method.

Acknowledgment

The authors are highly grateful to Dr. S. M. Junaid Zaidi, Rector CIIT, Islamabad, Pakistan, for providing excellent research environment and facilities.

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