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Semilinear hyperbolic boundary value problem for linear elasticity equations

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Abstract: Consider a semilinear hyperbolic boundary value problem associated to the linear elastic equations. Then, existence of a weak solution is established through compactness method. The uniqueness and the regularity of the solution have been gotten by eliminating some hypotheses that have been imposed by other authors for different particular problems.

Keywords: Compactness, Existence, *Gronwall's inequality*, linear elasticity, Uniqueness of solution, Regularity, Semilinear hyperbolic equation, Variational problem.

1. Introduction

In [8], *Lions* considered a semilinear boundary value problem associated to the *Laplace* operator with *Neumann* boundary conditions:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \Delta u + |u|^{\nu} u = f \;\; in\; \Omega \times (0,T), \\ u = 0 \;\; on\; \Gamma \times (0,T), \\ u(x,0) = u_0(x), u'(x,0) = u_1(x), \;\; x \in \Omega. \end{cases} \tag{1}$$

Using the compactness method and *Faedo Galerkin* techniques, the existence of a weak solution has been proved. Assuming that the condition $\nu \leq \dfrac{2}{n-2}$ holds, then it follows the uniqueness and the regularity of the solution.

In this work, we consider a semilinear hyperbolic boundary value problem governed by partial differential equations that describe the evolution of linear elastic materials with *Dirichlet* and *Neumann* boundary conditions as follows :

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - div\sigma(u) + |u|^{\nu} u = f, \; in\; \Omega \times (0,T), \\ \sigma(u) = F(\varepsilon(u)), \; in\; \Omega \times (0,T), \\ u = g \; on\; \Gamma_1 \times (0,T), \; \sigma(u)\eta = 0 \; on\; \Gamma_2 \times (0,T), \\ u(x,0) = u_0(x), u'(x,0) = u_1(x), \;\; x \in \Omega, \end{cases} \tag{2}$$

where F is a linear function. Assume certain hypotheses on the data functions. Then, by using *Faedo Galerkin* techniques and compactness method, we will prove the existence of the solution. Our main goal is, without taking into account the condition on ν, to prove the uniqueness and the regularity of the solution.

2. Problem statement

Let Ω be an open and bounded domain in $\mathbb{R}^n$, recall that the boundary Γ of Ω is assumed to be regular and is composed of two relatively closed parts : Γ_1, Γ_2, with mutually disjoint relatively open interiors. We assume that $meas\;(\Gamma_1) > 0$. We pose $\Sigma_i = \Gamma_i \times (0,T), i = 1,2$, where T is a finite real. To simplify the writing one will put $u' = \frac{\partial u}{\partial t}, u'' = \frac{\partial^2 u}{\partial t^2}$. $\sigma = (\sigma_{ij}), i,j = 1,2,...,n$ stands for the stress tensor field. To simplify the notations, we do not indicate explicitly the dependence of the functions u and σ with respect to $x \in \Omega$ and $t \in (0,T)$. Let η be the unit outward normal vector on Γ. Here and throughout this work, the summation convention over repeated indices is used. The classical formulation of the problem is as follows. Find a displacement field $u : \Omega \times (0,T) \to \mathbb{R}^n$, a stress field $\sigma : \Omega \times (0,T) \to S_n$, such that

$$u'' - div\sigma(u) + |u|^{\nu} u = f \; inQ, \nu \in \mathbb{N}, \tag{3}$$

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$$\sigma(u) = F(x, \varepsilon(u)) \ inQ, \tag{4}$$

$$\begin{cases} u = g \ on \ \Sigma_1, \\ \sigma(u)\eta = 0 \ on \ \Sigma_2, \end{cases} \tag{5}$$

$$\begin{cases} u(x,0) = u_0(x) in \Omega, \\ u'(x,0) = u_1(x) \ in \ \Omega. \end{cases} \tag{6}$$

Where $\mathcal{S}_n$ will denote the space of second-order symmetric tensors on $\mathbb{R}^n$. u , f and $\sigma(u)$ represent the displacement field, the density of volume forces and the tensor of constraints, respectively. div denotes the divergence operator of the tensor valued functions and $\sigma = (\sigma_{ij})$, stands for the stress tensor field. The latter is obtained from the displacement field by the constitutive law of linear elasticity defined by (4). F is a linear elastic constitutive law, and $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla^T u)$ is the linearized strain tensor. The equation (3), without the non linear term $|u|^\nu u$, describs the evolution of linear elastic materials, while (5) and (6) are the mixed boundary conditions on $\Sigma_i, i = 1,2$ and initial conditions, respectively.
We now define the space :

$$\mathcal{H} = L^2(\Omega)_s^{n\times n} = \left\{\sigma = (\sigma_{ij}) \in \mathcal{S}_n : \ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\}, \tag{7}$$

which is a *Hilbert* space endowed with the inner product

$$\langle \sigma, \tau \rangle = \int_\Omega \sigma_{ij}\tau_{ij} dx, \tag{8}$$

and the associated norm is denoted $\|.\|_{\mathcal{H}}$. When no ambiguousness is to fear, we will put :

$$\|v\|_{L^2(\Omega)} = |v| = \left(\int_\Omega v^2 dx\right)^{\frac{1}{2}}, \tag{9}$$

and we will use the notation $\|v\|_{L^2(\Omega)}$ in possible ambiguousness case.
In the study of mechanical problem involving elastic materials, we assume that the operator $F : \Omega \times S_n \to S_n$ satisfies the following conditions:

$$\begin{cases} (a) \ \exists \, m > 0; \ (F(x,\varepsilon), \varepsilon) \geq m \, \|\varepsilon\|^2, \\ \forall \varepsilon \in S_n, a.e. \ x \in \Omega. \\ (b) \ \ (F(x,\varepsilon), \tau) = (F(x,\tau), \varepsilon), \\ \forall \, \varepsilon, \tau \in S_n, a.e. \ x \in \Omega. \\ (c) \ F \ or \ any \ \varepsilon \in S_n, x \to F(x,\varepsilon) \\ is \ measurable \ on \ \Omega. \end{cases} \tag{10}$$

And we assume that the given data f, u_0 and u_1 and g verify

$$f \in L^2(Q), \tag{11}$$

$$u_0 \in V \cap L^p(\Omega), \ p = \nu + 2, \tag{12}$$

$$u_1 \in L^2(\Omega), \tag{13}$$

$$g \in H^{\frac{1}{2}}(\Sigma_1). \tag{14}$$

Referring to [4], it is easy to verify the following result.

Lemma 1. *Assume that hypotheses (10) holds. Then the function, still denoted by $F : \mathcal{H} \longrightarrow \mathcal{H}$, defined by*

$$F(\varepsilon(.)) = F(., \varepsilon(.)), \ a.e. on \ \Omega, \tag{15}$$

is continuous on $\mathcal{H}$

Lemma 2. *Assume that (10)-(14) hold. Then (3)-(6) is equivalent to the following variational problem:*

$$\begin{cases} Find \ u \in V \cap L^p(\Omega) such \ that \\ (u'', v) + a(u,v) - \int\limits_{\Gamma_1} \sigma(u(t))\eta g(t) \, d\Gamma_1 \\ + (|u|^\nu u, v) = (f, v), \ \forall v \in V \cap L^p(\Omega), \\ u(x,0) = u_0(x), \ u'(x,0) = u_1(x), \\ x \in \Omega, \ p = \nu + 2, \end{cases} \tag{P.V.}$$

where $V = \left\{v \in H^1(\Omega), v = g \ on \ \Sigma_1\right\}$ and $a(u,v) = \int_\Omega \sigma(u)\varepsilon(v)dx$.

3. Existence and Uniqueness

Our main existence and uniqueness result concerning problem (3)-(6), which we establish in this section, is the following.

3.1. Existence

Theorem 1. *Assume that (10)-(14) hold. Then there exists at least one solution to problem (3)-(6) and it satisfies*

$$u \in L^\infty(0,T; V \cap L^p(\Omega)), \ \ p = \nu + 2, \tag{16}$$

$$u' \in L^\infty(0,T; L^2(\Omega)). \tag{17}$$

Remark. Where as the problem is defined on the open interval $(0,T)$, the relations (6) don't have a sense, for that reason we must justify the definition of $u(t)$ and $u'(t)$ at point 0 in the initial conditions (6).

Using the result of the Theorem 1, we are going to demonstrate the following Lemma.

Lemma 3. *Assume that (10)-(14) hold. Then the initial conditions in (6) have a sense.*

Proof.
Using hypotheses (10)-(14), according to the Theorem 1, we have

$$u \in L^{\infty}\left(0,T;V\right) and\ u' \in L^{\infty}\left(0,T;L^2(\Omega)\right). \quad (18)$$

Referring to [8], it results

$$u : [0,T] \to L^2(\Omega), \quad (19)$$

is continuous, possibly after a modification on a subset of $[0,T]$ with zero measure, then $u\left(0\right)$ is well defined, therefore the first condition in (6) has a sense.
On the other hand, (16) implies that $\varepsilon(u) \in L^{\infty}\left(0,T;L^2(\Omega)\right)$. Thus, since F is continuous and $L^2(\Omega) \subset V'$, we have $F\left(\varepsilon(u)\right) \in L^{\infty}\left(0,T;L^2(\Omega)\right)$. Thus,

$$div\sigma\left(u\right) = divF\left(\varepsilon(u)\right) \in L^{\infty}\left(0,T;V'\right). \quad (20)$$

Also, we have

$$\begin{array}{l} \int_{\Omega}\left|(|u|^{\nu} u)\right|^{p'} dx \leq \int_{\Omega}|u|^{(\nu+1)p'} dx \\ = \int_{\Omega}|u|^{(\nu+1)\frac{p}{p-1}} dx \\ = \int_{\Omega}|u|^{(p-1)\frac{p}{p-1}} dx = \int_{\Omega}|u|^{p} dx \\ = \|u\|^{p}_{L^{P}(\Omega)},\ \frac{1}{p}+\frac{1}{p'}=1, \end{array} \quad (21)$$

which implies that

$$|u|^{\nu} u \in L^{\infty}\left(0,T;L^{p'}(\Omega)\right), \forall u \in L^{p}(\Omega). \quad (22)$$

Then, from (3) we have

$$u'' = f + div\sigma\left(u\right) - |u|^{\nu} u \in L^2\left(0,T;L^2(\Omega)\right) + L^{\infty}\left(0,T;V' + L^{p'}(\Omega)\right), \quad (23)$$

where V'denotes the dual of V and

$$V' + L^{p'}(\Omega) = \left\{u + v;\ u \in V'\ and\ v \in L^{p'}(\Omega)\right\}. \quad (24)$$

Since $L^2(\Omega) \subset V' + L^{p'}(\Omega)$, in particular case we have

$$u'' \in L^2\left(0,T;V' + L^{p'}(\Omega)\right). \quad (25)$$

Then, referring to [8] and using (17) we conclude that

$$u' : [0,T] \longrightarrow V' + L^{p'}(\Omega) \quad (26)$$

is continuous, possibly after a modification on a subset of $[0,T]$ with zero measure, then $u'\left(0\right)$ is well defined, therefore the second condition in (6) has a sense.
We turn now to prove Theorem 1.

Proof of Theorem 1 . It consists of four steps:

Step 1 : Approached solution.

It introduces a sequence (w_n) of functions having the following properties :

$* \forall j = 1,...,m; w_j \in V \cap L^p(\Omega)$;
$*$ The family $\{w_1, w_2, ..., w_m\}$ is linearly independent;
$*$ The $V_m = span\{w_1, w_2, ..., w_m\}$ generated by $\{w_1, w_2, ..., w_m\}$ is dense in $V \cap L^p(\Omega)$.
Let $u_m = u_m(t)$ be an approached solution such that

$$u_m(t) = \sum_{i=1}^{m} K_{jm}(t) w_i. \quad (27)$$

The K_{jm} being to be determined by the following expression :

$$\begin{cases} (u''_m(t), w_j) + a(u_m(t), w_j) - \int\limits_{\Gamma_1} \sigma(u_m(t))\eta g(t)\, d\Gamma_1 \\ \quad +(|u_m|^{\nu} u_m(t), w_j) = (f(t), w_j),\ 1 \leq j \leq m, \end{cases} \quad (28)$$

which is a nonlinear system of ordinary differential equations and will be completed by the following initial conditions

$$u_m(0) = u_{0m} = \sum_{i=1}^{m} \alpha_{im} w_i \overset{m\longrightarrow\infty}{\longrightarrow} u_0\ in\ V \cap L^p(\Omega), \quad (29)$$

$$u'_m(0) = u_{1m} = \sum_{i=1}^{m} \beta_{im} w_i \overset{m\longrightarrow\infty}{\longrightarrow} u_1\ in\ L^2(\Omega). \quad (30)$$

As the family $\{w_1, w_2, ..., w_m\}$ is linearly independent, the system (28), (29) and (30) admits at least one solution $u_m \in (0,T)$ having the following regularity

$$u_m\left(t\right) \in L^2\left(0,t_m;V_m\right),\ u'_m\left(t\right) \in L^2\left(0,t_m;V_m\right). \quad (31)$$

A priori, the time interval $(0,T)$ depends on m and thereafter we shall demonstrate that t_m does not depend on m based on the following a priori estimates.

Step 2 : A priori estimates. Let

$$\|u\|_1^2 = a(u,u) = \int_{\Omega} F\left(\varepsilon(u)\right)\varepsilon(u)dx. \quad (32)$$

Then, using (10), it can be shown that $\|u\|_1$ is a norm on V equivalent to the norm $\|u\|$ on $H^1(\Omega)$.

Multiplying the equation (28) by $K'_{jm}(t)$ and performing the summation over $j = 1$ to m, yields

$$\begin{cases} (u''_m(t), u'_m(t)) + a\left(u_m(t), u'_m(t)\right) - \int\limits_{\Gamma_1} \sigma(u_m(t))\eta g'(t)\, d\Gamma_1 \\ \quad + \left(|u_m|^{\nu} u_m(t), u'_m(t)\right) = \left(f\left(t\right), u'_m(t)\right). \end{cases} \quad (33)$$

Since $u_m \in L^2(0,t_m;V_m)$, $u'_m \in L^2(0,t_m;V_m)$, then $\varepsilon(u_m), \varepsilon(u'_m) \in L^2(0,T;L^2(\Omega))$. It follows from Lemma 1 that

$$F\varepsilon(u_m)),\ F(\varepsilon(u'_m) \in L^2\left(0,T;L^2\left(\Omega\right)\right). \quad (34)$$

On the other hand, we have

$$\frac{d}{dt}a\left(u_m(t),u_m(t)\right)=\left(F\left(\varepsilon(u_m(t))\right),\varepsilon(u_m'(t))\right)+\left(F\left(\varepsilon(u_m'(t))\right),\varepsilon(u_m(t))\right)=a\left(u_m(t),u_m'(t)\right)+a\left(u_m'(t),u_m(t)\right). \quad (35)$$

Then, using (10,b), we obtain

$$2a\left(u_m(t),u_m'(t)\right)=\frac{d}{dt}a\left(u_m(t),u_m(t)\right)=\frac{d}{dt}\left\|u_m(t)\right\|_1^2, \quad (36)$$

Also, we have

$$\frac{1}{2}\frac{d}{dt}\left|u_m'(t)\right|^2=\left(u_m''(t),u_m'(t)\right).$$
$$\frac{1}{p}\frac{d}{dt}\left\|u_m(x,t)\right\|_{L^p(\Omega)}^p=\left(\left|u_m\right|^\nu u_m(t),u_m'(t)\right),\ p=\nu+2. \quad (37)$$

As the function of the injection $u\to\sigma(u(t))$ of $H^1(\Omega)$ in $H^{\frac{-1}{2}}(\Gamma_2)$ is continuous and $g'(t)\in H^{\frac{1}{2}}(\Gamma_1)$, then

$$\left|\int_{\Gamma_1}\sigma(u_m(t))\eta g'(t)\,d\Gamma_1\right|\le C\left\|u_m(t)\right\|\le\frac{1}{2}C_2\left(1+\left\|u_m(t)\right\|^2\right). \quad (38)$$

Then from (33), by *Cauchy- Schwarz's* inequality we may conclude that

$$\frac{1}{2}\frac{d}{dt}\left[\left|u_m'(t)\right|^2+C_1\left\|u_m(t)\right\|^2\right]+\frac{1}{p}\frac{d}{dt}\left\|u_m(x,t)\right\|_{L^p(\Omega)}^p\le\left|(f(s)\right|\left|u_m'(s)\right|+\int_{\Gamma_1}\left|\sigma(u_m(t))\eta g'(t)\right|d\Gamma_1. \quad (39)$$

Therefore from(38) we obtain that

$$\frac{1}{2}\frac{d}{dt}\left[\left|u_m'(t)\right|^2+C_1\left\|u_m(t)\right\|^2\right]+\frac{1}{p}\frac{d}{dt}\left\|u_m(x,t)\right\|_{L^p(\Omega)}^p\le\left|(f(s)\right|\left|u_m'(s)\right|+\frac{1}{2}C_2\left\|u_m(t)\right\|^2+\frac{1}{2}C_2. \quad (40)$$

Now, integrating inequality (40) over $(0,T)$, we deduce that

$$\begin{aligned}&\frac{1}{2}\left(\left|u_m'(t)\right|^2+C_1\left\|u_m(t)\right\|^2\right)+\frac{1}{p}\left\|u_m(t)\right\|_{L^p(\Omega)}^p\\ &\le\frac{1}{2}\left|u_{1m}\right|^2+\frac{1}{2}C_1\left\|u_{0m}\right\|^2+\frac{1}{p}\left\|u_m(0)\right\|_{L^p(\Omega)}^p+\int_0^t\left|f(s)\right|\left|u_m'(s)\right|ds\\ &+\frac{1}{2}C_2\int_0^t\left\|u_m(s)\right\|^2ds+\frac{1}{2}TC_2.\end{aligned} \quad (41)$$

Then, from (41) by *Young's* inequality, we have that

$$\begin{aligned}&\frac{1}{2}\left(\left|u_m'(t)\right|^2+C_1\left\|u_m(t)\right\|^2\right)+\frac{1}{p}\left\|u_m(t)\right\|_{L^p(\Omega)}^p\le\frac{1}{2}\left|u_{1m}\right|^2+\\ &+\frac{1}{2}C_1\left\|u_{0m}\right\|^2+\frac{1}{p}\left\|u_{0m}\right\|_{L^p(\Omega)}^p+\frac{1}{2}\int_0^t\left|f(s)\right|^2ds+\frac{1}{2}\int_0^t\left|u_m'(s)\right|^2ds\\ &+\frac{1}{2}C_2\int_0^t\left\|u_m(s)\right\|^2ds+\frac{1}{2}TC_2.\end{aligned} \quad (42)$$

Since by assumptions, there exists a constant $C_3>0$ such that

$$\frac{1}{2}\left|u_{1m}\right|+\frac{1}{2}C_1\left\|u_{0m}\right\|^2+\frac{1}{p}\left\|u_{0m}\right\|_{L^p(\Omega)}^p+\frac{1}{2}\int_0^t\left|f(s)\right|^2ds+\frac{1}{2}TC_2\le C_3,\ \forall m\in\mathbb{N}^*. \quad (43)$$

It then follows from (42) that

$$\begin{cases}\frac{1}{2}\left(\left|u_m'(t)\right|^2+C_1\left\|u_m(t)\right\|^2\right)+\frac{1}{p}\left\|u_m(t)\right\|_{L^p(\Omega)}^p\\ \quad\le C_3+\frac{1}{2}\int_0^t\left(\left|u_m'(s)\right|^2+C_2\left\|u_m(s)\right\|^2\right)ds.\end{cases} \quad (44)$$

Hence

$$\left|u_m'(t)\right|^2+C_1\left\|u_m(t)\right\|^2\le 2C_3+\int_0^t\left(\left|u_m'(s)\right|^2+C_2\left\|u_m(s)\right\|^2\right)ds,\ \forall t\in(0,T). \quad (45)$$

Then, by *Gronwall's* inequality, we have that

$$\left|u_m'(t)\right|+\left\|u_m(t)\right\|\le C\ \ (independent\ of\ m). \quad (46)$$

Then, using (44), we arrive at

$$\left\|u_m(t)\right\|_{L^p(\Omega)}\le C\ (independent\ of\ m). \quad (47)$$

From where, we deduce that t_m is independent of m.
By passing to the limit where $m\longrightarrow\infty$, from (46) and (47) we conclude that

$$\begin{cases}(u_m)\ \ is\ bounded\ in\ L^\infty\left(0,T;V\cap L^p(\Omega)\right),\\ (u_m')\ \ is\ bounded\ in\ L^\infty\left(0,T;L^2(\Omega)\right).\end{cases} \quad (48)$$

Step 3 : Passage to the limit.
It follows from (48) that there exists a subsequence (u_μ) of (u_m) such that

$$u_\mu\longrightarrow u\ in\ L^\infty\left(0,T;V\cap L^p(\Omega)\right)\ weak\ star, \quad (49)$$

$$u_\mu'\longrightarrow u'\ in\ L^\infty\left(0,T;L^2(\Omega)\right)\ weak\ star. \quad (50)$$

From (48), it is obtained that sequences (u_m), (u_m') are bounded in $L^2(0,T;V)\subset L^2(0,T;L^2(\Omega))=L^2(Q)$, $L^2(Q)$, respectively.
Then, in particular, (u_m) is a bounded sequence in $H^1(Q)$.
It is known, see [8], that the injection of $H^1(Q)$ in $L^2(Q)$ is compact. Then, from (49) and (50) we have

$$u_\mu\longrightarrow u\ strongly\ in\ L^2(Q). \quad (51)$$

Setting $\frac{1}{p}+\frac{1}{p'}=1$, $p=\nu+2$, using (48) we have $(\left|u_m\right|^\nu u_m)$ is a bounded sequence in $L^\infty\left(0,T;L^{p'}(\Omega)\right)$.
Therefore, we have

$$\left|u_\mu\right|^\nu u_\mu\to\left|u\right|^\nu u\ in\ L^\infty\left(0,T;L^{p'}(\Omega)\right)\ weak\ star. \quad (52)$$

Let j be fixed and $\mu>j$. Then, by (28) we have

$$\left(u_\mu''(t),w_j\right)+a\left(u_\mu(t),w_j\right)-\int_{\Gamma_1}\sigma(u_\mu(t))\eta g(t)\,d\Gamma_1+\left(\left|u_\mu\right|^\nu u_\mu(t),w_j\right)=\left(f(t),w_j\right). \quad (53)$$

Therefore, (49), (50) imply

$$a(u_\mu,w_j)\longrightarrow a(u,w_j)\ in\ L^\infty(0,T)\ weak\ star, \quad (54)$$

$$(u_\mu',w_j)\longrightarrow(u',w_j)\ in\ L^\infty(0,T)\ weak\ star, \quad (55)$$

$$\int_{\Gamma_1}\sigma(u_\mu(t))\eta g(t)\,d\Gamma_1\longrightarrow\int_{\Gamma_1}\sigma(u(t))\eta g(t)\,d\Gamma_1 in L^\infty(0,T)\ weak\ star. \quad (56)$$

Hence

$$\left(u''_{\mu}(t), w_j\right) \longrightarrow \left(u''(t), w_j\right) \ in \ \mathcal{D}'(0,T). \tag{57}$$

Also, using (52) we have

$$\left(|u_\mu|^\nu u_\mu, w_j\right) \longrightarrow \left(|u|^\nu u, w_j\right) \ in \ L^\infty(0,T) \ weak \ star. \tag{58}$$

Then (53) takes the form

$$(u''(t), w_j) + a(u(t), w_j) - \int_{\Gamma_1} \sigma(u(t))\eta g(t)\, d\Gamma_1 + (|u|^\nu u(t), w_j) = (f(t), w_j). \tag{59}$$

Finally, be using the density of V_m in $V \cap L^p(\Omega)$ we obtain

$$(u''(t), v) + a(u(t), v) - \int_{\Gamma_1} \sigma(u(t))\eta g(t)\, d\Gamma_1 + (|u|^\nu u(t), v) = (f(t), v), \ \forall v \in V \cap L^p(\Omega). \tag{60}$$

Then u satisfies (3).

Step 4 : Initial condition verifications.
It follows from (49) and (50) that

$$u_\mu(0) \to u(0) \ weakly \ in \ L^2(\Omega). \tag{61}$$

Then, using (29) we deduce in particular that

$$u_\mu(0) = u_{0\mu} \to u_0 \ in \ V \cap L^p(\Omega). \tag{62}$$

Thus, the first condition in (6) is obtained.
On the other hand, by using (57) we have

$$\left(u''_{\mu}(t), w_j\right) \longrightarrow \left(u''(t), w_j\right) \ in \ L^\infty(0,T) \ weak \ star. \tag{63}$$

Hence

$$\left(u'_{\mu}(0), w_j\right) \longrightarrow \left(u'(0), w_j\right). \tag{64}$$

Since $\left(u'_\mu(0), w_j\right) \longrightarrow (u_1, w_j)$, we have $(u'(0), w_j) = (u_1, w_j), \forall j$. Then the second condition in (6) is satisfied.

3.2. Uniqueness

Many authors, for some particular problems have showed the uniqueness of the solution basing on the condition $\nu \leq \dfrac{2}{n-2}$. In this subsection the uniqueness of the solution will prove, by eliminating this condition.

Theorem 2. *Assume the conditions of Theorem 1 and also*

$$\nu \leq \tfrac{2k}{n-2}, \quad k \in \mathbb{N}^*, \ n \neq 2 \ (\nu \ any \ finished \ so \ n = 2). \tag{65}$$

Then, there exists a unique solution u to problem (3)-(6) and it satisfies (16), (17).

Proof.
Let u, v be two solutions of problem (3)-(6), to the sense of the Theorem 1.

Setting $w = u - v$, since F is linear we have

$$w'' - divF(\varepsilon(w)) + (|u|^\nu u - |v|^\nu v) = 0 \ in \ Q, \tag{66}$$

$$w(0) = w'(0) = 0 \ in \ \Omega, \tag{67}$$

$$w = 0 \ on \ \Sigma_1, \ \sigma(w)\eta = 0 \ on \ \Sigma_2, \tag{68}$$

$$w \in L^\infty(0,T; V \cap L^p(\Omega)), \ p = \nu + 2, \tag{69}$$

$$w' \in L^\infty(0,T; L^p(\Omega)), \ p = \nu + 2. \tag{70}$$

Multiplying the equation (66) by w' and integrating over Ω. Then, by using *Green*'s formula together with the conditions (67), (68), we obtain

$$\tfrac{1}{2}\tfrac{d}{dt}\left(|w'(t)|^2\right) + a\left(w(t), w'(t)\right) = \int_\Omega \left(|v|^\nu v - |u|^\nu u\right) w' dx. \tag{71}$$

Then by (10,b) we have

$$a\left(w(t), w'(t)\right) = \frac{d}{dt}a(w(t), w(t)) - \int_\Omega \frac{d}{dt}\left(F(\varepsilon(w))\right)\varepsilon(w)dx$$
$$\geq C_1\frac{d}{dt}\|w\|^2 - \int_\Omega \left(F(\varepsilon(w'))\right)\varepsilon(w)dx = C_1\frac{d}{dt}\|w\|^2 - a\left(w(t), w'(t)\right). \tag{72}$$

In this case (71) takes the form

$$\frac{1}{2}\frac{d}{dt}\left(|w'(t)|^2 + C_1\|w\|_1^2\right) \leq \int_\Omega (|v|^\nu v - |u|^\nu u) w' dx. \tag{73}$$

Also, we have

$$\left|\int_\Omega (|v|^\nu v - |u|^\nu u) w' dx\right| \leq (\nu+1)\int_\Omega \sup\left(|u|^\nu, |v|^\nu\right)|w|\,|w'|\,dx. \tag{74}$$

Next, by using *Holder's* inequality we have

$$\left|\int_\Omega (|v|^\nu v - |u|^\nu u) w' dx\right| \leq C_2\left(\||u|^\nu\|_{L^n(\Omega)} + \||v|^\nu\|_{L^n(\Omega)}\right)\|w(t)\|_{L^q(\Omega)}|w'(t)|, \tag{75}$$

where $\frac{1}{n} + \frac{1}{q} + \frac{1}{2} = 1$.
Also, by referring to [6] we have

$$\|v\|_{L^{kq}(\Omega)} = \left\||v|^k\right\|_{L^q(\Omega)}^{\frac{1}{k}} \quad \forall k, q \in \mathbb{N}^*. \tag{76}$$

Therefore by *(76)* we have $\||v|^\nu\|_{L^n(\Omega)} = \|v\|^\nu_{L^{\nu n}(\Omega)}$ and $\|v\|^\nu_{L^{kq}(\Omega)} = \left\||v|^k\right\|^{\frac{\nu}{k}}_{L^q(\Omega)}$ for all $\nu \in \mathbb{N}$. Using (65) we have $\nu n \leq kq$, then, this condition implies that $\|v\|^\nu_{L^{\nu n}(\Omega)} \leq \|v\|^\nu_{L^{kq}(\Omega)}$.
Since $\frac{1}{n} + \frac{1}{q} + \frac{1}{2} = 1$, by referring to [7] we have $H^1(\Omega) \subset L^q(\Omega)$, from where $\left\||v|^k\right\|^{\frac{\nu}{k}}_{L^q(\Omega)} \leq \left\||v|^k\right\|^{\frac{\nu}{k}} \leq C\|v\|^\nu$, therefore

$$\||v|^\nu\|_{L^n(\Omega)} \leq C\|v\|^\nu, \tag{77}$$

which implies that

$$\left|\int_\Omega (|v|^\nu v - |u|^\nu u) w' dx\right| \leq C_3\left(\|u\|^\nu + \|v\|^\nu\right)\|w\|\,|w'|. \tag{78}$$

Since $u, v \in L^{\infty}(0,T; V \cap L^{p}(\Omega))$ we obtain

$$\left| \int_{\Omega} (|v|^{\nu} v - |u|^{\nu} u) w' dx \right| \leq C_4 \|w\| \, |w'| \,. \tag{79}$$

Then, by *Young's* inequality from (73) we conclude that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(|w'(t)|^2 + C_1 \|w(t)\|^2\right) &\leq C_4 \|w\| \, |w'| \leq \\ &\leq \frac{1}{2} C_4 \left(|w'(t)|^2 + \|w(t)\|^2\right). \end{aligned} \tag{80}$$

Integrating equation above together with the initial conditions (67), we obtain

$$\left(|w'(t)|^2 + C_1 \|w(t)\|^2\right) \leq C_4 \int_0^t \left(|w'(s)|^2 + \|w(s)\|^2\right) ds. \tag{81}$$

Finally, use *Gronwall's* inequality to find $w = 0$. □

Lemma 4. *Assume the conditions of Theorem 1. Then, for all $\nu \in \mathbb{N}$ the solution u found to the Theorem 1. is unique.*

Proof.
For all $n > 2$, setting

$$k = E\left(\frac{\nu (n-2)}{2}\right) + 1, \tag{82}$$

where $E(x)$ denotes the integer part of x.
Then, we have

$$\nu \leq \frac{2k}{n-2}, \quad k \in \mathbb{N}^*, \; n \neq 2 (\nu any finished son = 2). \tag{83}$$

Thus, using Theorem 2, there exists a unique solution satisfies (16), (17). □

4. Regularity of the solution

Theorem 3. *Under the conditions stated in Theorem 1, and the additional assumptions*

$$f' \in L^2(Q), \tag{84}$$

$$u_0 \in V \cap H^2(\Omega), \tag{85}$$

$$u_1 \in V, \tag{86}$$

$$\nu \leq \frac{2k}{n-2}, \; k \in \mathbb{N}^* \; n \neq 2 (\nu \; any \; finished \; so \; n = 2). \tag{87}$$

Then, there exists a unique solution u to problem (3)-(6) and it satisfies the following regularities:

$$u \in L^{\infty}\left(0,T; V \cap H^2(\Omega)\right), \tag{88}$$

$$u' \in L^{\infty}(0,T;V), \tag{89}$$

$$u'' \in L^{\infty}\left(0,T; L^2(\Omega)\right). \tag{90}$$

Proof.
Consider the sequence of functions (w_n) such that
$*$ $\forall j = 1, ..., m; w_j \in V \cap H^2(\Omega)$;
$*$ The family $\{w_1, w_2, ..., w_m\}$ is linearly independent;
$*$ The $V_m = span\{w_1, w_2, ..., w_m\}$ generated by $\{w_1, w_2, ..., w_m\}$ is dense in $V \cap H^2(\Omega)$.
Let $u_m = u_m(t)$ be an approached solution satisfies (27) and (28).
Also, we assume that the initial data satisfy

$$u_{0m} \longrightarrow u_0 \;\; in \; V \cap H^2(\Omega), \tag{91}$$

$$u_{1m} \longrightarrow u_1 \;\; in \; V. \tag{92}$$

Then, it follows from (28) that

$$(u''_m(0), w_j) = \left(f(0) + divF(\varepsilon(u_{0m})) - |u_{0m}|^{\nu} u_{0m}, w_j\right), \; 1 \leq j \leq m. \tag{93}$$

Since F is continuous, then we conclude from (91) that

$$|divF(\varepsilon(u_{0m}))| \leq C. \tag{94}$$

By *Hölder's* inequality,

$$\int_{\Omega} \left| \left(|u_{0m}|^{\frac{\nu}{k}}\right)^{2k} u_{0m}^2 \right| dx \leq \left\| \left(|u_{0m}|^{\frac{\nu}{k}}\right)^{2k} \right\|_{L^n(\Omega)} \left\| u_{0m}^2 \right\|_{L^2(\Omega)} meas(\Omega)^{\frac{1}{q}} \leq \tag{95}$$

$$\leq C \left\| |u_{0m}|^{\frac{\nu}{k}} \right\|_{L^n(\Omega)}^{2k} |u_{0m}|^2, \; \frac{1}{n} + \frac{1}{q} + \frac{1}{2} = 1. \tag{96}$$

From (76) it follows $\left\| |u_{0m}|^{\frac{\nu}{k}} \right\|_{L^n(\Omega)}^{2k} = \|u_{0m}\|_{L^{\frac{\nu n}{k}}(\Omega)}^{2\nu}$.
Also, from (87) it results $\frac{\nu n}{k} \leq q$, then $\|u_{0m}\|_{L^{\frac{\nu n}{k}}(\Omega)}^{2\nu} \leq C \|u_{0m}\|^{2\nu}$. Consequently,

$$\int_{\Omega} \left| \left(|u_{0m}|^{\frac{\nu}{k}}\right)^{2k} u_{0m}^2 \right| dx \leq C \|u_{0m}\|^{2\nu} |u_{0m}|^2. \tag{97}$$

Then, from (91) we conclude that

$$\left(|u_{0m}|^{\nu} u_{0m}\right) \;\; is \; bounded \; in \;\; L^2(\Omega). \tag{98}$$

Multiplying the equation (93) by $K''_{jm}(0)$ and performing the summation over $j = 1$ to m, yields

$$(u''_m(0), u''_m(0)) = \left(f(0) + divF(\varepsilon(u_{0m})) - |u_{0m}|^{\nu} u_{0m}, u''_m(0)\right). \tag{99}$$

Then,

$$|u''_m(0)|^2 \leq \left(|f(0)| + |divF(\varepsilon(u_{0m}))| + |u_{0m}|^{\nu+1}\right) |u''_m(0)|. \tag{100}$$

Therefore, using (11) and (84), we obtain $f(0) \in L^2(\Omega)$.
Then,

$$|u''_m(0)| \leq C_2. \tag{101}$$

On the other hand, by derivating to time, (28) takes the form

$$(u'''_m(t), w_j) + a(u'_m(t), w_j) - \int_{\Gamma_1} \sigma(u'_m(t)) \eta g(t) \, d\Gamma_1 + (\nu + 1)\left(|u_m|^{\nu} u'_m(t), w_j\right) = \left(f'(t), w_j\right). \tag{102}$$

Multiplying (102) by $K''_{jm}(t)$ and performing the summation over $j = 1$ to m, yields

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\left|u''_m(t)\right|^2 + a\left(u'_m(t), u''_m\right) = \int\limits_{\Gamma_1} \sigma(u'_m(t))\eta g''(t)\, d\Gamma_1 + \left(f'(t), u''_m\right) \\ \qquad -(\nu+1)\left(\left|u_m(t)\right|^{\nu} u'_m(t), u''_m(t)\right). \end{cases} \tag{103}$$

As the function of the injection $u \to \sigma(u(t))$ of $H^1(\Omega)$ in $H^{\frac{-1}{2}}(\Gamma_2)$ is continuous and $g''(t) \in H^{\frac{1}{2}}(\Gamma_1)$, then

$$\left|\int\limits_{\Gamma_1} \sigma(u'_m(t))\eta g''(t)\, d\Gamma_1\right| \le C\left\|u'_m(t)\right\| \le \tfrac{1}{2}C_3\left(1 + \left\|u'_m(t)\right\|^2\right). \tag{104}$$

Since $\frac{1}{n} + \frac{1}{q} = \frac{1}{2}$, using *Hölder's* inequality we conclude that

$$\left|(\nu+1)\left(\left|u_m(t)\right|^{\nu} u'_m(t), u''_m(t)\right)\right| \le C\left\|\left|u_m(t)\right|^{\nu}\right\|_{L^n(\Omega)}\left\|u'_m(t)\right\|_{L^q(\Omega)}\left|u''_m(t)\right|. \tag{105}$$

Then, using (16) and as $\nu n \le kq$ by (77) we obtain

$$\left\|\left|u_m(t)\right|^{\nu}\right\|_{L^n(\Omega)} \le C\left\|u_m(t)\right\|^{\nu} \le C_3. \tag{106}$$

We also have

$$(\nu+1)\left|\left(\left|u_m(t)\right|^{\nu} u'_m(t), u''_m(t)\right)\right| \le C_4\left\|u'_m(t)\right\|\left|u''_m(t)\right|. \tag{107}$$

Since

$$\begin{aligned} \frac{d}{dt}a(u'_m(t), u'_m(t)) &= a(u''_m(t), u'_m(t)) + a(u'_m(t), u''_m(t)) = \\ &= 2a(u'_m(t), u''_m(t)) = \frac{d}{dt}\left\|u'_m(t)\right\|_1^2, \end{aligned} \tag{108}$$

we have

$$a(u'_m(t), u''_m(t)) = \frac{1}{2}\frac{d}{dt}\left\|u'_m(t)\right\|_1^2 \ge \tfrac{1}{2}C_1\tfrac{d}{dt}\left\|u'_m(t)\right\|^2. \tag{109}$$

Then, by using (108), (109) from (103) it follows

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\left|u''_m(t)\right|^2 + C_1\left\|u'_m(t)\right\|^2\right] \le \left(f'(t), u''_m\right) \\ &+ \int\limits_{\Gamma_1} \sigma(u'_m(t))\eta g''(t)\, d\Gamma_1 - (\nu+1)\left(\left|u_m(t)\right|^{\nu} u'_m(t), u''_m(t)\right). \end{aligned} \tag{110}$$

But by the inequalities of *Cauchy Schwartz* and *Young* the second member of (110) is raised in absolute value by

$$\begin{cases} \left|\left(f'(t), u''_m(t)\right) + \int\limits_{\Gamma_1} \sigma(u'_m(t))\eta g''(t)\, d\Gamma_1 - (\nu+1)\left(\left|u_m(t)\right|^{\nu} u'_m(t), u''_m(t)\right)\right| \\ \quad \le \left|f'(t)\right|\left|u''_m(t)\right| + \frac{1}{2}C_2\left(1 + \left\|u'_m(t)\right\|^2\right) + C_4\left\|u'_m(t)\right\|\left|u''_m(t)\right| \\ \quad \le \frac{1}{2}\left|f'(t)\right|^2 + \frac{1}{2}C_5\left(\left|u''_m(t)\right|^2 + \left\|u'_m(t)\right\|^2\right) + \frac{1}{2}C_2, \end{cases} \tag{111}$$

where $C_5 = 1 + C_2 + 2C_4$. It then follows from (110) that

$$\frac{1}{2}\frac{d}{dt}\left[\left|u''_m(t)\right|^2 + C_1\left\|u'_m(t)\right\|^2\right] \le \frac{1}{2}\left|f'(t)\right|^2 + \frac{1}{2}C_5\left(\left|u''_m(t)\right|^2 + \left\|u'_m(t)\right\|^2\right) + \frac{1}{2}C_2. \tag{112}$$

Integrating equation (112) over $(0, t)$, we obtain

$$\begin{aligned} &\frac{1}{2}\left[\left|u''_m(t)\right|^2 + C_1\left\|u'_m(t)\right\|^2\right] \le \frac{1}{2}\int_0^t \left|f'(s)\right|^2 ds + \frac{1}{2}\left|u''_m(0)\right|^2 \\ &+ \frac{1}{2}C_1\left\|u'_m(0)\right\|^2 + \frac{1}{2}C_2 T + \frac{1}{2}C_5\int_0^t \left(\left|u''_m(s)\right|^2 + \left\|u'_m(s)\right\|^2\right) ds. \end{aligned} \tag{113}$$

Then, by (84),(101),(92), it follows from (113) that

$$\left|u''_m(t)\right|^2 + C_1\left\|u'_m(t)\right\|^2 \le C_6\left(1 + \int_0^t \left(\left|u''_m(s)\right|^2 + \left\|u'_m(s)\right\|^2\right) ds\right), \tag{114}$$

where

$$C_6 = \max\left(\frac{1}{2}\int_0^t \left|f'(s)\right|^2 ds + \frac{1}{2}\left|u''_m(0)\right|^2 + \frac{1}{2}C_1\left\|u'_m(0)\right\|^2 + \frac{1}{2}C_2 T; \frac{1}{2}C_5\right).$$

Thus, by *Gronwall's* inequality, it follows that

$$\left|u''_m(t)\right| + \left\|u'_m(t)\right\| \le C \; (independent\ of\ m) \tag{115}$$

Therefore,

$$\begin{cases} (u'_m(t)) \;\; is\ bounded\ in\ L^{\infty}(0, T; V), \\ (u''_m(t)) \;\; is\ bounded\ in\ L^{\infty}\left(0, T; L^2(\Omega)\right). \end{cases} \tag{116}$$

Then, there exists a subsequence of $(u_m(t))$, denoted by (u_μ) such that

$$u''_\mu(t) \longrightarrow u''(t)\ in L^{\infty}\left(0, T; L^2(\Omega)\right)\ weak\ star. \tag{117}$$

But, by (51) we have that

$$u_\mu \longrightarrow u\ strongly\ in\ L^2(Q). \tag{118}$$

Also, by (52) we obtain

$$\left|u_\mu\right|^{\nu} u_\mu(t) \longrightarrow \left|u\right|^{\nu} u(t)\ in L^{\infty}\left(0, T; L^{p'}(\Omega)\right)\ weak\ star. \tag{119}$$

Let j be fixed and $\mu > j$. Then, using (49), (117) and (52), we deduce that

$$a(u_\mu(t), w_j) \longrightarrow a(u(t), w_j)\ in\ L^{\infty}(0,T)\ weak\ star, \tag{120}$$

$$\int\limits_{\Gamma_1} \sigma(u_\mu(t))\eta g(t)\, d\Gamma_1 \longrightarrow \int\limits_{\Gamma_1} \sigma(u(t))\eta g(t) d\Gamma_1\ in\ L^{\infty}(0,T)\ weak\ star, \tag{121}$$

$$\left(u''_\mu(t), w_j\right) \longrightarrow (u''(t), w_j)\ in\ L^{\infty}(0,T)\ weak\ star, \tag{122}$$

$$\left(\left|u_\mu\right|^{\nu} u_\mu, w_j\right) \longrightarrow \left(|u|^{\nu} u, w_j\right)\ in\ L^{\infty}(0,T)\ weak\ star. \tag{123}$$

Thus, it follows from (28) that

$$(u''(t), w_j) + a(u(t), w_j) - \int\limits_{\Gamma_1} \sigma(u(t))\eta g(t)\, d\Gamma_1 + \left(|u|^{\nu} u(t), w_j\right) = (f, w_j)\ \forall w_j. \tag{124}$$

Again, using the density of V_m in $V \cap H^2(\Omega)$ we find

$$(u''(t), v) + a(u(t), v) - \int\limits_{\Gamma_1} \sigma(u(t))\eta g(t)\, d\Gamma_1 + \left(|u|^{\nu} u, v\right) = (f(t), v),\ \forall v \in V \cap H^2(\Omega), \tag{125}$$

which implies that u satisfies (3), (89) and (90).
On the other hand, using (91), $|u|^{\nu} u \in L^{\infty}\left(0, T; L^2(\Omega)\right)$ and $f \in L^{\infty}\left(0, T; L^2(\Omega)\right)$, it then follows from (3) that

$$h = div\sigma(u) = u'' + |u|^{\nu} u - f \in L^{\infty}\left(0, T; L^2(\Omega)\right). \tag{126}$$

Since $div\sigma(u)$, see [7], is an isomorphism from V onto V'. Let G be its inverse. Then, since $u \in L^{\infty}(0, T; V)$ we have

$$u(t) = Gh(t). \tag{127}$$

As Ω is assumed regular. Then, by referring to [7] and [11] we have $G \in \mathcal{L}\left(V'; H^2(\Omega)\right)$, which implies (88).

5. Conclusion

Consider the following function

$$F(\varepsilon(u)) = 2\varepsilon(u) - Trace(\varepsilon(u))I, \qquad (128)$$

where I denotes the identity operator and $Trace$ denotes the trace operator. Then, problem (3)-(6), without the condition $\sigma(u)\eta = 0$ on Σ_2, is reduced to the following problem considered by *Lions* in [8] :

$$\begin{cases} u'' - \Delta u + |u|^{\nu} u = f \ in \ Q, \\ u = g \ on \ \Sigma, \\ u(x,0) = u_0(x); \ \ u'(x,0) = u_1(x)\ , a.e.\ x \in \Omega. \end{cases} \qquad (P)$$

Since F is linear and satisfies the hypotheses (10). Then, it is easy to verify that Theorems 1, 2 and 3 are verified for the problem (P).

References

[1] B. Benabderrahmane, B., Nouiri, Résolvante du système de Lamé dans un polygone, Anal. Univ. Oradea, fasc. Matematică, Tom XV (2008), 219-238.

[2] B. Benabderrahmane, B. Nouiri and Y. Boukhatem, Index of the elasticity operator with contact without friction boundary, Studia Univ." Babes-Bolyai", Mathematica, Vol. LIIII, No. 3, (2008), p. 3-11.

[3] Y. Boukhatem, B. Benabderrahmane, A. Rahmoune, *Méthode de Faedo-Galerkin pour un problème aux limites non linéaires*, Anal. Univ. Oradea, fasc. Matematică, Tom XVI (2009), p.167–181.

[4] H. Brézis, *Analyse fonctionnelle, théorie et application*, Paris, Masson, (1998).

[5] R. Dautry et J.L. Lions, *Analyse mathématique et calcul numérique pour les sciences et les techniques*, tome 3, Paris, Masson, (1985).

[6] Y. Choquet-Bruhat, C. Dewitt-Morette, M. Dillard-Bleick, *Analysis, Manifolds and physics*, North-Holland Publishing Company, Amsterdam, New York, Oxford (1977).

[7] P. Grisvard, *Boundary value problem in plan polygon. Instruction for use, E.D.F.*, Bulletin de la Direction des Etudes et Recherche, Serie C, Mathématiques No. 1, (1986), p. 21-59.

[8] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Paris, Dunod, (1969).

[9] J.L. Lions et E. Magenes, *Problèmes aux limites non homogènes et applications*, vols. **1**, **2**, Paris, Dunod, (1968).

[10] B. Merouani, *Solutions singulières du système de l'élasticité dans un polygone pour différentes conditions aux limites*, Maghreb Math. Rev., Vol. **5**, Nos 1 & 2, (1996), p. 95-112.

[11] M. Shillor, M. Sofonea, and J.J. Telega, Models and Analysis of Quasistatic Contact, Lecture Notes in Physics, 655, Springer, Berlin (2004).

[12] M. Sibony, *Analyse numérique III, Itérations et approximations*, Hermann (1988).

[13] Mircea Sofonea, Weimin Han, and Meir Shillor, *Analysis and Approximation of Contact Problems with Adhesion or Damage*, Chapman & Hall/CRC, Boca Raton London New York Singapore (2006).

[14] Rahmoune, A., Benabderrahmane, B., *Faedo-Galerkin's Method for a non Linear Boundary Value Problem*, Int. J. Open Probl. Comput. Sci. Math., IJOPCM, Vol. **4**, No. 4, (2011).

[15] W. Troy, *The existence of bounded solutions of a semilinear heat equation*, SIAM J. Math. Anal. **18** (1987), 332–336.

[16] F.B.Weissler, *Single point blow-up for a semilinear initial value problem*, J. Differential Equations **55** (1984), 204–224.

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