Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/040203

Properties of Mixed Hyperbolic B–Potential

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Received: 29 Jul. 2017, Revised: 9 Sep. 2017, Accepted: 3 Oct. 2017
Published online: 1 Apr. 2018

Abstract: In this article a theory of fractional powers of a singular hyperbolic operator on arbitrary spaces is discussed. Fourier–Hankel transform, semigroup property, boundedness in proper functional spaces and other properties of the mixed Riesz fractional integral generated by Bessel operator are obtained.

Keywords: Mixed hyperbolic Riesz B–potential, fractional power of a singular hyperbolic operator, Lorentz distance, singular Bessel differential operator, generalized translation, Fourier–Hankel transform, bounded operator.

1 Introduction

In this paper, we study the properties of the mixed hyperbolic Riesz B-potentials which is a fractional power of the operator

$$\frac{\partial^2}{\partial t^2} - \sum_{k=1}^{n} (B_{\gamma_i})_{x_i},$$

where $\gamma_1 > 0, ..., \gamma_n > 0$ and $(B_{\gamma_i})_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$ is the singular differential Bessel operator.

First we give a brief history of Riesz potentials and necessary definitions of spaces of basic and generalized functions adapted to study hyperbolic operators with Bessel operators instead of second derivatives.

1.1 Brief History

Potential theory is originating from the classical Isaac Newton mechanics, and was reshaped and developed by Marcel Riesz (see [1], [2]). The concept of Riesz potentials arose as generalizations of the Riemann–Liouville fractional integral for dealing with partial differential equations of the second order. Riesz introduced potentials with Euclidean and Lorentz distances which are applicable to calculating explicitly the potential for elliptic, hyperbolic and parabolic Cauchy problems. Further study, properties, and applications of classical Riesz potentials could be found in books (see [3] p. 49, 263; [4] p.117; [5] p. 131, [6] p. 483, 554; [7] p. 215; [8] p. 127, 341, 458, [9] p. 111, [10] p. 254, 316, [11] p. 144).

Along with operators representing fractional powers ordinary derivatives the theory of fractional powers of differential operators with the Bessel operator $B_\nu = D^2 + \frac{\nu}{x}D$, $D = \frac{d}{dx}$ develops. It is interesting that Bessel operator and a hyper–Bessel operator could be considered as generalized fractional derivatives too (see [12] p. 97). In [12], p. 18 hypergeometric fractional integral was investigated which is the fractional power of the Bessel operator in particular case. Fractional powers of Bessel operator have been also studied in [13]–[16].

The theory of fractional powers of elliptic operators with Bessel operators instead of all or some second derivatives is well developed. Such operators in the case of negative powers are analogues of Riesz potentials with Euclidean distance, and for them the term *elliptic B–potentials* is accepted. Elliptic B–potentials and its inverses have been studied in papers [17]–[29] and this list is not complete.

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The fractional powers of hyperbolic operators with Bessel operators instead of all or some second derivatives are much less investigated despite the fact that its study opens the widest possibilities for theoretical researchs and practical applications not only of singular differential equations but also differential geometry. When the power is negative such operators are called *hyperbolic B–potentials*. Some results for hyperbolic B–potentials could be found in [30]–[33].

In this paper we will study properties of a mixed hyperbolic B–potential for which the second derivative acts on the time variable, and the Bessel operators act on all the spatial variables. In our research we rely on methods developed in papers [34]–[42]. Namely, we use the methods of such transmutation operators as a generalized translation and the Poisson operator (see [34]–[38]), weighted spaces (see [39]–[42]) and methods of classical Riesz potentials (see [43]–[51]). In subsection 1.2, we give the basic definitions. In Section 2, we define the hyperbolic Riesz B-potential and write its properties. In Section 3 we provide some examples, and the conclusions are written in Section 4.

1.2 Basic Definitions

Suppose that $\mathbb{R}^{n+1}$ is the $n+1$–dimensional Euclidean space,

$$\mathbb{R}^{n+1}_+=\{(t,x)=(t,x_1,\ldots,x_n)\in\mathbb{R}^{n+1},\ x_1>0,\ldots,x_n>0\},$$

$\gamma=(\gamma_1,...,\gamma_n)$ is a multiindex consisting of positive fixed real numbers γ_i, $i=1,...,n$, and $|\gamma|=\gamma_1+\ldots+\gamma_n$. Let Ω be a finite or infinite open set in $\mathbb{R}^{n+1}$ symmetric with respect to each hyperplane $x_i=0$, $i=1,...,n$, $\Omega_+=\Omega\cap\mathbb{R}^{n+1}_+$ and $\overline{\Omega}_+=\Omega\cap\overline{\mathbb{R}}^{n+1}_+$ where $\overline{\mathbb{R}}^{n+1}_+=\{(t,x)=(t,x_1,\ldots,x_n)\in\mathbb{R}^{n+1},x_1\geq 0,\ldots,x_n\geq 0\}$. We deal with the class $C^m(\Omega_+)$ consisting of m times differentiable on Ω_+ functions and denote by $C^m(\overline{\Omega}_+)$ the subset of functions from $C^m(\Omega_+)$ such that all existing derivatives of these functions with respect to x_i for any $i=1,...,n$ are continuous up to $x_i=0$ and all existing derivative with respect to t are continuous for $t\in\mathbb{R}$. Class $C^m_{ev}(\overline{\Omega}_+)$ consists of all functions from $C^m(\overline{\Omega}_+)$ such that $\left.\frac{\partial^{2k+1}f}{\partial x_i^{2k+1}}\right|_{x=0}=0$ for all nonnegative integer $k\leq\frac{m-1}{2}$ and for $i=1,...,n$ (see [39] and [40], p. 21). In the following we will denote $C^m_{ev}(\overline{\mathbb{R}}^{n+1}_+)$ by C^m_{ev}. We set

$$C^\infty_{ev}(\overline{\Omega}_+)=\bigcap C^m_{ev}(\overline{\Omega}_+)$$

with intersection taken for all finite m. Let $C^\infty_{ev}(\overline{\mathbb{R}}^{n+1}_+)=C^\infty_{ev}$. Assuming that $\overset{\circ}{C}{}^\infty_{ev}(\overline{\Omega}_+)$ is the space of all functions $f\in C^\infty_{ev}(\overline{\Omega}_+)$ with a compact support. We will use the notation $\overset{\circ}{C}{}^\infty_{ev}(\overline{\Omega}_+)=\mathscr{D}_+(\overline{\Omega}_+)$.

Let $L^\gamma_p(\Omega_+)$, $1\leq p<\infty$ be the space of all measurable in Ω_+ functions such that

$$\int\limits_{\Omega_+}|f(t,x)|^p x^\gamma dtdx<\infty,$$

where and further

$$x^\gamma=\prod_{i=1}^n x_i^{\gamma_i}.$$

For a real number $p\geq 1$, the $L^\gamma_p(\Omega_+)$–norm of f is defined by

$$||f||_{L^\gamma_p(\Omega_+)}=\left(\int\limits_{\Omega_+}|f(t,x)|^p x^\gamma dtdx\right)^{1/p}.$$

Weighted measure of Ω_+ is denoted by $\mathrm{mes}_\gamma(\Omega_+)$ and is defined by the formula

$$\mathrm{mes}_\gamma(\Omega_+)=\int\limits_{\Omega_+}x^\gamma dtdx.$$

For every measurable function $f(x)$ defined on $\mathbb{R}^{n+1}_+$ we consider

$$\mu_\gamma(f,\sigma)=\mathrm{mes}_\gamma\{(t,x)\in\mathbb{R}^{n+1}_+:|f(t,x)|>\sigma\}=\int\limits_{\{(t,x):\,|f(t,x)|>\sigma\}^+}x^\gamma dtdx$$

where $\{(t,x):|f(t,x)|>\sigma\}^+=\{(t,x)\in\mathbb{R}^{n+1}_+:|f(t,x)|>\sigma\}$. We will call the function $\mu_\gamma=\mu_\gamma(f,\sigma)$ a *weighted distribution function* $|f(t,x)|$.

Let a space $L^\gamma_\infty(\Omega_+)$ be defined as a set of measurable on Ω_+ functions $f(t,x)$ such that

$$||f||_{L^\gamma_\infty(\Omega_+)}=\underset{(t,x)\in\Omega_+}{\mathrm{ess\,sup}_\gamma}|f(t,x)|=\inf_{\sigma\in\Omega_+}\{\mu_\gamma(f,\sigma)=0\}<\infty.$$

For $1\leq p\leq\infty$ the $L^\gamma_{p,loc}(\Omega_+)$ is the set of functions u defined almost everywhere in Ω_+ such that $uf\in L^\gamma_p(\Omega_+)$ for any $f\in\mathscr{D}_+(\overline{\Omega}_+)$.

Let define $\mathscr{D}'_+(\overline{\Omega}_+)$ as a set of continuous linear functionals on $\overline{\Omega}_+$. Each function $u\in L^\gamma_{1,loc}(\Omega_+)$ will be identified with the functional $u\in\mathscr{D}'_+(\overline{\Omega}_+)$ acting according to the formula

$$(u,f)_\gamma=\int\limits_{\Omega_+}u(t,x)\,f(t,x)\,x^\gamma\,dtdx,\qquad f\in\mathscr{D}_+(\overline{\Omega}_+).\tag{1}$$

Functionals $u\in\mathscr{D}'_+(\overline{\Omega}_+)$ acting by the formula (1) will be called *regular weighted functionals*. All other continuous linear functionals $u\in\mathscr{D}'_+(\overline{\Omega}_+)$ will be called *singular weighted functionals*. We will use the notation $D'_+=\mathscr{D}'_+(\overline{\mathbb{R}}_+)$.

Generalized function δ_γ is defined by the equality by analogy with ([40], p. 12)

$$(\delta_\gamma,\varphi)_\gamma=\varphi(0),\quad \varphi\in\mathscr{D}_+(\overline{\Omega}_+).$$

We will use the generalized convolution product defined by the formula

$$(f*g)_\gamma=\int\limits_{\mathbb{R}^{n+1}_+}f(\tau,y)({}^\gamma\mathbf{T}^y_x g)(t-\tau,x)y^\gamma d\tau dy,\tag{2}$$

where ${}^\gamma\mathbf{T}^y_x$ is multidimensional generalized translation

$$({}^\gamma\mathbf{T}^y_x f)(t,x)=({}^{\gamma_1}T^{y_1}_{x_1}...{}^{\gamma_n}T^{y_n}_{x_n}f)(t,x).\tag{3}$$

Each of one–dimensional generalized translations ${}^{\gamma_i}T^{y_i}_{x_i}$ is defined for $i=1,...,n$ by the next formula (see [34], p. 122, formula (5.19))

$$({}^{\gamma_i}T^{y_i}_{x_i}f)(t,x)=$$

$$=\frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int\limits_0^\pi f(t,x_1,...,x_{i-1},\sqrt{x_i^2+y_i^2-2x_iy_i\cos\varphi_i},x_{i+1},...,x_n)\,\sin^{\gamma_i-1}\varphi_i\,d\varphi_i.$$

As the space of basic functions we will use the subspace of rapidly decreasing functions:

$$S_{ev}(\mathbb{R}^{n+1}_+)=\left\{f\in C^\infty_{ev}:\sup_{(t,x)\in\mathbb{R}^{n+1}_+}\left|t^{\alpha_0}x^\alpha D^\beta f(t,x)\right|<\infty\right\},$$

where $\alpha=(\alpha_1,...,\alpha_n)$, $\beta=(\beta_0,\beta_1,...,\beta_n)$, $\alpha_0,\alpha_1,...,\alpha_n,\beta_0,\beta_1,...,\beta_n$ are arbitrary integer nonnegative numbers, $x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}$, $D^\beta=D_t^{\beta_0}D_{x_1}^{\beta_1}...D_{x_n}^{\beta_n}$, $D_t=\frac{\partial}{\partial t}$, $D_{x_j}=\frac{\partial}{\partial x_j}$, $j=1,...,n$. In the same way as $\mathscr{D}'_+$ we introduce the space S'_{ev}. In fact we identify S'_{ev} with a subspace of $\mathscr{D}'_+$ since $\mathscr{D}_+$ which is dense in S_{ev}.

2 Mixed Hyperbolic Riesz B–Potential and its Properties

Let $(B_\nu)_z=\frac{\partial^2}{\partial z^2}+\frac{\nu}{z}\frac{\partial}{\partial z}$ is the singular differential Bessel operator. In this section we define the negative fractional powers of the operator

$$(\Box_\gamma)_{t,x}=\frac{\partial^2}{\partial t^2}-\sum_{k=1}^n(B_{\gamma_i})_{x_i},\qquad \gamma_1>0,...,\gamma_n>0.\tag{4}$$

following M. Riesz (see [1], [2]) and prove its basic properties such as boundedness, semigroup property and connection with the iterated operator (4).

2.1 Definition of Mixed Hyperbolic Riesz B–Potential

Let $|x|=\sqrt{x_1^2+...+x_n^2}$. First for $(t,x)\in\mathbb{R}^{n+1}_+$, $\lambda\in\mathbb{C}$ we define function s^λ by the formula

$$s^\lambda(t,x)=\begin{cases}\frac{(t^2-|x|^2)^\lambda}{N(\alpha,\gamma,n)}, & \text{when } t^2\geq|x|^2 \text{ and } t\geq 0;\\ 0, & \text{when } t^2<|x|^2 \text{ or } t<0,\end{cases} \tag{5}$$

where

$$N(\alpha,\gamma,n)=\frac{2^{\alpha-n-1}}{\sqrt{\pi}}\prod_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{\alpha-n-|\gamma|+1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right). \tag{6}$$

Regular weighted generalized function corresponding to (5) we will denote by s_+^λ.

We introduce the *mixed hyperbolic Riesz B–potential* $I^\alpha_{s,\gamma}$ *of order* $\alpha>0$ as a generalized convolution product (2) with a weighted generalized function $s_+^{\frac{\alpha-n-|\gamma|-1}{2}}$ and $f\in S_{ev}$:

$$(I^\alpha_{s,\gamma}f)(t,x)=(s_+^{\frac{\alpha-n-|\gamma|-1}{2}}*f)_\gamma(t,x). \tag{7}$$

The precise definition of the constant $N(\alpha,\gamma,n)$ allows to obtain two key properties of the potentials (7). They are the semigroup property and its connection with the k–iterated operator (4).

We can rewrite formula (7) as

$$(I^\alpha_{s,\gamma}f)(t,x)=\int\limits_{\mathbb{R}^{n+1}_+}s_+^{\frac{\alpha-n-|\gamma|-1}{2}}(\tau,y)({}^\gamma\mathbf{T}^y_x)f(t-\tau,x)y^\gamma d\tau dy. \tag{8}$$

Along with the potential (7) we consider operator

$$(I^\alpha_{s,\gamma,\delta}f)(t,x)=\int\limits_{\delta t\geq|y|}s_+^{\frac{\alpha-n-|\gamma|-1}{2}}(\tau,y)({}^\gamma\mathbf{T}^y_x)f(t-\tau,x)y^\gamma d\tau dy,\qquad 0<\delta<1. \tag{9}$$

If $f\in S_{ev}$ and $n+|\gamma|-1<\alpha<n+|\gamma|+1$ then the integral (8) converges absolutely and

$$\lim_{\delta\to1}(I^\alpha_{s,\gamma,\delta}f)(t,x)=(I^\alpha_{s,\gamma}f)(x).$$

2.2 Boundedness

We first give the definitions of operators of strong and weak type adapted for our case, then we give Marcinkiewicz interpolation theorem and finally we prove the boundedness of mixed hyperbolic Riesz B–potential in proper spaces.

The operator A is said to be *quasilinear* (see [52], p. 41) if $A(f_1+f_2)$ is uniquely defined, Af_1 and Af_2 are defined, and if there exists a constant κ such that for all f_1 and f_2 at any point the next inequality is valid:

$$|A(f_1+f_2)|\leq\kappa(|Af_1|+|Af_2|).$$

A quasilinear operator A is of *strong type* $(p,q)_\gamma$, $1\leq p\leq\infty$, $1\leq q\leq\infty$ if it is defined from L^γ_p to L^γ_q and the following inequality is valid:

$$||Af||_{q,\gamma}\leq h||f||_{p,\gamma},\quad\forall f\in L^\gamma_p, \tag{10}$$

where constant h does not depend on f.

We say that a quasilinear operator A is an operator of *weak type* $(p,q)_\gamma$ $(1\leq p\leq\infty, 1\leq q<\infty)$ if

$$\mu_\gamma(Af,\lambda)\leq\left(\frac{h||f||_{p,\gamma}}{\lambda}\right)^q,\quad\forall f\in L^\gamma_p, \tag{11}$$

where h does not depend on f and λ, $\lambda>0$.

If $q=\infty$ then a quasilinear operator A is an operator of weak type $(p,q)_\gamma$ if it has strong type $(p,q)_\gamma$.

To prove the boundedness we shall use the Marcinkiewicz interpolation theorem in the following form (see [53]).

Theorem 1.*Let $1 \le p_i \le q_i < \infty$, $(i=1,2)$, $q_1 \neq q_2$, $0<\tau<1$, $\frac{1}{p}=\frac{1-\tau}{p_1}+\frac{\tau}{p_2}$, $\frac{1}{q}=\frac{1-\tau}{q_1}+\frac{\tau}{q_2}$. If a quasilinear operator A has simultaneously weak types $(p_1,q_1)_\gamma$ and $(p_2,q_2)_\gamma$ then an operator A has a strong type $(p,q)_\gamma$ and*

$$||Af||_{q,\gamma} \le M||f||_{p,\gamma}, \tag{12}$$

where a constant $M=M(\gamma,\tau,\kappa,p_1,p_2,q_1,q_2)$ and does not depend on f and A.

Theorem 2.*Let $n+|\gamma|-1<\alpha<n+|\gamma|+1$, $1\le p<\frac{n+|\gamma|+1}{\alpha}$. For the next estimate*

$$||I^\alpha_{s,\gamma}f||_{q,\gamma} \le M||f||_{p,\gamma}, \qquad f\in S_{ev} \tag{13}$$

to be valid it is necessary and sufficient that $q=\frac{(n+|\gamma|+1)p}{n+|\gamma|+1-\alpha p}$. Constant M does not depend on f.

Proof.Necessity. Let $n+|\gamma|-1<\alpha<n+|\gamma|+1$, $1<p<\frac{n+|\gamma|+1}{\alpha}$ and for some q an inequality

$$||I^\alpha_{s,\gamma}f||_{q,\gamma} \le M||f||_{p,\gamma}, \qquad f(x)\in S_{ev} \tag{14}$$

is hold. Let show that the inequality (14) is valid only for $q=\frac{(n+|\gamma|+1)p}{n+|\gamma|+1-\alpha p}$.

For the extension operator $\tau_\delta:(\tau_\delta f)(t,x)=f(\delta t,\delta x_1,...,\delta x_n)=f(\delta t,\delta x)$, $\delta>0$ we have

$$||\tau_\delta f||_{p,\gamma}=\delta^{-\frac{n+|\gamma|+1}{p}}||f||_{p,\gamma}, \tag{15}$$

$$(I^\alpha_{s,\gamma}f)(t,x)=\delta^\alpha\tau^{-1}_\delta I^\alpha_{s,\gamma}\tau_\delta f(t,x) \tag{16}$$

and

$$||\tau^{-1}_\delta I^\alpha_{s,\gamma}f(t,x)||_{q,\gamma}=\delta^{\frac{n+|\gamma|+1}{q}}||I^\alpha_{s,\gamma}f(t,x)||_{q,\gamma}. \tag{17}$$

From (14), (15), (16) and (17) we obtain

$$||I^\alpha_{s,\gamma}f(t,x)||_{q,\gamma}\le M\,\delta^{\frac{n+|\gamma|+1}{q}-\frac{n+|\gamma|+1}{p}+\alpha}||f(t,x)||_{p,\gamma}. \tag{18}$$

If $\frac{n+|\gamma|+1}{q}-\frac{n+|\gamma|+1}{p}+\alpha>0$ or $\frac{n+|\gamma|+1}{q}-\frac{n+|\gamma|+1}{p}+\alpha<0$ then tending in (18) to the limit with $\delta\to0$ or $\delta\to\infty$ respectively we find that

$$||I^\alpha_{s,\gamma}f(t,x)||_{q,\gamma}=0,$$

for all functions $f\in L^\gamma_p$ and yet this is obviously false. And that means that inequality (18) is valid only if $\frac{n+|\gamma|+1}{q}-\frac{n+|\gamma|+1}{p}+\alpha=0$, i.e. for $q=\frac{(n+|\gamma|+1)p}{n+|\gamma|+1-\alpha p}$. Thus the necessity is proved.

Sufficiency. Assume, without loss of generality, that $f(t,x)\ge0$, $(t,x)\in\mathbb{R}^{n+1}_+$. Let ω be a fixed real number. We define

$$G^0_{\delta,\omega}=\{(t,x)\in\mathbb{R}^{n+1}_+:\ \delta t^2\ge|x|^2,\ 0\le t\le\omega\},$$

$$G^\infty_{\delta,\omega}=\{(t,x)\in\mathbb{R}^{n+1}_+:\ \delta t^2\ge|x|^2,\ \omega<t\},$$

$$K^+_{0,\delta}(t,x)=\frac{1}{N(\alpha,\gamma,n)}\begin{cases}(t^2-|x|^2)^{\frac{\alpha-n-|\gamma|}{2}}, & (t,x)\in G^0_{\delta,\omega};\\ 0, & (t,x)\in\mathbb{R}^{n+1}_+\setminus G^0_{\delta,\omega},\end{cases}$$

$$K^+_{\infty,\delta}(t,x)=\frac{1}{N(\alpha,\gamma,n)}\begin{cases}(t^2-|x|^2)^{\frac{\alpha-n-|\gamma|}{2}}, & (t,x)\in G^\infty_{\delta,\omega};\\ 0, & (t,x)\in\mathbb{R}^{n+1}_+\setminus G^\infty_{\delta,\omega}.\end{cases}$$

Using these notations we obtain

$$(I^\alpha_{s,\gamma,\delta}f)(x)=(K^+_{0,\delta}*f)_\gamma+(K^+_{\infty,\delta}*f)_\gamma. \tag{19}$$

We have

$$N(\alpha,\gamma,n)||K^+_{0,\delta}||_{1,\gamma}=\int\limits_{G^0_{\delta,\omega}}(t^2-|x|^2)^{\frac{\alpha-n-|\gamma|-1}{2}}x^\gamma dt dx=$$

$$=\int\limits_0^\omega dt\int\limits_{|x|^2\leq\delta t^2}(t^2-|x|^2)^{\frac{\alpha-n-|\gamma|-1}{2}}x^\gamma dx=\{x=ty\}=$$

$$=\int\limits_0^\omega t^{\alpha-1}dt\int\limits_{|y|^2\leq\delta}(1-|y|^2)^{\frac{\alpha-n-|\gamma|-1}{2}}y^\gamma dy\leq\frac{\omega^\alpha}{\alpha}\int\limits_{|y|\leq 1}(1-|y|^2)^{\frac{\alpha-n-|\gamma|-1}{2}}y^\gamma dy=C^1_{\alpha,n,\gamma}\,\omega^\alpha,$$

where $C^1_{\alpha,n,\gamma}$ does not depend on δ and ω. Consequently $K^+_{0,\delta}\in L^\gamma_1$.

Lets take r,r' such that $\frac{1}{r}+\frac{1}{r'}=1$ and $\ell=\frac{(n+|\gamma|+1)r}{n+|\gamma|+1-r\alpha}$. We will estimate $||K^+_{\infty,\delta}||_{r',\gamma}$. Suppose first $r\neq 1$ (i.e. $r'\neq\infty$), so it follows

$$N(\alpha,\gamma,n)||K^+_{\infty,\delta}||_{r',\gamma}=\left(\int\limits_{G^\infty_{\delta,\omega}}(t^2-|x|^2)^{\frac{\alpha-n-|\gamma|-1}{2}r'}x^\gamma dx\right)^{1/r'}=$$

$$=\left(\int\limits_\omega^\infty dt\int\limits_{|x|^2\leq\delta t^2}(t^2-|x|^2)^{\frac{\alpha-n-|\gamma|-1}{2}r'}x^\gamma dx\right)^{1/r'}=\{x=ty\}=$$

$$=\left(\int\limits_\omega^\infty t^{(\alpha-n-|\gamma|-1)r'+n+|\gamma|}dt\right)^{1/r'}\left(\int\limits_{|y|^2\leq\delta}(1-|y|^2)^{\frac{\alpha-n-|\gamma|-1}{2}r'}y^\gamma dy\right)^{1/r'}\leq$$

$$\leq C^2_{\alpha,n,\gamma,r}(1-\delta)^{\frac{\alpha-n-|\gamma|-1}{2}}\omega^{-\frac{n+|\gamma|+1}{\ell}},$$

where $C^2_{\alpha,n,\gamma,r}$ does not depend on δ and ω. Passing to the limit $r'\to\infty$ we get an inequality for $r=1$. Then we derive that $K^+_{\infty,\delta}\in L^\gamma_{r'}$, $r'\leq\infty$.

Further, for any $\lambda>0$ we have

$$\mu_\gamma((1-\delta)^{\frac{n+|\gamma|+1-\alpha}{2}}(I^\alpha_{s,\gamma,\delta}f)(t,x),2\lambda)\leq$$

$$\leq\mu_\gamma((1-\delta)^{\frac{n+|\gamma|+1-\alpha}{2}}(K^+_{0,\delta}*f)_\gamma,\lambda)+\mu_\gamma((1-\delta)^{\frac{n+|\gamma|+1-\alpha}{2}}(K^+_{\infty,\delta}*f)_\gamma,\lambda).\tag{20}$$

Since

$$(1-\delta)^{\frac{n+|\gamma|+1-\alpha}{2}}||(K^+_{\infty,\delta}*f)_\gamma||_{\infty,\gamma}\leq C^3_{\alpha,n,\gamma}\,\omega^{-\frac{n+|\gamma|+1}{\ell}},$$

where $C^3_{\alpha,n,\gamma}$ does not depend on δ and ω, putting $\omega=(C^3_{\alpha,n,\gamma})^{\frac{\ell}{n+|\gamma|+1}}\lambda^{-\frac{\ell}{n+|\gamma|+1}}$ we obtain

$$\mu_\gamma((1-\delta)^{\frac{n+|\gamma|+1-\alpha}{2}}(K^+_{\infty,\delta}*f)_\gamma,\lambda)=0\tag{21}$$

and

$$\mu_\gamma((1-\delta)^{\frac{n+|\gamma|+1-\alpha}{2}}(K^+_{0,\delta}*f)_\gamma,\lambda)\leq\frac{C_{\alpha,n,\gamma,s}(1-\delta)^{\frac{n+|\gamma|+1-\alpha}{2}}}{\lambda^\ell}.\tag{22}$$

From (20), (21) and (22) it is clear that

$$\mu_\gamma((1-\delta)^{\frac{n+|\gamma|+1-\alpha}{2}}(I^\alpha_{s,\gamma,\delta}f)(x),\lambda)\leq\frac{C_{\alpha,n,\gamma}(1-\delta)^{\frac{n+|\gamma|+1-\alpha}{2}}}{\lambda^\ell},\tag{23}$$

where $C_{\alpha,n,\gamma}$ does not depend on λ, δ and f.

Putting $\ell=\frac{n+|\gamma|+1}{n+|\gamma|+1-\alpha}$ in (23) we obtain that the quasilinear operator $I^\alpha_{s,\gamma,\delta}$ has a weak type $\left(1,\frac{n+|\gamma|+1}{n+|\gamma|+1-\alpha}\right)_\gamma$. Similarly, assuming $\ell=\frac{(n+|\gamma|+1)p_1}{n+|\gamma|+1-\alpha p_1}$ in (23) we obtain that the quasilinear operator $I^\alpha_{s,\gamma,\delta}$ has a weak type $\left(p_1,\frac{(n+|\gamma|+1)p_1}{n+|\gamma|+1-\alpha p_1}\right)_\gamma$ where $1<p_1<\frac{n+|\gamma|+1}{\alpha}$. Lets take $p_1=\frac{p(1-\tau)}{1-\tau p}$, $\tau\in(0,1)$ so that $1<p_1<\frac{n+|\gamma|+1}{\alpha}$. Then an operator $I^\alpha_{s,\gamma,\delta}$ has weak types $(p_1,q_1)_\gamma$ and

$(p_2,q_2)_\gamma$ where $p_1=\frac{p(1-\tau)}{1-\tau p}$, $q_1=\frac{p(n+|\gamma|+1)(1-\tau)}{(n+|\gamma|+1)(1-\tau p)-\alpha p(1-\tau)}$, $p_2=1$, $q_2=\frac{n+|\gamma|+1}{n+|\gamma|+1-\alpha}$. Therefore by the Theorem 1 an operator $I^\alpha_{s,\gamma,\delta}$ has strong type $\left(p,\frac{(n+|\gamma|+1)p}{n+|\gamma|+1-\alpha p}\right)_\gamma$ and by (12)

$$||I^\alpha_{s,\gamma,\delta} f||_{q,\gamma} \leq M||f||_{p,\gamma}, \tag{24}$$

where $1\leq p<\frac{n+|\gamma|+1}{\alpha}$, $q=\frac{(n+|\gamma|+1)p}{n+|\gamma|+1-\alpha p}$, $n+|\gamma|-1<\alpha<n+|\gamma|+1$ where the constant M does not depend on f and $I^\alpha_{s,\gamma,\delta}$.

But because $f(t,x)\geq 0$ then for $0<\delta_1\leq\delta_2\leq...\leq\delta_m\leq...<1$ we have

$$(I^\alpha_{s,\gamma,\delta_1} f)(t,x)\leq (I^\alpha_{s,\gamma,\delta_2} f)(t,x)\leq ...\leq (I^\alpha_{s,\gamma,\delta_m} f)(t,x)\leq ...$$

Due to the fact that $\lim\limits_{\delta\to 1}(I^\alpha_{s,\gamma,\delta} f)(t,x)=(I^\alpha_{s,\gamma} f)(t,x)$ tending to the limit with $\delta\to 1$ in (24) we obtain

$$||(I^\alpha_{s,\gamma} f)(x)||_{q,\gamma}\leq M||f||_{p,\gamma}, \qquad 1\leq p<\frac{n+|\gamma|+1}{\alpha}, \qquad n+|\gamma|-1<\alpha<n+|\gamma|+1.$$

This proves the desired result.

Remark. By virtue of (13) there is unique extension of $I^\alpha_{s,\gamma}$ to all L^γ_p, $1<p<\frac{n+|\gamma|+1}{\alpha}$ preserving boundedness when $n+|\gamma|-1<\alpha<n+|\gamma|$. It follows that this extension is introduced by the integral (8) from its absolute convergence.

2.3 Fourier–Hankel Transform

The natural method for the investigation of operators associated with the Bessel differential operator $(B_\nu)_z=\frac{\partial^2}{\partial z^2}+\frac{\nu}{z}\frac{\partial}{\partial z}$ is to use the Hankel transform instead of Fourier transform by variables on which the Bessel operator acts. Let us define here the Fourier–Hankel integral transform and obtain an expression of mixed hyperbolic Riesz B–potential in the images of this transform.

The *Fourier–Hankel transform* of a function $f\in L^\gamma_1$ is expressed as

$$\mathscr{F}_\gamma[f](\tau,\xi)=\widehat{f}(\tau,\xi)=\int\limits_{\mathbb{R}^{n+1}_+} f(t,x)e^{-it\tau}\mathbf{j}_\gamma(x;\xi)x^\gamma dtdx,$$

where

$$\mathbf{j}_\gamma(x;\xi)=\prod_{i=1}^n j_{\frac{\gamma_i-1}{2}}(x_i\xi_i), \qquad \gamma_1>0,...,\gamma_n>0,$$

the symbol j_ν is used for the normalized Bessel function:

$$j_\nu(r)=\frac{2^\nu\Gamma(\nu+1)}{r^\nu}J_\nu(r) \tag{25}$$

and $J_\nu(r)$ is the Bessel function of the first kind of order ν.

For $f\in S_{ev}$ inverse Hankel transform is defined by

$$\mathscr{F}^{-1}_\gamma[\widehat{f}](t,x)=f(t,x)=\frac{2^{n-|\gamma|-1}}{\pi\prod\limits_{j=1}^n\Gamma^2\left(\frac{\gamma_j+1}{2}\right)}\int\limits_{\mathbb{R}^{n+1}_+}\widehat{f}(\tau,\xi)e^{-it\tau}\mathbf{j}_\gamma(x,\xi)\xi^\gamma d\tau d\xi.$$

Theorem 3. *For $f\in S_{ev}$ the Fourier–Hankel transform of mixed hyperbolic Riesz potential $I^\alpha_{s,\gamma}f$ is defined by the formula*

$$\mathscr{F}_\gamma[I^\alpha_{s,\gamma}f](\tau,\xi)=q\left|\tau^2-|\xi|^2\right|^{-\frac{\alpha}{2}}\mathscr{F}_\gamma[f(t,x)](\tau,\xi), \tag{26}$$

where

$$q=\begin{cases}1, & |\xi|^2\geq\tau^2;\\ e^{-\frac{\alpha\pi}{2}i}, & |\xi|^2<\tau^2,\ \tau\geq 0;\\ e^{\frac{\alpha\pi}{2}i}, & |\xi|^2<\tau^2,\ \tau<0.\end{cases}$$

*Proof.*Following the idea of L. Schwartz (see [3], p.263) first we are looking for $e^{-\varepsilon t}(I^{\alpha}_{s,\gamma}f)(t,x)$ and after that we pass to the limit $\varepsilon\to 0$. Using permutation property of generalized translation and the fact that

$$^{\gamma}\mathbf{T}^{y}\mathbf{j}_{\gamma}(x;\xi)=\mathbf{j}_{\gamma}(x;\xi)\mathbf{j}_{\gamma}(y;\xi) \tag{27}$$

(see [34]), we obtain

$$\mathscr{F}_{\gamma}[e^{-\varepsilon t}(I^{\alpha}_{s,\gamma}f)(t,x)](\tau,\xi)=$$

$$=\int\limits_{\mathbb{R}^{n+1}_{+}}\mathbf{j}_{\gamma}(x;\xi)e^{-it\tau-\varepsilon t}x^{\gamma}dtdx\int\limits_{\mathbb{R}^{n+1}_{+}}s_{+}^{\frac{\alpha-n-|\gamma|-1}{2}}(h,y)(^{\gamma}\mathbf{T}^{y}_{x}f)(t-h,x)y^{\gamma}dhdy=$$

$$=\int\limits_{\mathbb{R}^{n+1}_{+}}s_{+}^{\frac{\alpha-n-|\gamma|-1}{2}}(h,y)e^{-ih\tau-\varepsilon h}\mathbf{j}_{\gamma}(y;\xi)y^{\gamma}dhdy\int\limits_{\mathbb{R}^{n+1}_{+}}e^{-it\tau-\varepsilon t}\mathbf{j}_{\gamma}(x;\xi)f(t,x)x^{\gamma}dx=$$

$$=\mathscr{F}_{\gamma}[e^{-\varepsilon t}f(t,x)](\tau,\xi)\int\limits_{\mathbb{R}^{n+1}_{+}}s_{+}^{\frac{\alpha-n-|\gamma|-1}{2}}(h,y)e^{-ih\tau-\varepsilon h}\mathbf{j}_{\gamma}(y;\xi)y^{\gamma}dhdy=$$

$$=\frac{1}{N(\alpha,\gamma,n)}\mathscr{F}_{\gamma}[e^{-\varepsilon t}f(t,x)](\tau,\xi)\int\limits_{0}^{\infty}e^{-ih\tau-\varepsilon h}dh\int\limits_{\{|y|<h\}^{+}}(h^{2}-|y|^{2})^{\frac{\alpha-n-|\gamma|-1}{2}}\mathbf{j}_{\gamma}(y,\xi)y^{\gamma}dy.$$

Applying the linear change of variables $y=h\eta$ in inner integral we get to

$$\mathscr{F}_{\gamma}[e^{-\varepsilon t}(I^{\alpha}_{s,\gamma}f)(t,x)](\tau,\xi)=$$

$$=\frac{1}{N(\alpha,\gamma,n)}\mathscr{F}_{\gamma}[e^{-\varepsilon t}f(t,x)](\tau,\xi)\int\limits_{0}^{\infty}e^{-ih\tau-\varepsilon h}h^{\alpha-1}dh\int\limits_{\{|\eta|<1\}^{+}}(1-|\eta|^{2})^{\frac{\alpha-n-|\gamma|-1}{2}}\mathbf{j}_{\gamma}(h\eta,\xi)\eta^{\gamma}d\eta.$$

In the inner integral we pass to spherical coordinates $\eta=r\theta$:

$$\mathscr{F}_{\gamma}[e^{-\varepsilon t}(I^{\alpha}_{s,\gamma}f)(t,x)](\tau,\xi)=\frac{1}{N(\alpha,\gamma,n)}\mathscr{F}_{\gamma}[e^{-\varepsilon t}f(t,x)](\tau,\xi)\times$$

$$\times\int\limits_{0}^{\infty}e^{-ih\tau-\varepsilon h}h^{\alpha-1}dh\int\limits_{0}^{1}(1-r^{2})^{\frac{\alpha-n-|\gamma|-1}{2}}r^{n+|\gamma|-1}dr\int\limits_{S_{1}^{+}(n)}\mathbf{j}_{\gamma}(hr\theta,\xi)\theta^{\gamma}dS.$$

Integral $\int\limits_{S_{1}^{+}(n)}\mathbf{j}_{\gamma}(\omega\theta,\xi)\theta^{\gamma}dS$ was calculated in [54] and equals to

$$\int\limits_{S_{1}^{+}(n)}\mathbf{j}_{\gamma}(\omega\theta,\xi)\theta^{\gamma}dS=\frac{\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}j_{\frac{n+|\gamma|}{2}-1}(\omega|\xi|). \tag{28}$$

Then using (30) and formula $j_{\nu}(r)=\frac{2^{\nu}\Gamma(\nu+1)}{r^{\nu}}J_{\nu}(r)$ we obtain

$$\mathscr{F}_{\gamma}[e^{-\varepsilon t}(I^{\alpha}_{s,\gamma}f)(t,x)](\tau,\xi)=\frac{1}{N(\alpha,\gamma,n)}\mathscr{F}_{\gamma}[e^{-\varepsilon t}f(t,x)](\tau,\xi)\frac{2^{\frac{|\gamma|-n}{2}}\prod\limits_{i=1}^{n}\Gamma\left(\frac{\gamma_i+1}{2}\right)}{|\xi|^{\frac{n+|\gamma|}{2}-1}}\times$$

$$\times\int\limits_{0}^{\infty}e^{-ih\tau-\varepsilon h}h^{\alpha-\frac{n+|\gamma|}{2}}dh\int\limits_{0}^{1}(1-r^{2})^{\frac{\alpha-n-|\gamma|-1}{2}}J_{\frac{n+|\gamma|}{2}-1}(|\xi|hr)r^{\frac{n+|\gamma|}{2}}dr.$$

Applying formula 2.12.4.6 from [55] which has the form

$$\int_0^a t^{\nu+1}(a^2-t^2)^{\beta-1}J_\nu(ct)dt=\frac{2^{\beta-1}a^{\beta+\nu}}{c^\beta}\Gamma(\beta)J_{\beta+\nu}(ac),$$

for $\alpha > n+|\gamma|-2$ we can write

$$\mathscr{F}_\gamma[e^{-\varepsilon t}(I^\alpha_{s,\gamma}f)(t,x)](\tau,\xi)=\frac{1}{N(\alpha,\gamma,n)}\mathscr{F}_\gamma[e^{-\varepsilon t}f(t,x)](\tau,\xi)\frac{2^{\frac{\alpha-1}{2}-n}\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{|\xi|^{\frac{\alpha-1}{2}}}\times$$

$$\times\Gamma\left(\frac{\alpha-n-|\gamma|+1}{2}\right)\int_0^\infty e^{-ih\tau-\varepsilon h}h^{\frac{\alpha-1}{2}}J_{\frac{\alpha-1}{2}}(|\xi|h)dh.$$

By using formula 2.12.8 from [55] which has the form

$$\int_0^\infty e^{-pt}t^\nu J_\nu(ct)dt=\frac{(2c)^\nu}{\sqrt{\pi}}(p^2+c^2)^{-\nu-\frac{1}{2}}\Gamma\left(\nu+\frac{1}{2}\right),\qquad \mathrm{Re}\,\nu>-\frac{1}{2},\qquad \mathrm{Re}\,p>|\mathrm{Im}\,c|$$

final representation is obtained

$$\mathscr{F}_\gamma[e^{-\varepsilon t}(I^\alpha_{s,\gamma}f)(t,x)](\tau,\xi)=\frac{1}{N(\alpha,\gamma,n)}\mathscr{F}_\gamma[e^{-\varepsilon t}f(t,x)](\tau,\xi)\times$$

$$\times\frac{2^{\alpha-n-1}}{\sqrt{\pi}}\prod_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)\Gamma\left(\frac{\alpha-n-|\gamma|+1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)\left((\varepsilon+i\tau)^2+|\xi|^2\right)^{-\frac{\alpha}{2}}=$$

$$=\left((\varepsilon+i\tau)^2+|\xi|^2\right)^{-\frac{\alpha}{2}}\mathscr{F}_\gamma[e^{-\varepsilon t}f(t,x)](\tau,\xi).$$

Let find $\lim\limits_{\varepsilon\to0}\mathscr{F}_\gamma[e^{-\varepsilon t}(I^\alpha_{s,\gamma}f)(t,x)](\tau,\xi)$. We have

$$\arg\left(\varepsilon^2-\tau^2+|\xi|^2+2i\varepsilon\tau\right)=\begin{cases}\arctan\frac{2\varepsilon\tau}{\varepsilon^2-\tau^2+|\xi|^2}, & \varepsilon^2-\tau^2+|\xi|^2\geq0;\\ \arctan\frac{2\varepsilon\tau}{\varepsilon^2-\tau^2+|\xi|^2}+\pi, & \varepsilon^2-\tau^2+|\xi|^2<0,\ \tau\geq0;\\ \arctan\frac{2\varepsilon\tau}{\varepsilon^2-\tau^2+|\xi|^2}-\pi, & \varepsilon^2-\tau^2+|\xi|^2<0,\ \tau<0\end{cases}$$

and

$$\left|\varepsilon^2-\tau^2+|\xi|^2+2i\varepsilon\tau\right|=\sqrt{(\tau^2-|\xi|^2-\varepsilon^2)+4\varepsilon^2\tau^2}.$$

Then

$$\mathscr{F}_\gamma[(I^\alpha_{s,\gamma}f)(t,x)](\tau,\xi)=\lim_{\varepsilon\to0}\mathscr{F}_\gamma[e^{-\varepsilon t}(I^\alpha_{s,\gamma}f)(t,x)](\tau,\xi)=$$

$$=\mathscr{F}_\gamma[f(t,x)](\tau,\xi)\begin{cases}|\tau^2-|\xi|^2|^{-\frac{\alpha}{2}}, & |\xi|^2\geq\tau^2;\\ |\tau^2-|\xi|^2|^{-\frac{\alpha}{2}}e^{-\frac{\alpha\pi}{2}i}, & |\xi|^2<\tau^2,\ \tau\geq0;\\ |\tau^2-|\xi|^2|^{-\frac{\alpha}{2}}e^{\frac{\alpha\pi}{2}i}, & |\xi|^2<\tau^2,\ \tau<0.\end{cases}$$

2.4 Semigroup Properties

In this subsection we establish semigroup property for the operator $I^\alpha_{\square_\gamma}$ and obtain formula for $(\square_\gamma)^k I^{\alpha+2k}_{\square_\gamma}$.

Theorem 4. *The mixed hyperbolic Riesz B–potentials satisfy the following semigroup property for $f\in S_{ev}$:*

$$I^\beta_{s,\gamma}I^\alpha_{s,\gamma}f=I^{\alpha+\beta}_{s,\gamma}f. \tag{29}$$

*Proof.*Using (26) we obtain

$$\mathscr{F}_\gamma[I^\beta_{s,\gamma}I^\alpha_{s,\gamma}f](\tau,\xi)=\mathscr{F}_\gamma[I^\alpha_{s,\gamma}f](\tau,\xi)\begin{cases}|\tau^2-|\xi|^2|^{-\frac{\beta}{2}}, & |\xi|^2\geq\tau^2;\\ |\tau^2-|\xi|^2|^{-\frac{\beta}{2}}e^{-\frac{\beta\pi}{2}i}, & |\xi|^2<\tau^2,\ \tau\geq 0;\\ |\tau^2-|\xi|^2|^{-\frac{\beta}{2}}e^{\frac{\beta\pi}{2}i}, & |\xi|^2<\tau^2,\ \tau<0\end{cases}=$$

$$=\mathscr{F}_\gamma[f](\tau,\xi)\begin{cases}|\tau^2-|\xi|^2|^{-\frac{\alpha+\beta}{2}}, & |\xi|^2\geq\tau^2;\\ |\tau^2-|\xi|^2|^{-\frac{\alpha+\beta}{2}}e^{-\frac{(\alpha+\beta)\pi}{2}i}, & |\xi|^2<\tau^2,\ \tau\geq 0;\\ |\tau^2-|\xi|^2|^{-\frac{\beta}{2}}e^{\frac{(\alpha+\beta)\pi}{2}i}, & |\xi|^2<\tau^2,\ \tau<0\end{cases}=$$

$$=\mathscr{F}_\gamma[I^{\alpha+\beta}_{s,\gamma}f](\tau,\xi).$$

Applying inverse Fourier–Hankel transform we get (29).

Theorem 5.*For $f\in S_{ev}$, $k\in\mathbb{N}$ and $n+|\gamma|-1<\alpha<n+|\gamma|+1$ the next formula is valid*

$$(\Box_\gamma)^k_{t,x}(I^{\alpha+2k}_{s,\gamma}f)(t,x)=(I^\alpha_{s,\gamma}f)(t,x),$$

where $(\Box_\gamma)_{t,x}=\frac{\partial^2}{\partial t^2}-\sum\limits_{i=1}^n(B_{\gamma_i})_{x_i}$.

*Proof.*Using commutativity of ${}^{\gamma_i}T^{y_i}_{x_i}$ and $(B_{\gamma_i})_{x_i}$ as well as formula 1.8.3 from [40] we obtain

$$(\Box_\gamma)^k_{t,x}(I^{\alpha+2k}_{s,\gamma}f)(t,x)=(\Box_\gamma)^k_{t,x}\int\limits_{\mathbb{R}^{n+1}_+}s_+^{\frac{\alpha+2k-n-|\gamma|-1}{2}}(h,y)({}^\gamma\mathbf{T}^y_x f)(t-h,x)y^\gamma dhdy=$$

$$=(\Box_\gamma)^k_{t,x}\int\limits_{\mathbb{R}^{n+1}_+}({}^\gamma\mathbf{T}^y_x s_+^{\frac{\alpha+2k-n-|\gamma|-1}{2}})(t+h,x)f(h,y)y^\gamma dhdy=$$

$$=\int\limits_{\mathbb{R}^{n+1}_+}({}^\gamma\mathbf{T}^y_x(\Box_\gamma)^k_{t,x}s_+^{\frac{\alpha+2k-n-|\gamma|-1}{2}}(t+h,x))f(h,y)y^\gamma dhdy.$$

A direct calculation yields

$$(\Box_\gamma)^k s_+^{\frac{\alpha+2k-n-|\gamma|-1}{2}}(t+h,x)=2^{2k}\frac{\Gamma\left(\frac{\alpha-n-|\gamma|-1}{2}+k+1\right)}{\Gamma\left(\frac{\alpha-n-|\gamma|-1}{2}+1\right)}\frac{\Gamma\left(\frac{\alpha}{2}+k\right)}{\Gamma\left(\frac{\alpha}{2}\right)}s_+^{\frac{\alpha-n-|\gamma|-1}{2}}(t+h,x).$$

Since

$$2^{2k}\frac{\Gamma\left(\frac{\alpha-n-|\gamma|}{2}+k+1\right)}{\Gamma\left(\frac{\alpha-n-|\gamma|}{2}+1\right)}\frac{\Gamma\left(\frac{\alpha}{2}+k\right)}{\Gamma\left(\frac{\alpha}{2}\right)}\cdot\frac{1}{N(\alpha+2k,\gamma,n)}=\frac{1}{N(\alpha,\gamma,n)}$$

we have

$$(\Box_\gamma)^k(I^{\alpha+k}_{s,\gamma}f)(t,x)=$$

$$=\int\limits_{\mathbb{R}^{n+1}_+}({}^\gamma\mathbf{T}^y s_+^{\frac{\alpha-n-|\gamma|-1}{2}})(t+h,x)f(h,y)y^\gamma dhdy=(I^\alpha_{s,\gamma}f)(t,x).$$

It completes the proof of the theorem.

3 Examples

In this section we give an example of mixed hyperbolic Riesz B–potential for particular function f.

Let $f(t,x)=\varphi(t)\mathbf{j}_\gamma(x;b)$, where $b=(b_1,...,b_n)$, $f(t)\in S$. Applying (27) we get

$$I^\alpha_{s,\gamma}\varphi(t)\mathbf{j}_\gamma(x;b)=\frac{1}{N(\alpha,\gamma,n)}\int\limits_0^\infty\varphi(t-\tau)d\tau\int\limits_{\{|y|<\tau\}^+}(\tau^2-|y|^2)^{\frac{\alpha-n-|\gamma|-1}{2}}({}^\gamma\mathbf{T}^y_x)\mathbf{j}_\gamma(x;b)y^\gamma dy=$$

$$=\frac{\mathbf{j}_\gamma(x;b)}{N(\alpha,\gamma,n)}\int\limits_0^\infty\varphi(t-\tau)d\tau\int\limits_{\{|y|<\tau\}^+}(\tau^2-|y|^2)^{\frac{\alpha-n-|\gamma|-1}{2}}\mathbf{j}_\gamma(y;b)y^\gamma dy=\{y=\tau\eta\}=$$

$$=\frac{\mathbf{j}_\gamma(x;b)}{N(\alpha,\gamma,n)}\int\limits_0^\infty\varphi(t-\tau)\tau^{\alpha-1}d\tau\int\limits_{\{|\eta|<1\}^+}(1-|\eta|^2)^{\frac{\alpha-n-|\gamma|-1}{2}}\mathbf{j}_\gamma(\tau\eta;b)\eta^\gamma d\eta=\{\eta=\rho\theta\}=$$

$$=\frac{\mathbf{j}_\gamma(x;b)}{N(\alpha,\gamma,n)}\int\limits_0^\infty\varphi(t-\tau)\tau^{\alpha-1}d\tau\int\limits_0^1(1-\rho^2)^{\frac{\alpha-n-|\gamma|-1}{2}}\rho^{n+|\gamma|-1}d\rho\int\limits_{S_1^+(n)}\mathbf{j}_\gamma(\tau\rho\theta;b)\theta^\gamma dS.$$

The integral $\int\limits_{S_1^+(n)}\mathbf{j}_\gamma(\tau\rho\theta;b)\theta^\gamma dS$ was calculated in [54] and has the form

$$\int\limits_{S_1^+(n)}\mathbf{j}_\gamma(\tau\rho\theta;b)\theta^\gamma dS=\frac{\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}j_{\frac{n+|\gamma|}{2}-1}(\tau\rho|b|). \tag{30}$$

Using (25) and formula 2.12.4.6 from [55] we obtain

$$I^\alpha_{s,\gamma}\varphi(t)\mathbf{j}_\gamma(x;b)=\frac{\mathbf{j}_\gamma(x;b)}{N(\alpha,\gamma,n)}\frac{\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\frac{2^{\frac{n+|\gamma|}{2}-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}{|b|^{\frac{n+|\gamma|}{2}-1}}\times$$

$$\times\int\limits_0^\infty\varphi(t-\tau)\tau^{\alpha-\frac{n+|\gamma|}{2}}d\tau\int\limits_0^1(1-\rho^2)^{\frac{\alpha-n-|\gamma|-1}{2}}\rho^{\frac{n+|\gamma|}{2}}J_{\frac{n+|\gamma|}{2}-1}(\tau\rho|b|)d\rho=$$

$$=\frac{\mathbf{j}_\gamma(x;b)}{N(\alpha,\gamma,n)}\frac{\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{2^{n-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}\frac{2^{\frac{n+|\gamma|}{2}-1}\Gamma\left(\frac{n+|\gamma|}{2}\right)}{|b|^{\frac{n+|\gamma|}{2}-1}}\times$$

$$\times\frac{2^{\frac{\alpha-n-|\gamma|-1}{2}}}{|b|^{\frac{\alpha-n-|\gamma|+1}{2}}}\Gamma\left(\frac{\alpha-n-|\gamma|+1}{2}\right)\int\limits_0^\infty\varphi(t-\tau)\tau^{\frac{\alpha-1}{2}}J_{\frac{\alpha-1}{2}}(|b|\tau)d\tau=$$

$$=\frac{\mathbf{j}_\gamma(x;b)}{N(\alpha,\gamma,n)}\frac{2^{\frac{\alpha-1}{2}-n}\prod\limits_{i=1}^n\Gamma\left(\frac{\gamma_i+1}{2}\right)}{|b|^{\frac{\alpha-1}{2}}}\Gamma\left(\frac{\alpha-n-|\gamma|+1}{2}\right)\int\limits_0^\infty\varphi(t-\tau)\tau^{\frac{\alpha-1}{2}}J_{\frac{\alpha-1}{2}}(|b|\tau)d\tau=$$

$$=\frac{\sqrt{\pi}\mathbf{j}_\gamma(x;b)}{(2|b|)^{\frac{\alpha-1}{2}}\Gamma\left(\frac{\alpha}{2}\right)}\int\limits_0^\infty\varphi(t-\tau)\tau^{\frac{\alpha-1}{2}}J_{\frac{\alpha-1}{2}}(|b|\tau)d\tau.$$

Now using formula

$$I^\alpha_{s,\gamma}\varphi(t)\mathbf{j}_\gamma(x;b)=\frac{\sqrt{\pi}\mathbf{j}_\gamma(x;b)}{(2|b|)^{\frac{\alpha-1}{2}}\Gamma\left(\frac{\alpha}{2}\right)}\int\limits_0^\infty\varphi(t-\tau)\tau^{\frac{\alpha-1}{2}}J_{\frac{\alpha-1}{2}}(|b|\tau)d\tau \tag{31}$$

we can calculate $I^{\alpha}_{s,\gamma}f$ for different functions $\varphi(t)$. First we take $\varphi(t)=\theta(t)$, where θ is the Heaviside step function, when

$$I^{\alpha}_{s,\gamma}\theta(t)\mathbf{j}_{\gamma}(x;b)=\frac{\sqrt{\pi}\mathbf{j}_{\gamma}(x;b)}{(2|b|)^{\frac{\alpha-1}{2}}\Gamma\left(\frac{\alpha}{2}\right)}\int\limits_0^t \tau^{\frac{\alpha-1}{2}}J_{\frac{\alpha-1}{2}}(|b|\tau)d\tau=$$

$$=\frac{\sqrt{\pi}\mathbf{j}_{\gamma}(x;b)}{(2|b|)^{\frac{\alpha-1}{2}}\Gamma\left(\frac{\alpha}{2}\right)}\frac{t^{\frac{\alpha+1}{2}}}{2^{\frac{\alpha+1}{2}}}(|b|t)^{\frac{\alpha-1}{2}}\Gamma\left(\frac{\alpha}{2}\right){}_1F_2\left(\frac{\alpha}{2};\frac{\alpha+1}{2},\frac{\alpha+2}{2};-\frac{|b|^2t^2}{4}\right)=$$

$$=\frac{\sqrt{\pi}t^{\alpha}\mathbf{j}_{\gamma}(x;b)}{2^{\alpha}}{}_1F_2\left(\frac{\alpha}{2};\frac{\alpha+1}{2},\frac{\alpha+2}{2};-\frac{|b|^2t^2}{4}\right).\ t>0.$$

Let $n=1$, $\gamma=5$, $b=1$, $\alpha=5.5$. The plot of

$$I^{\alpha}_{s,\gamma}\theta(t)j_2(x)=\frac{\sqrt{\pi}t^{5.5}j_2(x)}{2^{5.5}}{}_1F_2\left(\frac{11}{4};\frac{13}{4},\frac{15}{4};-\frac{t^2}{4}\right)$$

is given on figure 1 obtained through the Wolfram|Alpha.

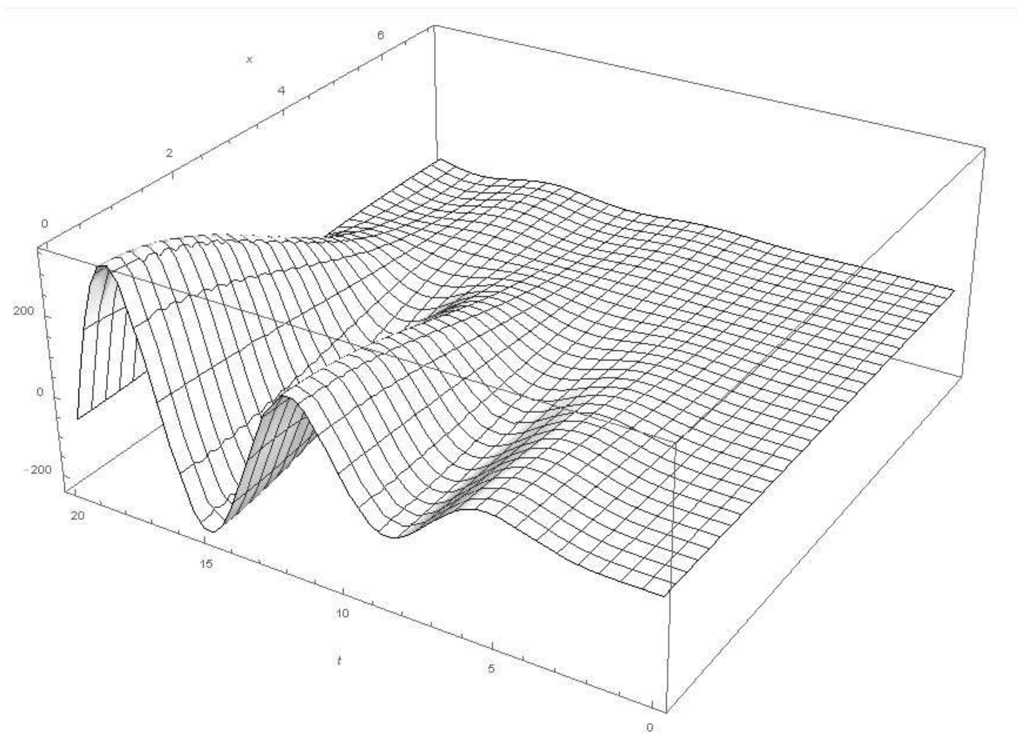

Fig. 1: $I^{\alpha}_{s,\gamma}\theta(t)j_{\frac{\gamma-1}{2}}(x),\gamma=5,\alpha=5.5.$

Now let $\varphi(t)=t^{\beta-1}\theta(t)$, where θ is the Heaviside step function, $\beta>0$. So for $f(t,x)=t^{\beta-1}\theta(t)\mathbf{j}_{\gamma}(x;b)$ we have

$$(I^{\alpha}_{s,\gamma}f)(t,x)=\frac{\sqrt{\pi}\mathbf{j}_{\gamma}(x;b)}{(2|b|)^{\frac{\alpha-1}{2}}\Gamma\left(\frac{\alpha}{2}\right)}\int\limits_0^t(t-\tau)^{\beta-1}\tau^{\frac{\alpha-1}{2}}J_{\frac{\alpha-1}{2}}(|b|\tau)d\tau.$$

Using formula 2.12.3.1 from [55] we obtain

$$(I^{\alpha}_{s,\gamma}f)(t,x)=\frac{B(\alpha,\beta)}{\Gamma(\alpha)}\mathbf{j}_{\gamma}(x;b)t^{\alpha+\beta-1}{}_1F_2\left(\frac{\alpha}{2};\frac{\alpha+\beta}{2},\frac{\alpha+\beta+1}{2};-\frac{t^2|b|^2}{4}\right),\ t\geq 0.$$

Let $n = 1$, $\gamma = \frac{1}{2}$, $\beta = 1$, $b = 1$. The plot of

$$I^{\alpha}_{s,\gamma}\theta(t)j_{\frac{\gamma-1}{2}}(x) = \frac{\Gamma\left(\frac{3}{4}\right)}{2^{\frac{1}{4}}\Gamma(\alpha+1)}x^{\frac{1}{4}}J_{-\frac{1}{4}}(x)t^{\alpha}\,{}_1F_2\left(\frac{\alpha}{2};\frac{\alpha+1}{2},\frac{\alpha+2}{2};-\frac{t^2}{4}\right)$$

for $\alpha = 0.7$ is given on figure 2 obtained through the Wolfram|Alpha.

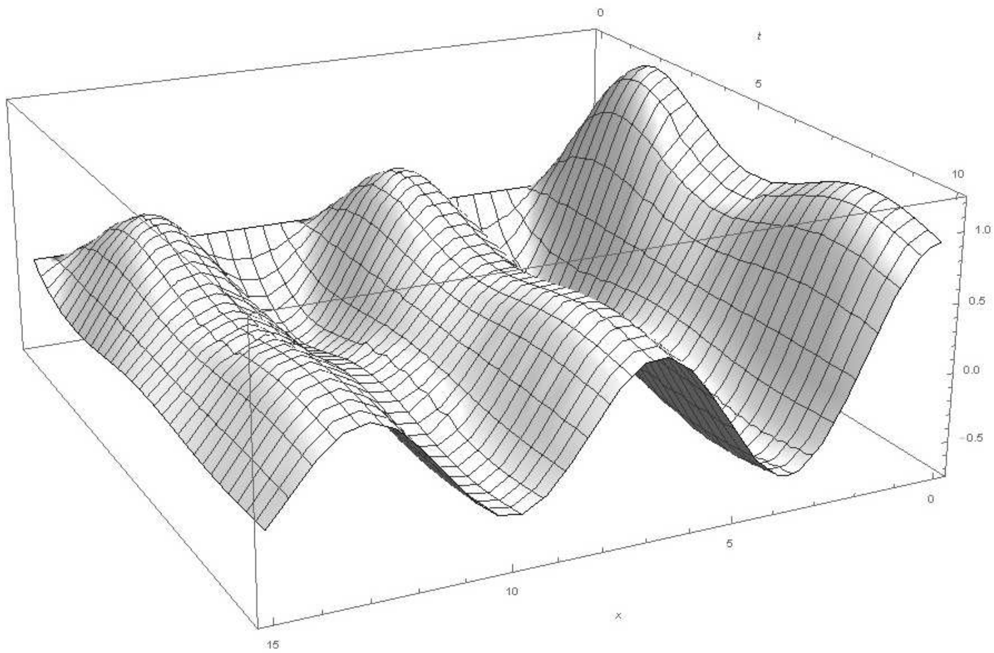

Fig. 2: $I^{\alpha}_{s,\gamma}\theta(t)j_{\frac{\gamma-1}{2}}(x), \gamma = \frac{1}{2}, \alpha = 0.7.$

Finally let $\varphi(t) = \sin(t)\theta(t)$ then using the Wolfram|Alpha we obtain

$$I^{\alpha}_{s,\gamma}\sin(t)\theta(t)\mathbf{j}_{\gamma}(x;b) = \frac{\sqrt{\pi}\mathbf{j}_{\gamma}(x;b)}{(2|b|)^{\frac{\alpha-1}{2}}\Gamma\left(\frac{\alpha}{2}\right)}\int\limits_0^t \sin(t-\tau)\tau^{\frac{\alpha-1}{2}}J_{\frac{\alpha-1}{2}}(|b|\tau)d\tau =$$

$$= \frac{\sqrt{\pi}\mathbf{j}_{\gamma}(x;b)}{(2|b|)^{\frac{\alpha-1}{2}}\Gamma\left(\frac{\alpha}{2}\right)}\frac{t^{\frac{\alpha}{2}}}{\alpha}\left(\frac{2^{\frac{1-\alpha}{2}}t^{\frac{\alpha}{2}}\sin(t)\,{}_2F_3\left(\frac{\alpha}{4}+\frac{1}{2},\frac{\alpha}{4};\frac{1}{2},\frac{\alpha}{2}+\frac{1}{2},\frac{\alpha}{2}+1;-t^2\right)}{\Gamma\left(\frac{\alpha+1}{2}\right)}+\right.$$

$$\left.+\frac{\cos(t)\left(t\sin(t)J_{\frac{\alpha+3}{2}}(t)+J_{\frac{\alpha+1}{2}}(t)(t\cos(t)-(\alpha+1)\sin(t))\right)}{\sqrt{t}}\right).$$

The plot of $I^{\alpha}_{s,\gamma}\sin(t)\theta(t)j_{\frac{\gamma-1}{2}}(x)$ when $n = 1$, $\gamma = \frac{1}{2}$, $\beta = 1$, $\alpha = 1$ is given on figure 3 obtained through the Wolfram|Alpha.

4 Conclusion

In the present work, a fractional powers of a singular hyperbolic operator on arbitrary spaces was studied. Such fractional powers are named mixed hyperbolic Riesz B–potentials. For the mixed hyperbolic Riesz B–potential such properties as Fourier–Hankel transform, semigroup property, boundedness in proper functional spaces and others were obtained.

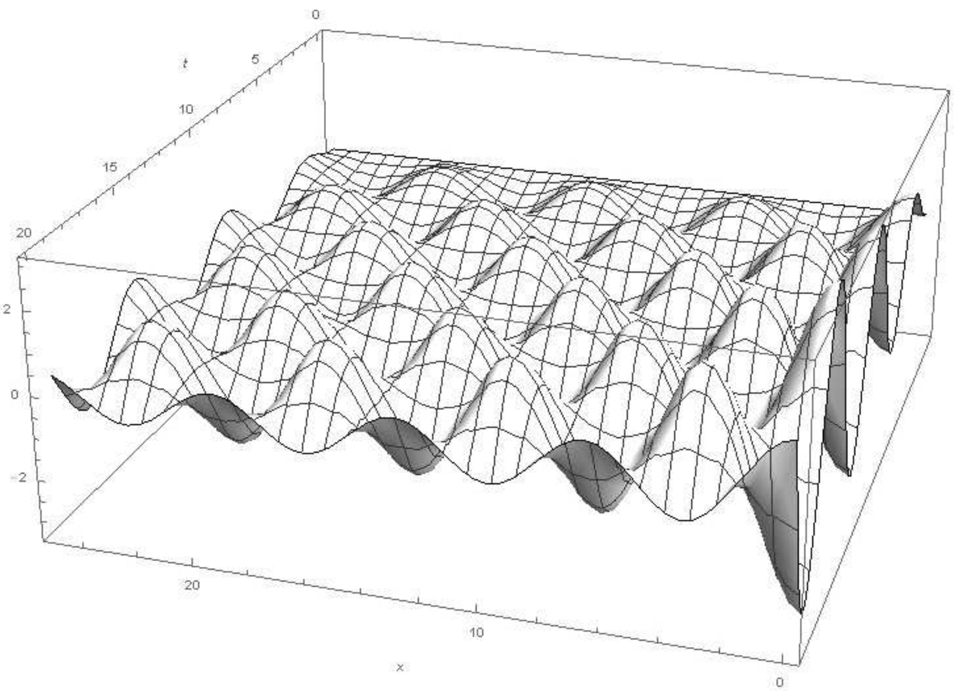

Fig. 3: $I^{\alpha}_{S,\gamma}\sin(t)\theta(t)j_{\frac{\gamma-1}{2}}(x), \gamma=\frac{1}{2}, \alpha=1.$

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