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On the Generalized (k,ρ)-Fractional Derivative

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Abstract: We generalize a fractional derivative type, the so-called (k,ρ)-fractional derivative and establish some properties of the resulting generalized operator. As an application we show that with this new definition, the Cauchy problem is equivalent to a Volterra integral equation of the second kind. We discuss, for this problem, some particular cases.

Keywords: k-gamma function, k-Mittag-Leffler function, (k,ρ)-Riemann-Liouville fractional integral, generalized (k,ρ)-fractional derivative.

1 Introduction

There are several problems in which fractional derivatives (non-integer order derivatives) play a central role [1,2,3,4,5]. It should be emphasized that fractional derivatives are presented in many ways, in particular, characterizing three distinct ways, which we will mention in order to develop the work in one of them.

Each classical fractional derivative is usually defined in terms of a specific integral. Among the most well-known definitions of fractional derivatives we may mention the Riemann-Liouville, Caputo, Grünwald-Letnikov and Hadamard derivatives [6,7], whose formulations involve integrals with singular kernels and which are used to study, for example, problems involving the memory effect [8]. On the other hand, in the years 2010, other formulations of fractional derivatives have appeared in the literature [9]. These new formulations differ from the classical ones in several aspects. For instance, classical fractional derivatives are defined in such a way that in the limit in which the order of the derivative is an integer, one recovers the classical derivatives in the sense of Newton and Leibniz. There has also been recently proposed a new fractional derivative [10,11,12] with a corresponding integral whose kernel can be a non-singular function, for instance, a Mittag-Leffler function [13]. Also in such cases, integer order derivatives are recovered by considering adequate limits for the values of its parameters.

On the other hand, there are many ways to obtain a generalization of classical fractional derivatives. Some authors introduce parameters in classical definitions or in some particular special function [3,14,15,16,17,18,19,20], as we shall do below. Also, in a recent paper [21], the authors introduce a parameter and discuss a generalization for fractional derivatives on two particular spaces, which they call generalized fractional derivative, and further propose a Caputo modification of this generalization.

Furthermore, also recent is the paper [22] in which, due to the proliferation of definitions, the authors propose a method (algorithm) to characterize a fractional derivative by imposing some requirements for a particular definition to be considered a good definition of a fractional (non-integer order) derivative.

In this paper we are interested in the so-called k-fractional and (k,ρ)-fractional derivative types, which generalize the classical fractional derivatives. Specifically, we propose a new generalization of fractional derivatives and discuss a general Cauchy problem in order to study the existence and uniqueness of its solutions and their dependence on initial conditions. As a by product, we recover a wide class of fractional derivatives.

This paper is organized as follows: In Section 2, we present some definitions aiming at our main result; in particular, the definition of k-Mittag-Leffler functions, the spaces in which we work and the k-fractional integrals in the senses of Riemann-Liouville and Hadamard. In Section 3, we present some properties of the so-called (k,ρ)-fractional operator

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and in Section 4, our main result, we introduce the generalized (k,ρ)-fractional derivative and we demonstrate that, using adequate parameters, we are able to recover a wide list of definitions of fractional derivatives. As an application, introduced in the previous section by means of theorems, we approach linear fractional differential equations by studying the Cauchy problem and discuss the existence and uniqueness of its solution and its dependence on initial conditions. Concluding remarks close the paper.

2 Preliminaries

Let us introduce the concept of k-$*$, where $*$ denotes the gamma function, the beta function or the Pochhammer symbol, in order to introduce the k-Mittag-Leffler function. Díaz and Pariguan [16] have been the first ones to define the k-gamma function, the k-beta function and the k-Pochhammer symbol. We start explaining the k-gamma function, defined by

$$\Gamma_k(x)=\int_0^\infty t^{x-1}\mathrm{e}^{-\frac{t^k}{k}}dt, \quad \text{with} \quad x,k>0, \tag{1}$$

for which the following relations hold:

$$\Gamma_k(x)=k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right) \qquad \text{and} \qquad \Gamma_k(k)=1. \tag{2}$$

In the limit $k\to 1$, we have $\Gamma_k(x)=\Gamma(x)$. The k-Pochhammer symbol is defined by

$$(x)_{n,k}=\begin{cases}1, & \text{for} \quad n=0\\ x(x+k)\cdots(x+(n-1)k), & \text{for} \quad n\in\mathbb{N}, x\in\mathbb{R}, k>0,\end{cases} \tag{3}$$

or in terms of a quotient of k-gamma functions,

$$(x)_{n,k}=\frac{\Gamma_k(x+nk)}{\Gamma_k(x)}. \tag{4}$$

Finally, the k-beta function is defined by

$$B_k(x,y)=\frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt, \quad x>0, \quad y>0, \quad k>0. \tag{5}$$

Notice that, when $k\to 1$ we have $B_k(x,y)=B(x,y)$. The k-beta function can be written in terms of k-gamma functions and in terms of the common beta function as follows:

$$B_k(x,y)=\frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)} \quad \text{and} \quad B_k(x,y)=\frac{1}{k}B\left(\frac{x}{k},\frac{y}{k}\right).$$

Mittag-Leffler functions play a very important role in the solution of linear fractional differential equations and integral equations, [23,24,25,26,27]. In order to generalize such functions, Dorrego and Cerutti [28] defined the so-called k-Mittag-Leffler function as follows:

$$E^{\delta}_{k,\beta,\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\delta)_{n,k}}{\Gamma_k(\beta n+\gamma)}\frac{z^n}{n!}, \quad z\in\mathbb{R}, \quad \beta>0, \quad \gamma>0, \tag{6}$$

where $n\in\mathbb{N}$, $(\delta)_{n,k}$ is the k-Pochhammer symbol defined in Eq.(3) and $\Gamma_k(x)$ is the k-gamma function, Eq.(1). In the case $k=1$ we recover the three-parameters Mittag-Leffler function [29]. Gupta and Parihar [30] defined the so-called k-new generalized Mittag-Leffler function using the following series:

$$E_{k,\xi,\sigma}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma_k(\xi n+\sigma)}, \quad z\in\mathbb{R}, \quad \xi>0, \quad \sigma>0, \tag{7}$$

where $n\in\mathbb{N}$. Again, in order to recover the two-parameters Mittag-Leffler function introduced by Wiman [31] one just has to consider $k=1$. Before presenting the definition of k-fractional integrals and their generalizations, we define the adequate function spaces for such definitions, as well as the Lipschitz condition for a function $f(x,\varphi)$.

Definition 1.[23]
Let $[a,b]$ be a finite or infinite interval on the real axis $\mathbb{R}=(-\infty,\infty)$. We denote by $L_p(a,b)$ the set of those Lebesgue complex-valued measurable function φ on $[a,b]$ defined by

$$L_p(a,b)=\left\{\varphi : \|\varphi\|_p=\left(\int_a^b|\varphi(x)|^p dx\right)^{1/p}<+\infty\right\}, \quad 1\leq p<\infty. \tag{8}$$

In the case $p=1$, we denote $L_1(a,b)=L(a,b)$.

Definition 2.[32] *Assume that $f(x,\varphi)$ is defined on the set $(a,b]\times G$, $G\subset\mathbb{R}$. A function $f(x,\varphi)$ satisfies Lipschitz condition with respect to φ, if for all $x\in(a,b]$ and for $\varphi_1,\varphi_2\in\mathbb{R}$,*

$$|f(x,\varphi_1)-f(x,\varphi_2)|\le A|\varphi_1-\varphi_2|,$$

where $A>0$ does not depend on x.

Once such functions and the adequate space were defined, Mubeen and Habibullah [18] introduced the so-called k-Riemann-Liouville fractional integral, a generalization of the Riemann-Liouville fractional integral, obtained for $k=1$. Such integral is defined here, for the left-sided only, as

$$({}_k\mathscr{I}^{\alpha}_{a^+}\varphi)(x)=\frac{1}{k\Gamma_k(\alpha)}\int_a^x(x-t)^{\frac{\alpha}{k}-1}\varphi(t)dt,\quad \alpha>0,\quad x>a, \tag{9}$$

where $\varphi\in L(a,b)$. When $k\to1$, we have $\Gamma_k(\alpha)=\Gamma(\alpha)$ and ${}_k\mathscr{I}^{\alpha}_{a^+}=\mathscr{I}^{\alpha}_{a^+}$, where $\mathscr{I}^{\alpha}_{a^+}$ is the classical Riemann-Liouville fractional integral. Similarly, in order to generalize the Hadamard fractional integral, the k-Hadamard fractional integral [33] was introduced. The definition of this operator, for the left-sided only, is given by

$$({}_k\mathscr{H}^{\alpha}_{a^+}\varphi)(x)=\frac{1}{k\Gamma_k(\alpha)}\int_a^x\left(\ln\frac{x}{t}\right)^{\frac{\alpha}{k}-1}\varphi(t)\frac{dt}{t},\quad \alpha>0,\quad x>a, \tag{10}$$

where $k>0$ and $\varphi\in L(a,b)$. When $k\to1$, we have ${}_k\mathscr{H}^{\alpha}_{a^+}\to\mathscr{H}^{\alpha}_{a^+}$, where $\mathscr{H}^{\alpha}_{a^+}$ is the Hadamard fractional integral. Recently, Sarikaya et al. [17] proposed the so-called (k,ρ)-fractional integral which, at adequate limits, recovers the k-Riemann-Liouville and k-Hadamard fractional integrals. This operator is defined —left-sided only— by

$$({}^{\rho}_{k}\mathscr{I}^{\alpha}_{a^+}\varphi)(x)=\frac{1}{k\Gamma_k(\alpha)}\int_a^x\left(\frac{x^\rho-t^\rho}{\rho}\right)^{\frac{\alpha}{k}-1}t^{\rho-1}\varphi(t)dt,\quad \alpha>0,\quad x>a, \tag{11}$$

with $n-1<\alpha\le n$, $n\in\mathbb{N}$, $k>0$, $\rho>0$ and $\varphi\in L(a,b)$. When $k\to1$, we have $\Gamma_k(\alpha)\to\Gamma(\alpha)$ and ${}^{\rho}_{k}\mathscr{I}^{\alpha}_{a^+}\to{}^{\rho}\mathscr{I}^{\alpha}_{a^+}$, where ${}^{\rho}\mathscr{I}^{\alpha}_{a^+}$ is the generalized fractional integral defined in [34]. When $\rho\to1$, we obtain the $k-$Riemann-Liouville fractional integral, Eq.(9). On the other hand, considering $\rho\to0^+$, we obtain the k-Hadamard fractional integral, Eq.(10).

3 Auxiliary Results

We now present some properties of the fractional integrals defined in the previous section, in order to use them throughout this work. We start by presenting the semigroup property for the (k,ρ)-fractional operator and an application to the power function; both results are theorems found in [17].

Theorem 1.*Let $\alpha>0$, $\beta>0$, $k>0$, $\rho>0$ and $\varphi\in L_p(a,b)$, then*

$$({}^{\rho}_{k}\mathscr{I}^{\alpha}_{a^+}\,{}^{\rho}_{k}\mathscr{I}^{\beta}_{a^+}\varphi)(x)=({}^{\rho}_{k}\mathscr{I}^{\alpha+\beta}_{a^+}\varphi)(x)=({}^{\rho}_{k}\mathscr{I}^{\beta}_{a^+}\,{}^{\rho}_{k}\mathscr{I}^{\alpha}_{a^+}\varphi)(x).$$

Theorem 2.*Let $\alpha,\beta>0$ and $k,\rho>0$. Then, we have*

$$[{}^{\rho}_{k}\mathscr{I}^{\alpha}_{a^+}(t^\rho-a^\rho)^{\frac{\beta}{k}-1}](x)=\frac{\Gamma_k(\beta)}{\rho^{\frac{\alpha}{k}}\Gamma_k(\alpha+\beta)}(x^\rho-a^\rho)^{\frac{\alpha+\beta}{k}-1}.$$

The following lemma shows that the (k,ρ)-fractional operator is bounded in the space $L(a,b)$.

Lemma 1.[17] *Let $\varphi\in L(a,b)$; then, the (k,ρ)-Riemann-Liouville fractional integral of order $\alpha>0$ is bounded in the space $L(a,b)$, i.e.*

$$\|{}^{\rho}_{k}\mathscr{I}^{\alpha}_{a^+}\varphi\|_1\le M\|\varphi\|_1, \tag{12}$$

where

$$M=\frac{1}{\alpha\Gamma_k(\alpha)}\left(\frac{b^\rho-a^\rho}{\rho}\right)^{\frac{\alpha}{k}}.$$

Most fractional differentiation operators are defined in terms of some corresponding fractional integral. We now present the definition of Hilfer fractional derivative, which is associated with the Riemann-Liouville fractional integral.

Definition 3.[35] *The Hilfer fractional derivative of order $0<\alpha<1$ and type $0\leq\beta\leq 1$ with respect to x is defined by*

$$(\mathscr{D}_{a^+}^{\alpha,\beta}\varphi)(x)=\left(\mathscr{I}_{a^+}^{\beta(1-\alpha)}\frac{d}{dx}\mathscr{I}_{a^+}^{(1-\beta)(1-\alpha)}\varphi\right)(x) \tag{13}$$

for functions for which the expression on the right-hand side exists.

Similarly, we have the Hilfer-Hadamard fractional derivative, which is associated with the Hadamard fractional integral.

Definition 4.[36] *The Hilfer-Hadamard fractional derivative of order $0<\alpha<1$ and type $0\leq\beta\leq 1$ with respect to x is defined by*

$$({}_{\mathscr{H}}\mathscr{D}_{a^+}^{\alpha,\beta}\varphi)(x)=\left(\mathscr{H}_{a^+}^{\beta(1-\alpha)}\left(x\frac{d}{dx}\right)\mathscr{H}_{a^+}^{(1-\beta)(1-\alpha)}\varphi\right)(x) \tag{14}$$

for functions for which the expression on the right hand side exists.

In order to obtain a more general derivative than the one proposed by Hilfer, that is, a fractional derivative of order $\alpha\in\mathbb{R}^+$ with $n-1<\alpha\leq n$, where $n\in\mathbb{N}$, Hilfer, Luchko and Tomovski [37] proposed the generalized Riemann-Liouville fractional derivative, which is associated with the Riemann-Liouville fractional integral.

Definition 5.[37] *Let $\alpha,\beta\in\mathbb{R}$ such that $n-1<\alpha\leq n$, $n\in\mathbb{N}$, $0\leq\beta\leq 1$, where α is the order and β is the type of generalized Riemann-Liouville fractional derivative, then*

$$({}^{n}\mathscr{D}_{a^+}^{\alpha,\beta}\varphi)(x)=\left(\mathscr{I}_{a^+}^{\beta(n-\alpha)}\frac{d^n}{dx^n}\mathscr{I}_{a^+}^{(1-\beta)(n-\alpha)}\varphi\right)(x) \tag{15}$$

for functions for which the expression on the right-hand side exists.

Recently, Nisar et al. [3] proposed the (k,ρ)-fractional derivative, which is associated with the (k,ρ)-fractional integral, Eq.(11).

Definition 6.[3] *Let $\mu,\nu,k\in\mathbb{R}$ such that $0<\mu<1$, $0\leq\nu\leq 1$ and $k>0$. The (k,ρ)-fractional derivative is defined by*

$$({}_{k}^{\rho}\mathscr{D}_{a^+}^{\mu,\nu}\varphi)(x)=\left({}_{k}^{\rho}\mathscr{I}_{a^+}^{\nu(k-\mu)}\left(x^{1-\rho}\frac{d}{dx}\right)(k\,{}_{k}^{\rho}\mathscr{I}_{a^+}^{(1-\nu)(k-\mu)}\varphi)\right)(x), \tag{16}$$

for functions for which the expression on the right hand side exists.

4 Generalized (k,ρ)-Fractional Derivative

In this section we propose, as our main result, a generalization for the fractional derivative proposed in [3]. The definition in that work considers the order of derivative to be $0<\mu<1$, but here we consider $\alpha\in\mathbb{R}^+$, with $n-1<\alpha\leq n$ and $n\in\mathbb{N}$. We call our definition generalized (k,ρ)-fractional derivative. The fractional integral associated with this differentiation operator is the (k,ρ)-fractional integral, Eq.(11). In this section we also prove some properties of this operator.

Definition 7.*Let $\alpha,\nu\in\mathbb{R}$ such that $n-1<\alpha\leq n$, $n\in\mathbb{N}$, $0\leq\nu\leq 1$, $\rho>0$ and $k>0$. We define the generalized (k,ρ)-fractional derivative by*

$$({}_{k}^{\rho}\mathscr{D}_{a^+}^{\alpha,\nu}\varphi)(x)=\left({}_{k}^{\rho}\mathscr{I}_{a^+}^{\nu(nk-\alpha)}\left(x^{1-\rho}\frac{d}{dx}\right)^n(k^n\,{}_{k}^{\rho}\mathscr{I}_{a^+}^{(1-\nu)(kn-\alpha)}\varphi)\right)(x) \tag{17}$$

$$=\left({}_{k}^{\rho}\mathscr{I}_{a^+}^{\nu(nk-\alpha)}\delta_\rho^n(k^n\,{}_{k}^{\rho}\mathscr{I}_{a^+}^{(1-\nu)(kn-\alpha)}\varphi)\right)(x), \tag{18}$$

where $\delta_\rho^n=\left(x^{1-\rho}\frac{d}{dx}\right)^n$.

With adequate choices of parameters in Definition 7, we recover well-known operators of fractional differentiation, namely:

–if $n=1$, we obtain the (k,ρ)-fractional derivative [3];
–if $k=1$ and $n=1$, we have the Hilfer-Katugampola fractional derivative proposed in [1];
–if $k=1$ and $\rho=1$, we obtain the so-called generalized Riemann-Liouville fractional derivative [37];

–if $k=1$, $\rho\to 0^+$ and $n=1$, we have the Hilfer-Hadamard fractional derivative [36];
–if $k=1$, $\rho=1$ and $n=1$, we obtain the well-known Hilfer derivative [38];
–if $k=1$, $\rho=1$ and $\nu=0$, we obtain the Riemann-Liouville fractional derivative [23, p. 70];
–if $k=1$, $\rho=1$ and $\nu=1$, we obtain the Caputo derivative [23, p. 92];
–if $k=1$, $\rho\to 0^+$ and $\nu=0$, we obtain the Hadamard fractional derivative [23, p. 111];
–if $k=1$, $\rho\to 0^+$ and $\nu=1$, we have the Caputo-Hadamard fractional derivative [39].

It is also possible to recover, for particular extreme values of integration, the fractional derivative in the Liouville sense [23, p. 87] and in the Weyl [38] sense.
The generalized (k,ρ)-fractional derivative ${}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}$ is the inverse operator of the (k,ρ)-fractional integral, ${}^{\rho}_{k}\mathscr{J}^{\alpha}_{a^+}$. We prove this result by means of the following lemma.

Lemma 2. *Let $\alpha\in\mathbb{R}^*$ and $\rho>0$, $k>0$. If $1\le p\le\infty$, then for $\varphi\in L_p(a,b)$ we have*

$$({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\,{}^{\rho}_{k}\mathscr{J}^{\alpha}_{a^+}\varphi)(x)=\varphi(x). \tag{19}$$

Proof. In order to simplify the development and the notation, we define

$$\Psi=\frac{\nu(nk-\alpha)}{k}\quad\text{and}\quad \Phi=n-\Psi. \tag{20}$$

From Definition 7 and Theorem 1, we can write

$$\begin{aligned}({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\,{}^{\rho}_{k}\mathscr{J}^{\alpha}_{a^+}\varphi)(x) &= \left({}^{\rho}_{k}\mathscr{J}^{\nu(nk-\alpha)}_{a^+}\delta^n_\rho(k^n\,{}^{\rho}_{k}\mathscr{J}^{(1-\nu)(kn-\alpha)+\alpha}_{a^+}\varphi)\right)(x)\\ &=\frac{k^{n-2}\rho^{2-\Psi-\Phi}}{\Gamma_k[k\Psi]\Gamma_k[k\Phi]}\int_a^x(x^\rho-t^\rho)^{\Psi-1}t^{\rho-1}\delta^n_\rho\underbrace{\left[\int_a^t(t^\rho-u^\rho)^{\Phi-1}u^{\rho-1}\varphi(u)du\right]}_{F(t)}dt.\end{aligned} \tag{21}$$

Solving by parts the integral within brackets, and choosing $w=\varphi(u)$ and $dv=(t^\rho-u^\rho)^{\Phi-1}$, we obtain

$$F(t)=\frac{\rho^{-1}}{\Phi}\left\{\varphi(a)(t^\rho-a^\rho)^\Phi+\int_a^t(t^\rho-u^\rho)^\Phi\varphi'(u)du\right\}. \tag{22}$$

Applying the differential operator δ^n_ρ to Eq.(22) we obtain, by mathematical induction, the following expression

$$\delta^n_\rho F(t)=\frac{\rho^{n-1}\Gamma(\Phi+1)}{\Phi\Gamma(\Phi-n+1)}\left\{\varphi(a)(t^\rho-a^\rho)^{\Phi-n}+\int_a^t(t^\rho-u^\rho)^{\Phi-n}\varphi'(u)du\right\} \tag{23}$$

We substitute Eq.(23) into Eq.(21) and use the first expression of Eq.(2) to get

$$\begin{aligned}({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\,{}^{\rho}_{k}\mathscr{J}^{\alpha}_{a^+}\varphi)(x) &= \frac{\rho}{k\,k^{\Psi-1}\Gamma[\Psi]\,k^{(1-\Psi)-1}\Gamma[1-\Psi]}\left\{\varphi(a)\int_a^x(x^\rho-t^\rho)^{\Psi-1}t^{\rho-1}(t^\rho-a^\rho)^{-\Psi}dt\right.\\ &\quad\left.+\int_a^x\varphi'(u)du\int_u^x(x^\rho-t^\rho)^{\Psi-1}t^{\rho-1}(t^\rho-u^\rho)^{-\Psi}dt\right\}.\end{aligned}$$

Introducing in the integral from a to x the change of variable $u=(t^\rho-a^\rho)/(x^\rho-a^\rho)$ and doing the same in the integral from u to x, we have

$$({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\,{}^{\rho}_{k}\mathscr{J}^{\alpha}_{a^+}\varphi)(x)=\frac{1}{\Gamma_k[k\Psi]\,\Gamma_k[k(1-\Psi)]}\left[\varphi(a)+\int_a^x\varphi'(u)du\right]\left\{\frac{1}{k}\int_0^1(1-u)^{\Psi-1}u^{(1-\Psi)-1}du\right\}.$$

We then use the two expressions in Eq.(2) to obtain

$$\begin{aligned}({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\,{}^{\rho}_{k}\mathscr{J}^{\alpha}_{a^+}\varphi)(x) &= \frac{1}{\Gamma_k[k\Psi]\,\Gamma_k[k(1-\Psi)]}\left[\varphi(a)+\int_a^x\varphi'(u)du\right]\frac{\Gamma_k[k\Psi]\,\Gamma_k[k(1-\Psi)]}{\Gamma_k[k]}\\ &=\varphi(a)+\int_a^x\varphi'(u)du.\end{aligned}$$

Finally, we use the fundamental theorem of calculus, whence it immediately follows that

$$({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\,{}^{\rho}_{k}\mathscr{J}^{\alpha}_{a^+}\varphi)(x)=\varphi(x).$$

The following result yields the composition between the (k,ρ)-fractional integral and the generalized (k,ρ)-fractional derivative.

Lemma 3. *Let $\alpha,\beta\in\mathbb{R}$ such that $\alpha>\beta>0$, $k,\rho>0$, $n-1<\alpha\leq n$ and $n\in\mathbb{N}$. If $1\leq p\leq\infty$, then for $\varphi\in L_p(a,b)$, we have*

$$({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\,{}^{\rho}_{k}\mathscr{I}^{\beta}_{a^+}\varphi)(x)=({}^{\rho}_{k}\mathscr{I}^{\beta-\alpha}_{a^+}\varphi)(x). \tag{24}$$

Proof. The proof is analogous to the previous lemma [40].

Again, in order to simplify the development and notation, we introduce the parameter Λ:

$$\Lambda=\frac{\nu(nk-\alpha)+\alpha}{k}. \tag{25}$$

Lemma 4. *Let $\alpha>0$, $n=[\alpha]+1$, where $n\in\mathbb{N}$. If $\varphi\in L_p(a,b)$ and $({}^{\rho}_{k}\mathscr{I}^{(1-\nu)(kn-\alpha)-k(n-j)}_{a^+}\varphi))(x)\in AC^n_\delta[a,b]$, then*

$$({}^{\rho}_{k}\mathscr{I}^{\alpha}_{a^+}\,{}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\varphi)(x)=\varphi(x)-\sum_{j=1}^{n}\frac{({}^{\rho}_{k}\mathscr{I}^{(1-\nu)(kn-\alpha)-k(n-j)}_{a^+}\varphi)(a)}{\Gamma_k[k(\Lambda-j+1)]}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Lambda-j}. \tag{26}$$

In particular, if $0<\alpha<1$, then

$$({}^{\rho}_{k}\mathscr{I}^{\alpha}_{a^+}\,{}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\varphi)(x)=\varphi(x)-\frac{({}^{\rho}_{k}\mathscr{I}^{(1-\nu)(k-\alpha)-k(1-j)}_{a^+}\varphi)(a)}{\Gamma_k[\nu(k-\alpha)+\alpha-k(j-1)]}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Lambda-j}. \tag{27}$$

Proof. From Definition 7, we can write

$$\begin{aligned}
({}^{\rho}_{k}\mathscr{I}^{\alpha}_{a^+}\,{}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\varphi)(x)&=\left({}^{\rho}_{k}\mathscr{I}^{\alpha}_{a^+}\,{}^{\rho}_{k}\mathscr{I}^{\nu(nk-\alpha)}_{a^+}\delta^n_\rho(k^n\,{}^{\rho}_{k}\mathscr{I}^{(1-\nu)(kn-\alpha)}_{a^+}\varphi)\right)(x)\\
&=\left({}^{\rho}_{k}\mathscr{I}^{\nu(nk-\alpha)+\alpha}_{a^+}\delta^n_\rho(k^n\,{}^{\rho}_{k}\mathscr{I}^{(1-\nu)(kn-\alpha)}_{a^+}\varphi)\right)(x)\\
&=\frac{\rho^{1-\Lambda}}{\Gamma_k[k\Lambda]}\int_a^x(x^\rho-t^\rho)^{\Lambda-1}t^{\rho-1}\left\{\delta^n_\rho(k^n\,{}^{\rho}_{k}\mathscr{I}^{(1-\nu)(kn-\alpha)}_{a^+}\varphi)(t)\right\}dt.
\end{aligned}$$

Integrating by parts the last expression, we obtain

$$\begin{aligned}
({}^{\rho}_{k}\mathscr{I}^{\alpha}_{a^+}\,{}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\varphi)(x)&=\frac{-\rho^{1-\Lambda}(x^\rho-a^\rho)^{\Lambda-1}}{k^\Lambda\Gamma(\Lambda)}[\delta^{n-1}_\rho(k^n\,{}^{\rho}_{k}\mathscr{I}^{(1-\nu)(kn-\alpha)}_{a^+}\varphi)(a)]\\
&\quad+\frac{\rho^{2-\Lambda}}{k^\Lambda\Gamma(\Lambda-1)}\int_a^x(x^\rho-t^\rho)^{\Lambda-1}t^{\rho-1}\delta^n_\rho(k^n\,{}^{\rho}_{k}\mathscr{I}^{(1-\nu)(kn-\alpha)}_{a^+}\varphi)(t)dt.
\end{aligned}$$

Thus, integrating by parts $(n-1)$ times, we have

$$\begin{aligned}
({}^{\rho}_{k}\mathscr{I}^{\alpha}_{a^+}\,{}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\varphi)(x)&=-\sum_{j=0}^{n-1}\frac{\delta^{n-j-1}_\rho(k^n\,{}^{\rho}_{k}\mathscr{I}^{(1-\nu)(kn-\alpha)}_{a^+}\varphi)(a)}{k^{j+1}\Gamma_k[k(\Lambda-j)]}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Lambda-j-1}\\
&\quad+\frac{1}{k\Gamma_k[k(\Lambda-n)]}\int_a^x\left(\frac{x^\rho-t^\rho}{\rho}\right)^{\Lambda-n-1}t^{\rho-1}({}^{\rho}_{k}\mathscr{I}^{(1-\nu)(kn-\alpha)}_{a^+}\varphi)(t)dt\\
&=-\sum_{j=0}^{n-1}\frac{\delta^{n-j-1}_\rho(k^n\,{}^{\rho}_{k}\mathscr{I}^{(1-\nu)(kn-\alpha)}_{a^+}\varphi)(a)}{k^{j+1}\Gamma_k[k(\Lambda-j)]}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Lambda-j-1}\\
&\quad+({}^{\rho}_{k}\mathscr{I}^{\nu(kn-\alpha)+\alpha-nk}_{a^+}\,{}^{\rho}_{k}\mathscr{I}^{(1-\nu)(kn-\alpha)}_{a^+}\varphi)(x)\\
&=\varphi(x)-\sum_{j=1}^{n}\frac{\delta^{n-j}_\rho(k^n\,{}^{\rho}_{k}\mathscr{I}^{(1-\nu)(kn-\alpha)}_{a^+}\varphi)(a)}{k^j\Gamma_k[k(\Lambda-j+1)]}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Lambda-j}\\
&=\varphi(x)-\sum_{j=1}^{n}\frac{({}^{\rho}_{k}\mathscr{I}^{(1-\nu)(kn-\alpha)-k(n-j)}_{a^+}\varphi)(a)}{\Gamma_k[k(\Lambda-j+1)]}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Lambda-j}.
\end{aligned}$$

Next, we show that the generalized (k,ρ)-fractional derivative of order α of the polynomial function $(t^\rho - a^\rho)^{\Lambda-j}$ is null, i.e., $\left[{}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}(t^\rho - a^\rho)^{\Lambda-j}\right](x) = 0$.

Lemma 5. *Let $\alpha,\nu \in \mathbb{R}$ such that $n-1<\alpha\le n$, $n\in\mathbb{N}$, $0\le\nu\le 1$, $\rho>0$ and $k>0$. Then, for all $j=1,2,\ldots,n$, we have*

$$\left[{}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}(t^\rho - a^\rho)^{\Lambda-j}\right](x) = 0. \tag{28}$$

Proof. Again, in order to simplify the development and the notation in what follows, we use Eq.(25) and we define

$$\Omega = \frac{(1-\nu)(kn-\alpha)}{k}. \tag{29}$$

Thus, from Definition 7 and Eq.(11), we have

$$\left(k^n\,{}^{\rho}_{k}\mathscr{J}^{(1-\nu)(kn-\alpha)}_{a^+}(t^\rho - a^\rho)^{\Lambda-j}\right)(x) = \frac{k^n\rho^{1-\Omega}}{k\Gamma_k[k\Omega]}\int_a^x (x^\rho - t^\rho)^{\Omega-1}(t^\rho - a^\rho)^{\Lambda-j}t^{\rho-1}dt.$$

We introduce the change of variable $u=(t^\rho - a^\rho)/(x^\rho - a^\rho)$, and use the definition of k-beta function, Eq.(5), to obtain

$$\begin{aligned}\left(k^n\,{}^{\rho}_{k}\mathscr{J}^{(1-\nu)(kn-\alpha)}_{a^+}(t^\rho - a^\rho)^{\Lambda-j}\right)(x) &= \frac{k^n\rho^{-\Omega}}{\Gamma_k[k\Omega]}(x^\rho - a^\rho)^{n-j}\left\{\frac{1}{k}\int_0^1(1-u)^{\Omega-1}u^{\Lambda-j}du\right\}\\ &= \frac{k^n\rho^{-\Omega}\Gamma_k[k(\Lambda-j+1)]}{\Gamma_k[k(n-j-1)]}(x^\rho - a^\rho)^{n-j}.\end{aligned}$$

Next, we calculate $\delta^n_\rho(k^n\,{}^{\rho}_{k}\mathscr{J}^{(1-\nu)(kn-\alpha)}_{a^+}(t^\rho - a^\rho)^{\Lambda-j})(x)$, that is,

$$\begin{aligned}\left(x^{1-\rho}\frac{d}{dx}\right)^n(x^\rho - a^\rho)^{n-j} &= \left(x^{1-\rho}\frac{d}{dx}\right)^{n-1}\left(x^{1-\rho}\frac{d}{dx}\right)(x^\rho - a^\rho)^{n-j}\\ &= \rho(n-j)\left(x^{1-\rho}\frac{d}{dx}\right)^{n-1}(x^\rho - a^\rho)^{n-j-1}.\end{aligned}$$

Differentiating more $(n-1)$ times, we obtain

$$\left(x^{1-\rho}\frac{d}{dx}\right)^n(x^\rho - a^\rho)^{n-j} = \rho^n(n-j)(n-j-1)\cdots(2-j)(1-j)(x^\rho - a^\rho)^{-j} = 0. \tag{30}$$

As $j=1,2,\ldots,n$, then for each j there is one null term in the product given by Eq.(30); this completes the proof.

Finally, we show the equivalence between the Cauchy problem and a Volterra integral equation of the second kind.

Theorem 3. *Let $\alpha>0$ and $n=[\alpha]+1$ where $n\in\mathbb{N}$. Let G be an open set in $\mathbb{R}$ and $f:(a,b]\times G\to\mathbb{R}$ be a function such that $f(x,\varphi(x))\in L(a,b)$ for any $\varphi\in G$. If $\varphi\in L(a,b)$, then φ satisfies the relations*

$$({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\varphi)(x) = f(x,\varphi(x)), \tag{31}$$

$$({}^{\rho}_{k}\mathscr{J}^{(1-\nu)(kn-\alpha)-k(n-j)}_{a^+}\varphi)(a^+) = b_j,\ b_j\in\mathbb{R},\ (j=1,2,\ldots,n), \tag{32}$$

if, and only if, φ satisfies the Volterra integral equation

$$\varphi(x) = \sum_{j=1}^n \frac{b_j}{\Gamma_k[k(\Lambda-j+1)]}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda-j} + \frac{1}{k\Gamma_k(\alpha)}\int_a^x\left(\frac{x^\rho - t^\rho}{\rho}\right)^{\frac{\alpha}{k}-1}t^{\rho-1}f(t,\varphi(t))\,dt, \tag{33}$$

with Λ defined in Eq.(25).

Proof. $(\Rightarrow)$ We consider $\varphi\in L(a,b)$ satisfying Eq.(31) and Eq.(32). As $\varphi\in L(a,b)$, then Eq.(31) exists and $({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\varphi)(x)\in L(a,b)$. Applying operator ${}^{\rho}_{k}\mathscr{J}^{\alpha}_{a^+}$ on both sides of Eq.(31) and using Lemma 4 and Eq.(32), we obtain

$$\begin{aligned}({}^{\rho}_{k}\mathscr{J}^{\alpha}_{a^+}\,{}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\varphi)(x) &= ({}^{\rho}_{k}\mathscr{J}^{\alpha}_{a^+}f(t,\varphi(t)))(x)\\ \varphi(x) - \sum_{j=1}^n\frac{({}^{\rho}_{k}\mathscr{J}^{(1-\nu)(kn-\alpha)-k(n-j)}_{a^+}\varphi)(a)}{\Gamma_k[k(\Lambda-j+1)]}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda-j} &= ({}^{\rho}_{k}\mathscr{J}^{\alpha}_{a^+}f(t,\varphi(t)))(x)\\ \varphi(x) = \sum_{j=1}^n\frac{b_j}{\Gamma_k[k(\Lambda-j+1)]}\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda-j} &+ ({}^{\rho}_{k}\mathscr{J}^{\alpha}_{a^+}f(t,\varphi(t)))(x).\end{aligned}$$

From Lemma 1, the integral $({}^{\rho}_{k}\mathscr{J}^{\alpha}_{a^+}\, f(t,\varphi(t)))(x) \in L(a,b)$, thus Eq.(33) follows.

$(\Leftarrow)$ Assume that $\varphi \in L(a,b)$ satisfies Eq.(33). Applying operator ${}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}$ on both sides of Eq.(33), we obtain

$$({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\varphi)(x) = \sum_{j=1}^{n} \frac{b_j\,\rho^{j-\Lambda}}{\Gamma_k[k(\Lambda-j+1)]}\left[{}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}(t^{\rho}-a^{\rho})^{\Lambda-j}\right](x) + ({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\,{}^{\rho}_{k}\mathscr{J}^{\alpha}_{a^+}\, f(t,\varphi(t)))(x).$$

From Lemma 2 and Lemma 5, Eq.(31) follows. Next, we prove the validity of Eq.(32). Therefore, we apply the operator ${}^{\rho}_{k}\mathscr{J}^{(1-\nu)(kn-\alpha)-k(n-m)}_{a^+}$, with $m = 1,2,\ldots,n$, on both sides of Eq.(33), in order to obtain

$$\begin{aligned}
({}^{\rho}_{k}\mathscr{J}^{(1-\nu)(kn-\alpha)-k(n-j)}_{a^+}\varphi)(x) &= \sum_{j=1}^{n} \frac{b_j}{\Gamma_k[k(\Lambda-j+1)]}\left[{}^{\rho}_{k}\mathscr{J}^{(1-\nu)(kn-\alpha)-k(n-m)}_{a^+}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\Lambda-j}\right] \\
&+ ({}^{\rho}_{k}\mathscr{J}^{(1-\nu)(kn-\alpha)-k(n-m)}_{a^+}\,{}^{\rho}_{k}\mathscr{J}^{\alpha}_{a^+}\, f(t,\varphi(t)))(x) \\
&= \sum_{j=1}^{m} \frac{b_j}{\Gamma_k[k(m-j+1)]}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{m-j} \\
&+ ({}^{\rho}_{k}\mathscr{J}^{k(m-\nu n)+\alpha\nu}_{a^+}\, f(t,\varphi(t)))(x).
\end{aligned}$$

Letting $x \to a^+$, we finally have

$$({}^{\rho}_{k}\mathscr{J}^{(1-\nu)(kn-\alpha)-k(n-j)}_{a^+}\varphi)(a^+) = b_m, \quad \text{with} \quad m = 1,2,\ldots,n.$$

5 Linear Fractional Differential Equations

In this section we analyze some particular cases of function $f(x,\varphi(x))$ appearing in Theorem 3. We propose, as our result, apply the method of successive approximations in order to obtain an analytical solution of the resulting linear fractional differential equations. Let us first consider $f(x,\varphi(x)) = \lambda\varphi(x)$ in Theorem 3.

Theorem 4. *Let $\alpha,\lambda \in \mathbb{R}^*$ such that $n-1 < \alpha \leq n$, where $n \in \mathbb{N}$. If $\varphi \in L(a,b)$, then the Cauchy problem*

$$({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\varphi)(x) = \lambda\varphi(x) \tag{34}$$

$$({}^{\rho}_{k}\mathscr{J}^{(1-\nu)(kn-\alpha)-k(n-j)}_{a^+}\varphi)(a^+) = b_j,\ b_j \in \mathbb{R},\ (j = 1,2,\ldots,n), \tag{35}$$

admits a unique solution in the space $L(a,b)$, given by

$$\varphi(x) = \sum_{j=1}^{n} b_j\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\Lambda-j} E_{k,\alpha,k(\Lambda-j+1)}\left[\lambda\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\frac{\alpha}{k}}\right], \tag{36}$$

where $E_{k,\xi,\sigma}(\cdot)$ is defined in Eq.(7).

Proof. According to Theorem 3, we just need to solve the Volterra integral equation, Eq.(33), with $f(t,\varphi(t)) = \lambda\varphi(t)$. As the Volterra integral equation of the second kind admits a unique solution [41], the uniqueness of Eq.(33) is guaranteed. In order to find the exact solution, we use the method of successive approximations, that is, we consider

$$\varphi_0 = \sum_{j=1}^{n} \frac{b_j}{\Gamma_k[k(\Lambda-j+1)]}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\Lambda-j} \tag{37}$$

$$\varphi_i(x) = \varphi_0(x) + \frac{\lambda}{k\Gamma_k(\alpha)}\int_a^x \left(\frac{x^{\rho}-t^{\rho}}{\rho}\right)^{\frac{\alpha}{k}-1} t^{\rho-1}\varphi_{i-1}(t)\,dt. \tag{38}$$

We define a parameter Λ_m by

$$\Lambda_m = \frac{\nu(nk-\alpha)+\alpha m}{k} \quad \text{with} \quad m = 1,2,\ldots,i+1. \tag{39}$$

In case $m=1$, we have $\Lambda_1=\Lambda$ given by Eq.(25). Thus, from Eq.(37) and Eq.(38), we can write

$$\begin{aligned}\varphi_1(x) &= \varphi_0(x)+\frac{\lambda}{k\Gamma_k(\alpha)}\int_a^x\left(\frac{x^\rho-t^\rho}{\rho}\right)^{\frac{\alpha}{k}-1}t^{\rho-1}\varphi_0(t)\,dt\\ &= \varphi_0(x)+\sum_{j=1}^{n}\frac{\lambda\, b_j}{\Gamma_k[k(\Lambda-j+1)]}\left[{}^{\rho}_{k}\mathscr{I}^{\alpha}_{a^+}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\Lambda-j}\right](x).\end{aligned}$$

Using Theorem 2, we obtain

$$\begin{aligned}\varphi_1(x) &= \sum_{j=1}^{n}\frac{b_j}{\Gamma_k[k(\Lambda-j+1)]}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Lambda-j}+\sum_{j=1}^{n}\frac{b_j}{\Gamma_k[k(\Lambda_2-j+1)]}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Lambda_2-j}\\ &= \sum_{j=1}^{n}b_j\sum_{m=1}^{2}\frac{\lambda^{m-1}}{\Gamma_k[k(\Lambda_m-j+1)}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Lambda_m-j}. \end{aligned}\tag{40}$$

Similarly, using Eq.(37), Eq.(40) and Theorem 2, we obtain the expression for $\varphi_2(x)$, that is,

$$\begin{aligned}\varphi_2(x) &= \varphi_0(x)+\frac{\lambda}{k\Gamma_k(\alpha)}\int_a^x\left(\frac{x^\rho-t^\rho}{\rho}\right)^{\frac{\alpha}{k}-1}t^{\rho-1}\varphi_1(t)\,dt\\ &= \varphi_0(x)+\lambda\sum_{j=1}^{n}b_j\sum_{m=1}^{2}\frac{\lambda^{m-1}}{\Gamma_k[k(\Lambda_m-j+1)]}\left[{}^{\rho}_{k}\mathscr{I}^{\alpha}_{a^+}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\Lambda_m-j}\right](x)\\ &= \sum_{j=1}^{n}b_j\sum_{m=1}^{3}\frac{\lambda^{m-1}}{\Gamma_k[k(\Lambda_m-j+1)}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Lambda_m-j}.\end{aligned}$$

Repeating this process, we obtain the expression for $\varphi_i(x)$, with $i\in\mathbb{N}$:

$$\varphi_i(x)=\sum_{j=1}^{n}b_j\sum_{m=1}^{i+1}\frac{\lambda^{m-1}}{\Gamma_k[k(\Lambda_m-j+1)}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Lambda_m-j}.$$

Taking $i\to\infty$, we obtain the explicit solution for $\varphi(x)$

$$\varphi(x)=\sum_{j=1}^{n}b_j\sum_{m=1}^{\infty}\frac{\lambda^{m-1}}{\Gamma_k[k(\Lambda_m-j+1)}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Lambda_m-j}.$$

Changing the summation index, $m\to m+1$, we have

$$\varphi(x)=\sum_{j=1}^{n}b_j\sum_{m=0}^{\infty}\frac{\lambda^{m}}{\Gamma_k[k(\Lambda_{m+1}-j+1)]}\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Lambda_{m+1}-j}.$$

Moreover, we can rewrite this last expression in terms of k-new generalized Mittag-Leffler function, that is,

$$\varphi(x)=\sum_{j=1}^{n}b_j\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Lambda-j}E_{k,\alpha,k(\Lambda-j+1)}\left[\lambda\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\frac{\alpha}{k}}\right].\tag{41}$$

As another application, we consider $f(x,\varphi(x))=\lambda({}^{\rho}_{k}\mathscr{D}^{\beta,\nu}_{a^+}\varphi)(x)$ in Theorem 3.

Theorem 5. *Let $\alpha,\beta\in\mathbb{R}$, $\alpha>\beta>0$, $n-1<\alpha\le n$, $n\in\mathbb{N}$ and $\lambda\in\mathbb{R}$. Then, the Cauchy problem*

$$({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\varphi)(x)=\lambda({}^{\rho}_{k}\mathscr{D}^{\beta,\nu}_{a^+}\varphi)(x)$$

$$({}^{\rho}_{k}\mathscr{I}^{(1-\nu)(kn-\alpha)-k(n-j)}_{a^+}\varphi)(a^+)=b_j,\ b_j\in\mathbb{R},\ (j=1,2,\ldots,n),$$

admits a unique solution in the space $L(a,b)$, given by

$$\varphi(x)=\sum_{j=1}^{n}b_j\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\Theta-j}E_{k,\alpha-\beta,k(\Theta-j+1)}\left[\lambda\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\frac{\alpha-\beta}{k}}\right],$$

where $\Theta=\dfrac{\alpha+\nu(nk-\alpha+\beta)}{k}$.

*Proof.*Suppose the solution $\varphi = ({}^{\rho}_{k}\mathscr{J}^{\beta}_{a^+}g)(x) \in L(a,b)$, then

$({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\,{}^{\rho}_{k}\mathscr{J}^{\beta}_{a^+}g)(x) = \lambda({}^{\rho}_{k}\mathscr{D}^{\beta,\nu}_{a^+}\,{}^{\rho}_{k}\mathscr{J}^{\beta}_{a^+}g)(x).$

By Lemma 2, we can write

$({}^{\rho}_{k}\mathscr{D}^{\alpha,\nu}_{a^+}\,{}^{\rho}_{k}\mathscr{J}^{\beta}_{a^+}g)(x) = \lambda g(x),$

and by Lemma 3, we have

$({}^{\rho}_{k}\mathscr{J}^{\beta-\alpha}_{a^+}g)(x) = \lambda g(x) \quad \text{or} \quad ({}^{\rho}_{k}\mathscr{D}^{\alpha-\beta,\nu}_{a^+}g)(x) = \lambda g(x).$

We shall use the second expression. Thus, let $\Upsilon_m = \dfrac{(\alpha-\beta)m+\alpha+\beta+\nu(nk-\alpha+\beta)}{k}$; in the case $m=1$ we denote $\Upsilon_m = \Upsilon$. Taking $\alpha \to \alpha-\beta$ in Theorem 4, we can write

$$g(x) = \sum_{j=1}^{n} b_j \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Upsilon-j} E_{k,\alpha-\beta,k(\Upsilon-j+1)}\left[\lambda\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\frac{\alpha-\beta}{k}}\right]. \tag{42}$$

As $\varphi(x) = ({}^{\rho}_{k}\mathscr{J}^{\beta}_{a^+}g)(x)$, we apply the operator $({}^{\rho}_{k}\mathscr{J}^{\beta}_{a^+})$ on both sides of Eq.(42), in order to obtain

$$({}^{\rho}_{k}\mathscr{J}^{\beta}_{a^+}g)(x) = \sum_{j=1}^{n} b_j \sum_{m=0}^{\infty} \frac{\lambda^m}{\Gamma_k[k(\Upsilon_m - j+1)]}\left[{}^{\rho}_{k}\mathscr{J}^{\beta}_{a^+}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\Upsilon_m-j}\right](x).$$

Using Theorem 2 and rewriting the expression, we obtain

$$\varphi(x) = \sum_{j=1}^{n} b_j \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Theta-j} E_{k,\alpha-\beta,k(\Theta-j+1)}\left[\lambda\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\frac{\alpha-\beta}{k}}\right].$$

In the next theorem we consider a sequence of linear fractional differential equations of order αn. This theorem generalizes the results presented in [24].

Theorem 6.*Let $\alpha,\beta \in \mathbb{R}$, $\alpha > \beta > 0$, $n-1 < \alpha \le n$, $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$. Then, the Cauchy problem*

$$({}^{\rho}_{k}\mathscr{D}^{\alpha n,\nu}_{a^+}\varphi)(x) = \lambda\varphi(x) \tag{43}$$

$$({}^{\rho}_{k}\mathscr{J}^{(1-\nu)(kn-\alpha n)-k(n-j)}_{a^+}\varphi)(a^+) = b_j,\ b_j \in \mathbb{R},\ (j=1,2,\ldots,n), \tag{44}$$

admits a unique solution in the space $L(a,b)$, given by

$$\varphi(x) = \sum_{j=1}^{n} b_j \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda_n-j} E_{k,\alpha n,k(\Lambda_n-j+1)}\left[\lambda_n\left(\frac{x^\rho-a^\rho}{\rho}\right)^{\frac{\alpha n}{k}}\right], \tag{45}$$

where $\Lambda_n = \dfrac{\nu(nk-\alpha n)+\alpha n}{k}$.

*Proof.*We consider $\alpha \to \alpha n$ in Theorem 4; we thus obtain the solution, Eq.(45).

6 Dependence on Initial Conditions

In this section, we present the changes in a solution entailed by small changes in initial conditions. Consider Eq.(31) with the following changes in the initial conditions shown in Eq.(32):

$$({}^{\rho}_{k}\mathscr{J}^{(1-\nu)(kn-\alpha)-k(n-j)}_{a^+}\varphi)(a^+) = b_j + \eta_j, \quad b_j \in \mathbb{R},\ (j=1,2,\ldots,n), \tag{46}$$

where η_j $(j=1,\ldots,n)$ are arbitrary constants.

Theorem 7. *Suppose that the hypotheses of Theorem 3 are satisfied. Let $\varphi(x)$ and $\tilde{\varphi}(x)$ be solutions of the initial value problems Eq.(31)-Eq.(32) and Eq.(31)-Eq.(46), respectively. Then,*

$$|\varphi(x)-\tilde{\varphi}(x)| \leq \sum_{j=1}^{n} |\eta_j| \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\frac{v(nk-\alpha)+\alpha}{k}-j} E_{k,\alpha,\alpha+v(nk-\alpha)-k(j-1)}\left[A\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\frac{\alpha}{k}}\right],$$

with $x \in (a,b]$, where $E_{k,\xi,\sigma}(z)$ is the k-Mittag-Leffler function, Eq.(7).

Proof. According to Theorem 4, we have

$$\varphi(x) = \lim_{i\to\infty} \varphi_i(x)$$

where $\varphi_0(x)$ is given by Eq.(37) and

$$\varphi_i(x) = \varphi_0(x) + \frac{1}{k\Gamma_k[\alpha]} \int_a^x \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\frac{\alpha}{k}-1} t^{\rho-1} f(t,\varphi_{i-1}(t))dt. \tag{47}$$

We also have

$$\tilde{\varphi}(x) = \lim_{i\to\infty} \tilde{\varphi}_i(x), \tag{48}$$

$$\tilde{\varphi}_0(x) = \sum_{j=1}^{n} \frac{(b_j+\eta_j)}{\Gamma_k[k(\Lambda-j+1)]} \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda-j}, \tag{49}$$

$$\tilde{\varphi}_i(x) = \tilde{\varphi}_0(x) + \frac{1}{k\Gamma_k[\alpha]} \int_a^x \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\frac{\alpha}{k}-1} t^{\rho-1} f(t,\tilde{\varphi}_{i-1}(t))dt, \quad i = 1,2,\ldots \tag{50}$$

From Eq.(37) and Eq.(49), we can write

$$|\varphi_0(x)-\tilde{\varphi}_0(x)| \leq \sum_{j=1}^{n} \frac{|\eta_j|}{\Gamma_k[k(\Lambda-j+1)]} \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda-j}. \tag{51}$$

We finally consider Eq.(47) and Eq.(50) with $i=1$, the Lipschitz condition for function $f(t,\varphi)$, Definition 2, the inequality Eq.(51) and Theorem 2, in order to obtain

$$\begin{aligned}
|\varphi_1(x)-\tilde{\varphi}_1(x)| &\leq \sum_{j=1}^{n} \frac{|\eta_j|}{\Gamma_k[k(\Lambda-j+1)]} \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda-j} \\
&+ \frac{A}{k\Gamma_k[\alpha]} \int_a^x \left(\frac{x^\rho - t^\rho}{\rho}\right)^{\frac{\alpha}{k}-1} t^{\rho-1} |f(t,\varphi_0(t)) - f(t,\tilde{\varphi}_0(t))|dt \\
&\leq \sum_{j=1}^{n} \frac{|\eta_j|}{\Gamma_k[k(\Lambda-j+1)]} \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda-j} \\
&+ \frac{A}{k\Gamma_k[\alpha]} \int_a^x \left(\frac{x^\rho - t^\rho}{\rho}\right)^{\frac{\alpha}{k}-1} t^{\rho-1} |\varphi_0(t) - \tilde{\varphi}_0(t)|dt \\
&\leq \sum_{j=1}^{n} \frac{|\eta_j|}{\Gamma_k[k(\Lambda-j+1)]} \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda-j} \\
&+ \frac{A}{k\Gamma_k[\alpha]} \sum_{j=1}^{n} \frac{|\eta_j|}{\Gamma_k[k(\Lambda-j+1)]} \int_a^x \left(\frac{x^\rho - t^\rho}{\rho}\right)^{\frac{\alpha}{k}-1} t^{\rho-1} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\Lambda-j} \\
&= \sum_{j=1}^{n} |\eta_j| \sum_{m=1}^{2} \frac{A^{m-1}}{\Gamma_k[k(\Lambda_m-j+1)]} \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda_m-j},
\end{aligned}$$

where Λ_m is given by Eq.(39). Thus, continuing this procedure, we obtain

$$|\varphi_i(x)-\tilde{\varphi}_i(x)| \leq \sum_{j=1}^{n} |\eta_j| \sum_{m=1}^{i+1} \frac{A^{m-1}}{\Gamma_k[k(\Lambda_m-j+1)]} \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda_m-j}.$$

Taking $i \to \infty$ and $m \to m+1$, it follows that

$$|\varphi(x) - \tilde{\varphi}(x)| \leq \sum_{j=1}^{n} |\eta_j| \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\Lambda - j} E_{k,\alpha,k(\Lambda-j+1)}\left[A\left(\frac{x^\rho - a^\rho}{\rho}\right)^{\frac{\alpha}{k}}\right].$$

7 Conclusion

We presented a generalization for a fractional derivative recently discussed in [3], obtained by inserting a new parameter in its definition. This generalization, for adequate values of its parameters, recovers a wide list of definitions of classical fractional derivatives. We presented some properties of this generalized (k,ρ)-fractional derivative. Furthermore, we discussed the equivalence between a Cauchy problem, using this operator of fractional differentiation, and a Volterra integral equation of the second kind. Finally, we considered some particular cases for this Cauchy problem, and proved that small changes on initial conditions entail small changes in the solution of the problem.

A natural continuation of this paper consists in verifying the Leibniz's rule, or product rule, for the generalized (k,ρ)-fractional derivative and to investigate the validity of the fundamental theorem of fractional calculus. Studies in this direction have already begun [40].

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