Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/040303

A New Approach for Solving Fractional Optimal Control Problems Using Shifted Ultraspherical Polynomials

Hoda F. Ahmed[1] *and Marina B. Melad*[2,*]

[1] Department of Mathematics, Faculty of Science, Minia University, Minia, Egypt
[2] Department of Mathematics, Faculty of Science, Assuit University Branch of New- Vally, Assuit, Egypt

Received: 28 Jul. 2017, Revised: 6 Oct. 2017, Accepted: 4 Nov. 2017
Published online: 1 Jul. 2018

Abstract: We propose a numerical scheme for solving the one- and two-dimensional fractional optimal control problems (FOCPs). The suggested scheme is established by using the operational matrix (OM) of the Riemann-Liouville fractional integral (RLFI) of the shifted Gegenbauer polynomials (SGPs). These polynomials generalize the shifted Legendre and shifted Chebyshev polynomials, and are special cases of the Jacobi polynomials. By employing the proposed technique, the FOCP is converted into a variational problem. The Gegenbauer- Gauss quadrature method (GGQM) and the Rayleigh-Ritz method (RRM) are implemented to convert the obtained variational problem into a system of algebraic equations (AEs) which is easy to solve. Numerical results of some examples including the one- and two-dimensional FOCPs are shown to prove the validity of the investigated technique.

Keywords: Riemann-Liouville fractional integral operator, fractional optimal control problems, operational matrix, shifted Gegenbauer polynomials, Rayleigh-Ritz method.

1 Introduction

Recently, numerous applications in diverse areas scientific areas of engineering and science have been expressed in the form of fractional differential equations (FDEs) or fractional functional equations (FFEs). This is the reason why the fractional derivatives give more precise performance of these applications [1,2,3,4].

The optimal control (OC) theory is a mathematical branch which has been under progress for years, however the FOC theory is a novel subject. The FOCPs are those optimal control problems with constraints expressed by FDEs. FOCPs are specified according to the used fractional derivatives. Those familiar fractional derivatives are the Riemann-Liouville (RL) and Caputo fractional derivatives. FOCPs have also received intensive consideration in various applications. Materials with memory and hereditary effects, dynamical processes containing gas diffusion, and heat conduction in fractal porous media are sufficiently displayed by fractional-order models than by integer-order models [5]. Other applications of FOCPs are given in Refs. [6,7,8]. Several numerical schemes have been established to solve these problems because most of these problems don't have exact solutions.

FOCPs with RL fractional derivatives were first presented in Ref. [9] by using the fractional variational principle and the Lagrange multiplier technique; while the FOCPs were expressed by the Caputo fractional derivatives [10,11]. Also the polynomial and rational approximations were utilized to solve such problems [12,13]. The optimal solutions for multiple control problems of Sobolev type with nonlocal nonlinear FDEs were investigated in Ref. [14]. The existence of OCs for linear time-invariant neutral control systems with different fractional orders is discussed in Ref. [15].

Direct numerical techniques based on the OMs of fractional integral of various orthogonal polynomials have been derived and applied to solve different kinds of FDEs; such as Jacobi polynomials [16], shifted orthonormal Jacobi polynomials [17], Legendre polynomials [18,19], Laguerre polynomials [20], Bernstein polynomials [21], and Bernoulli polynomials [22]. Gegenbauer polynomials have many useful properties; they achieve rapid rates of convergence for small ranges of the spectral expansion terms. Therefor some studies are interested in using these polynomials for solving various kinds of DEs (more details are found in Refs. [23,24,25,26,27]). To the best of our knowledge, little studies deal

* Corresponding author e-mail: marina.benyamin.melad@gmail.com

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with the application of GPs in handling FDEs [28,29]. This encourages us for using such kind of polynomials wishing to employ them in numerous practical applications. Another motivation is that the Chebyshev and Legendre polynomials can be considered as special cases of the GPs. During this paper, we investigate a new OM of the RL fractional integral of the SGPs and utilize it to solve numerically the following FOCPs with the RL fractional derivative

$$Min.J = \int_0^t f(t,x(t),u(t))dt, \tag{1}$$

under the constraint,

$$D^{(\nu)}x(t) = g(t,x(t)) + b(t)u(t), \tag{2}$$

with the initial condition,

$$D^{(i)}x(0) = x_i, \quad i = 0,1,\dots,m-1,$$

where $m-1 < \nu \leq m$ and $b(t) \neq 0$.

The proposed technique can be briefed in the subsequent steps:

1. Using the SGPs in approximating "$D^\nu(x)$" with unidentified coefficients.
2. Using the OM of fractional integrals and couple the resultant equation of dynamic constraint (2) with the performance index (1) to create a new variational problem.
3. Using SGQM to approximate the integration in the obtained variational problem, which may be not easy to compute.
4. By using the RRM, the new variational problem is transformed into a system of AEs which is easily solved.

The central importance of the suggested method is that by using a few numbers of GPs, acceptable results are attainted.

This paper is organized as follows. In Section 2, some preliminaries of fractional calculus and GPs are given. In Section 3, the SGOM of RLFI is derived. In Section 4, the convergence of the suggested technique is discussed. In Section 5, the proposed technique of applying SGOM of fractional integration for solving FOCPs is presented. In Section 6, some explanatory examples are shown. The last section is devoted to a conclusion.

2 Preliminaries and Used Formulae

2.1 Fractional calculus

Definition 1.

One of the popular definitions of fractional integral is the RL, which is determined as

$$\begin{aligned} &I^\nu f(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-\xi)^{\nu-1} f(\xi)d\xi, \quad m-1 < \nu \leq m, \quad m \in N, \quad \nu > 0, \quad t > 0, \\ &I^0 f(t) = f(t). \end{aligned} \tag{3}$$

The operator I^ν has properties, according to Ref. [30], we just recall the next property

$$I^\nu t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\nu+\beta+1)} t^{\nu+\beta}. \tag{4}$$

Definition 2.

D^ν is the RL fractional derivative of order ν which is given as

$$D^\nu f(t) = \frac{d^m}{dx^m}(I^{m-\nu} f(t)), \quad m-1 < \nu \leq m, \quad m \in N, \quad \nu \in R, \tag{5}$$

where m is the smallest integer order greater than ν.

Lemma 1.

If $m-1 < \nu \leq m$, $m \in N$, then

$$D^\nu I^\nu f(t) = f(t),$$

$$I^\nu D^\nu f(t) = f(t) - \sum_{i=0}^{m-1} f^{(i)}(0^+)\frac{t^i}{i!}, \quad t > 0. \tag{6}$$

For more details see Ref. [30].

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2.2 Shifted ultraspherical (Gegenbauer) polynomials and their properties

The ultraspherical (Gegenbauer) polynomials $C_j^{(\alpha)}(t)$, of degree $j \in \mathbb{Z}^+$, and associated with the parameter $\alpha > \frac{-1}{2}$ are a sequence of real polynomials in the finite domain $[-1,1]$. They are a set of orthogonal polynomials which have many applications [23].

Definition 3.

The GPs are the Jacobi polynomials, $P_j^{(\alpha,\beta)}$, with $\alpha = \beta = \alpha - \frac{1}{2}$ so that

$$C_j^{(\alpha)}(t) = \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(j+2\alpha)}{\Gamma(2\alpha)\Gamma(j+\alpha+\frac{1}{2})} P_j^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(t), \quad j = 0,1,2,\ldots$$

- GPs have useful relations to the Chebyshev polynomials of the first, second kind, and the Legendre polynomials as follows

$$T_j(t) \equiv \frac{j}{2} \lim_{\alpha\to 0} \alpha^{-1} C_j^{(\alpha)}(t), \quad j \geq 1,$$

$$C_j^{(1)}(t) \equiv \frac{1}{j+1} U_j(t),$$

and

$$L_j(t) \equiv C_j^{(\frac{1}{2})}(t),$$

respectively.

- The GPs can be created from the next recurrence equation

$$(j+2\alpha)C_{j+1}^{(\alpha)}(t) = 2(j+\alpha)tC_j^{(\alpha)}(t) - jC_{j-1}^{(\alpha)}(t), \quad j = 1,2,\ldots$$

with

$$C_0^{(\alpha)}(t) = 1, \quad C_1^{(\alpha)}(t) = t.$$

- The orthogonality relation of the GPs is given by the weighted inner product

$$\left\langle C_i^{(\alpha)}(t), C_j^{(\alpha)}(t) \right\rangle = \int_{-1}^{1} C_i^{(\alpha)}(t) C_j^{(\alpha)}(t) \omega^{(\alpha)}(t) dt = \lambda_j^{(\alpha)} \delta_{i,j},$$

where $\omega^{(\alpha)}(x)$ is the weight function, it is an even function given from the relation

$$\omega^{(\alpha)}(t) = (1-t^2)^{\alpha-\frac{1}{2}},$$

and

$$\lambda_j^{(\alpha)} = \|C_j^{(\alpha)}(t)\|^2 = \frac{2^{1-2\alpha}\pi\Gamma(j+2\alpha)}{j!(j+\alpha)\Gamma^2(\alpha)}, \tag{7}$$

is the normalization factor and $\delta_{i,j}$ is the Kronecker delta function.

- These polynomials will be used in the interval $[0,L]$, so the SGPs are formed by replacing the variable t with $\frac{2t}{L} - 1$, $0 \leq t \leq L$, can be written as

$$C_{S,j}^{(\alpha)}(t) = C_j^{(\alpha)}(\frac{2t}{L} - 1), \quad C_{S,0}^{(\alpha)}(t) = 1, \quad C_{S,1}^{(\alpha)}(t) = \frac{2t}{L} - 1.$$

- The analytical form of the SGP is given by

$$\begin{aligned} C_{S,j}^{(\alpha)}(t) &= \sum_{k=0}^{j} (-1)^{j-k} \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(j+k+2\alpha)}{\Gamma(k+\alpha+\frac{1}{2})\Gamma(2\alpha)(j-k)!k!L^k} t^k, \\ C_{S,j}^{(\alpha)}(0) &= (-1)^j \frac{\Gamma(j+2\alpha)}{\Gamma(2\alpha)j!}. \end{aligned} \tag{8}$$

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•The orthogonal relation of SGPs is obtained from

$$\left\langle C^{(\alpha)}_{S,i}(t), C^{(\alpha)}_{S,j}(t)\right\rangle = \int_0^L C^{(\alpha)}_{S,i}(t)C^{(\alpha)}_{S,j}(t)\omega^{(\alpha)}_{S}(t)dt = \lambda^{(\alpha)}_{S,j}\delta_{i,j}, \tag{9}$$

where $\omega^{(\alpha)}_{S}(t)$ is the weight function, it is an even function given from the relation

$$\omega^{(\alpha)}_{S}(t) = (tL-t^2)^{\alpha-\frac{1}{2}},$$

and

$$\lambda^{(\alpha)}_{S,j} = \left(\frac{L}{2}\right)^{2\alpha}\lambda^{(\alpha)}_{j}.$$

•This polynomial recovers the shifted Chebyshev polynomial of the first kind $T_{S,j}(t) \equiv C^{(0)}_{S,j}(t)$, the shifted Legendre polynomial $L_{S,j}(t) \equiv C^{(\frac{1}{2})}_{S,j}(t)$, and the shifted Chebyshev polynomial of the second kind $C^{(1)}_{S,j}(t) \equiv \frac{1}{j+1}U_{S,j}(t)$.

•The square integrable function $y(t)$ in $[0,L]$ can be approximated by SGPs as:

$$y(t) = \sum_{j=0}^{N}\tilde{y}_j C^{(\alpha)}_{S,j}(t),$$

where the coefficients $\tilde{y}_j$ are obtained from

$$\tilde{y}_j = (\lambda^{(\alpha)}_{S,j})^{-1}\int_0^L y(t)\omega^{(\alpha)}_{S}(t)C^{(\alpha)}_{S,j}(t)dt. \tag{10}$$

•The approximation of function $y(t)$ in the vector form is defined by

$$y(t) = Y^T\phi(t), \tag{11}$$

where $Y^T = [\tilde{y}_0, \tilde{y}_1, \ldots, \tilde{y}_N]$ is the shifted Gegenbauer coefficient vector, and

$$\phi(t) = \left[C^{(\alpha)}_{S,0}(t), C^{(\alpha)}_{S,1}(t), \ldots, C^{(\alpha)}_{S,N}(t)\right]^T \tag{12}$$

is the shifted Gegenbauer vector.

•The q times repeated integration of Gegenbauer vector can be extracted from

$$I^q\phi(t) \simeq P^{(q)}\phi(t), \tag{13}$$

where $P^{(q)}$ is called the OM of the integration of $\phi(t)$.

3 Fractional Shifted Gegenbauer Operational Matrix (SGOM) of Integration

At this section, SGOM of RL fractional integral is proved.

Theorem(1)

Let $\phi(t)$ be the shifted Gegenbauer vector and $\nu > 0$ then

$$I^\nu\phi(t) \simeq P^{(\nu)}\phi(t), \tag{14}$$

where $t \in [0,1]$ and $P^{(\nu)}$ is called OM of fractional integration of order ν in the RL sense, it is a square matrix of order $(N+1)\times(N+1)$ is written as follows:

$$P^{(\nu)} = \begin{pmatrix} \sum_{k=0}^{0}\xi_{0,0,k} & \sum_{k=0}^{0}\xi_{0,1,k} & \cdots & \sum_{k=0}^{0}\xi_{0,N,k} \\ \sum_{k=0}^{1}\xi_{1,0,k} & \sum_{k=0}^{1}\xi_{1,1,k} & \cdots & \sum_{k=0}^{1}\xi_{1,N,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=0}^{i}\xi_{i,0,k} & \sum_{k=0}^{i}\xi_{i,1,k} & \cdots & \sum_{k=0}^{i}\xi_{i,N,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=0}^{N}\xi_{N,0,k} & \sum_{k=0}^{N}\xi_{N,1,k} & \cdots & \sum_{k=0}^{N}\xi_{N,N,k} \end{pmatrix} \tag{15}$$

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where $\xi_{i,j,k}$ is given by:

$$\xi_{i,j,k} = \Xi \times \Upsilon,$$

where

$$\Xi = \sum_{k=0}^{i} (-1)^{i-k} \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(i+k+2\alpha)}{\Gamma(k+\alpha+\frac{1}{2})\Gamma(2\alpha)\Gamma(k+\nu+1)(i-k)!},$$

$$\Upsilon = \sum_{f=0}^{j} (-1)^{j-f} \frac{j!(j+\alpha)\Gamma^2(\alpha)\Gamma^2(\alpha+\frac{1}{2})\Gamma(2\alpha+j+f)\Gamma(\nu+k+f+\alpha+\frac{1}{2})}{2^{(1-4\alpha)}\pi\Gamma(2\alpha+j)\Gamma(2\alpha)\Gamma(\alpha+f+\frac{1}{2})(j-f)!f!\Gamma(\nu+k+f+2\alpha+1)} \tag{16}$$

Proof
From relation (8) and by using Eqs. (3) and (4), we can write

$$I^{\nu} C_{S,i}^{(\alpha)}(t) = \sum_{k=0}^{i} (-1)^{i-k} \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(i+k+2\alpha)}{\Gamma(k+\alpha+\frac{1}{2})\Gamma(2\alpha)(i-k)!k!} I^{\nu}(t^k), \quad t \in [0,1]$$

$$= \sum_{k=0}^{i} (-1)^{i-k} \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(i+k+2\alpha)}{\Gamma(k+\alpha+\frac{1}{2})\Gamma(2\alpha)(i-k)!\Gamma(\nu+k+1)} t^{k+\nu}, \quad i = 0,1,2,\ldots,N. \tag{17}$$

The function $t^{k+\nu}$ can be written as a series of $N+1$ terms of Gegenbauer polynomials,

$$t^{k+\nu} = \sum_{j=0}^{N} \tilde{t}_j C_{S,j}^{(\alpha)}(t), \tag{18}$$

Where

$$\tilde{t}_j = \sum_{f=0}^{j} (-1)^{j-f} \frac{j!(j+\alpha)\Gamma^2(\alpha)\Gamma^2(\alpha+\frac{1}{2})\Gamma(2\alpha+j+f)\Gamma(\nu+k+f+\alpha+\frac{1}{2})}{2^{(1-4\alpha)}\pi\Gamma(2\alpha+j)\Gamma(\alpha+f+\frac{1}{2})(j-f)!f!\Gamma(2\alpha)\Gamma(\nu+k+f+2\alpha+1)}. \tag{19}$$

Now, by employing equations (17)- (19) we obtain:

$$I^{\nu} C_{S,i}^{(\alpha)}(t) = \sum_{k=0}^{i} \sum_{j=0}^{N} (-1)^{i-k} \frac{\Gamma(\alpha+\frac{1}{2})\Gamma(i+k+2\alpha)}{\Gamma(k+\alpha+\frac{1}{2})\Gamma(2\alpha)(i-k)!\Gamma(\nu+k+1)} \tilde{t}_j C_{S,j}^{(\alpha)}(t),$$

$$= \sum_{j=0}^{N} \left(\sum_{k=0}^{i} \xi_{i,j,k} \right) C_{S,j}^{(\alpha)}(t), \quad i = 0,1,\ldots,N, \tag{20}$$

where $\xi_{i,j,k}$ is obtained from Eq. (16).
Writing the last equation in a vector form gives

$$I^{\nu} C_{S,i}^{(\alpha)}(t) \simeq \left[\sum_{k=0}^{i} \xi_{i,0,k}, \sum_{k=0}^{i} \xi_{i,1,k}, \ldots, \sum_{k=0}^{i} \xi_{i,N,k} \right] \phi(t), \quad i = 0,1,\ldots,N, \tag{21}$$

which ends our proof.

4 Error Estimation and Convergence Analysis

4.1 Error estimation

In the following theorem, the error estimation for the approximated functions is obtained in terms of Gram determinant [31].

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Theorem(2):
For the Hilbert space $H = L^2[0,1]$, suppose that Y be a closed subspace of H such that $Y = Span\left\{C^{(\alpha)}_{S,0}(t), C^{(\alpha)}_{S,1}(t), \ldots, C^{(\alpha)}_{S,N}(t)\right\}$. Let $y(t)$ be an arbitrary element of H and $y^*(t)$ be the unique best approximation of $y(t)$ out of Y, then

$$\| y(t) - y^*(t) \|^2 = \frac{Gram(y(t), C^{(\alpha)}_{S,0}(t), C^{(\alpha)}_{S,1}(t), \ldots, C^{(\alpha)}_{S,N}(t))}{Gram(C^{(\alpha)}_{S,0}(t), C^{(\alpha)}_{S,1}(t), \ldots, C^{(\alpha)}_{S,N}(t))} \tag{22}$$

where $Gram(y(t), C^{(\alpha)}_{S,0}(t), C^{(\alpha)}_{S,1}(t), \ldots, C^{(\alpha)}_{S,N}(t))$

$$= \begin{vmatrix} < y(t), y(t) > & < y(t), C^{(\alpha)}_{S,0}(t) > & . & . & . & < y(t), C^{(\alpha)}_{S,N}(t) > \\ < C^{(\alpha)}_{S,0}(t), y(t) > & < C^{(\alpha)}_{S,0}(t), C^{(\alpha)}_{S,0}(t) > & . & . & . & < C^{(\alpha)}_{S,0}(t), C^{(\alpha)}_{S,N}(t) > \\ < C^{(\alpha)}_{S,1}(t), y(t) > & < C^{(\alpha)}_{S,1}(t), C^{(\alpha)}_{S,0}(t) > & . & . & . & < C^{(\alpha)}_{S,1}(t), C^{(\alpha)}_{S,N}(t) > \\ . & . & & & & . \\ . & . & . & . & . & . \\ . & . & & & & . \\ < C^{(\alpha)}_{S,N}(t), y(t) > & < C^{(\alpha)}_{S,N}(t), C^{(\alpha)}_{S,0}(t) > & . & . & . & < C^{(\alpha)}_{S,N}(t), C^{(\alpha)}_{S,N}(t) > \end{vmatrix}$$

4.2 Convergence analysis

Suppose that the error, E_{I^ν} of the integration OM in RL sense as

$$E_{I^\nu} = P^\nu \Phi(t) - I^\nu \Phi(t),$$

where

$$E_{I^\nu} = \left[E_{I^\nu,0}, E_{I^\nu,1}, ., ., ., E_{I^\nu,N}\right]^T,$$

is an error vector. From Eq. (17), we had approximated $t^{k+\nu}$ as $\sum_{j=0}^{N} \tilde{t}_j C^{\alpha}_{S,j}(t)$. From above theorem we have

$$\left\| t^{k+\nu} - \sum_{j=0}^{N} \tilde{t}_j C^{\alpha}_{S,j}(t) \right\|_2 = \left(\frac{Gram(t^{k+\nu}, C^{(\alpha)}_{S,0}(t), C^{(\alpha)}_{S,1}(t), \ldots, C^{(\alpha)}_{S,N}(t))}{Gram(C^{(\alpha)}_{S,0}(t), C^{(\alpha)}_{S,1}(t), \ldots, C^{(\alpha)}_{S,N}(t))} \right)^{\frac{1}{2}} \tag{23}$$

From Eq. (20), we obtain the upper bound of the integration OM as follows

$$\|E_{I^\nu,i}\|_2 = \left\| I^\nu C^{\alpha}_{S,i}(t) - \sum_{j=0}^{N} \left(\sum_{k=0}^{i} \xi_{i,j,k} \right) C^{(\alpha)}_{S,j}(t) \right\|, i = 0, ..., N, \tag{24}$$

$$\leq \sum_{k=0}^{i} \left| \frac{\Gamma(\alpha + \frac{1}{2})\Gamma(i+k+2\alpha)}{\Gamma(k+\alpha+\frac{1}{2})\Gamma(2\alpha)(i-k)!\Gamma(\nu+k+1)} \right| \left(\frac{Gram(t^{k+\nu}, C^{(\alpha)}_{S,0}(t), C^{(\alpha)}_{S,1}(t), \ldots, C^{(\alpha)}_{S,N}(t))}{Gram(C^{(\alpha)}_{S,0}(t), C^{(\alpha)}_{S,1}(t), \ldots, C^{(\alpha)}_{S,N}(t))} \right)^{\frac{1}{2}} \tag{25}$$

The following theorem illustrates that with increasing the number of Gegenbauer polynomials the error tend to zero.

Theorem (3):
Assume that function $y(t) \in L^2[0,1]$ is estimated by $g_N(t)$ as follows

$$g_N(t) = \mu_0 C^{\alpha}_{S,0}(t) + \mu_1 C^{\alpha}_{S,1}(t) + \ldots + \mu_N C^{\alpha}_{S,N}(t),$$

where

$$\mu_i = \int_0^1 C^{\alpha}_{S,i}(t) y(t) dt, \quad i = 0, \ldots, N.$$

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Consider

$$s_N(y) = \int_0^1 [y(t) - g_N(t)]^2 \, dt,$$

then we have

$$\lim_{N\to\infty} s_N(y) = 0.$$

For the proof see Ref. [32].

5 Application of SGOM of Fractional Integration for FOCPs

In this section, we use SGOM of integration to solve problem (1) with the dynamic constraint (2) as in the following.

5.1 Shifted Gegenbauer approximation

Firstly, approximating $D^\nu x(t)$ by SGPs, $C_{S,j}^{(\alpha)}(t)$, as

$$D^\nu x(t) \simeq X^T \phi(t), \tag{26}$$

where X is an unknown coefficients matrix which takes the form

$$\begin{pmatrix} \tilde{x}_0 \\ \tilde{x}_1 \\ . \\ . \\ . \\ \tilde{x}_N \end{pmatrix}$$

By using (6),we have

$$I^\nu D^\nu x(t) = x(t) - \sum_{i=0}^{m-1} x^{(i)}(0^+) \frac{t^i}{i!}, \tag{27}$$

From Eq.(14) together with Eq.(26), we get

$$I^\nu D^\nu x(t) \simeq X^T P^\nu \phi(t), \tag{28}$$

From Eq.(28) and Eq. (27), we obtain

$$x(t) \simeq X^T P^\nu \phi(t) + \sum_{i=0}^{m-1} x^{(i)}(0) \frac{t^i}{i!}, \tag{29}$$

Using the Eqs. (26)- (29), the dynamic constraint (2) takes the form

$$X^T \phi(t) = g\left(t, X^T P^\nu \phi(t) + \sum_{i=0}^{m-1} x^{(i)}(0) \frac{t^i}{i!}\right) + b(t)u(t),$$

$$u(t) = \frac{1}{b(t)} \left(X^T \phi(t) - g\left(t, X^T P^\nu \phi(t) + \sum_{i=0}^{m-1} x^{(i)}(0) \frac{t^i}{i!}\right)\right). \tag{30}$$

By using the Eqs. (29) and (30), the performance index (1) takes the form

$$J_N[\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_N] = \int_0^t f\left(t, X^T P^\nu \phi(t) + \sum_{i=0}^{m-1} x^{(i)}(0) \frac{t^i}{i!}, \frac{1}{b(t)} \left(X^T \phi(t) - g\left(t, X^T P^\nu \phi(t) + \sum_{i=0}^{m-1} x^{(i)}(0) \frac{t^i}{i!}\right)\right)\right) dt \tag{31}$$

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5.2 Gegenbauer-Gauss quadrature

Secondly, the integral in Eq. (31) is more difficult to compute, so the Gegenbauer- Gauss quadrature formula is used for approximating it as:

$$J_N[\tilde{x}_0,\tilde{x}_1,\ldots,\tilde{x}_N]=\sum_{j=0}^{N}\tilde{\varpi}_j^{(\alpha)}f(\tilde{t}_j^{(\alpha)}),0\leq j\leq N \tag{32}$$

where $\tilde{t}_j^{(\alpha)}$ are the zeros of Gegenbauer- Gauss quadrature in the interval (0,1), and

$$\tilde{\varpi}_j^{(\alpha)}=\left(\frac{L}{2}\right)^{(2\alpha)}\varpi_j^{(\alpha)},$$

is the Christoffel numbers, where $\varpi_j^{(\alpha)}$ is obtained from the relation

$$(\varpi_j^{(\alpha)})^{-1}=\sum_{k=0}^{N}(\lambda_k^{(\alpha)})^{-1}(C_k^{(\alpha)}(t_j^{(\alpha)}))^2,$$

where $\lambda_k^{(\alpha)}$ calculated from the Eq. (7).

The necessary optimality conditions of Eq. (32) are attained by applying RRM as

$$\frac{\partial J_N}{\partial\tilde{x}_0}=\frac{\partial J_N}{\partial\tilde{x}_1}=\ldots=\frac{\partial J_N}{\partial\tilde{x}_N}=0, \tag{33}$$

By using Newton iterative method, this system of AEs can be solved for the unknown coefficients of the vector X.

5.3 Approximation of our problem

Here, the set of Gegenbauer polynomials, $C_{S,N}^{\alpha}(t)$ is used for a basis form the space $D_1[0,1]=\{y(t):y$ is continuously differentiable on interval $[0,1]\}$, with uniform norm $\parallel y\parallel=\parallel y\parallel_\infty+\parallel\acute{y}\parallel_\infty$. Let us consider $M_n=\theta_0C_{S,0}^{\alpha}(t)+\theta_1C_{S,1}^{\alpha}(t)+\ldots+\theta_nC_{S,n}^{\alpha}(t)$, where M_n is the n-dimensional subspace of $D_1[0,1]$ and $\theta_0,\theta_1,\ldots,\theta_n$ are arbitrary real numbers. If we choose $\theta_0,\theta_1,...,\theta_n$ in such a way that M_n minimizes J, denoting the minimum by σ_n. Then, we should have $M_n\subset M_{n+1}$, this implies $\sigma_n\geq\sigma_{n+1}$.

Theorem (4):
Consider the functional J then $\lim_{n\to\infty}\sigma_n=\sigma$ where $\sigma=\inf_{x\in D_1[0,1]}J$.
(Check [33], [34] for the proof).

6 Illustrative Problems

Problem (1)

Consider the following FOCP [34]

$$Min.J=\frac{1}{2}\int_0^1(x^2(t)+u^2(t))dt,$$

under the dynamic constraints

$$\begin{aligned}D^{\nu}x(t)&=-x(t)+u(t),\quad 0\leq\nu\leq 1\\ x(0)&=1.\end{aligned}$$

The exact solution of this problem at $\nu=1$ is

$$\begin{aligned}x(t)&=\cosh(\sqrt{2}t)+\beta\sinh(\sqrt{2}t),\\ u(t)&=(1+\sqrt{2}\beta)\cosh(\sqrt{2}t)+(\sqrt{2}+\beta)\sinh(\sqrt{2}t),\end{aligned}$$

where

$$\beta=-\frac{\cosh(\sqrt{2})+\sqrt{2}\sinh(\sqrt{2})}{\sqrt{2}\cosh(\sqrt{2})+\sinh(\sqrt{2})}.$$

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Table 1: The absolute errors of the state variable x(t) for problem (1) at different values of N.

t	Absolute errors (N=3)	Absolute errors (N=5)	Absolute errors (N=8)	Absolute errors (N=10)
0	1.25437×10^{-3}	6.25467×10^{-6}	6.26213×10^{-10}	1.98861×10^{-9}
0.1	3.30159×10^{-4}	2.39179×10^{-6}	1.22351×10^{-10}	6.29444×10^{-10}
0.2	4.86069×10^{-4}	1.21248×10^{-6}	3.57127×10^{-11}	5.40695×10^{-10}
0.3	7.78748×10^{-5}	1.7249×10^{-6}	1.11152×10^{-10}	2.81709×10^{-10}
0.4	3.34676×10^{-4}	6.82411×10^{-7}	1.53137×10^{-10}	2.13957×10^{-10}
0.5	4.57932×10^{-4}	1.93055×10^{-6}	6.82487×10^{-12}	5.58041×10^{-10}
0.6	2.30996×10^{-4}	3.10922×10^{-7}	1.46338×10^{-10}	3.82721×10^{-10}
0.7	2.02962×10^{-4}	1.9004×10^{-6}	1.17524×10^{-10}	1.53133×10^{-10}
0.8	5.2129×10^{-4}	9.16645×10^{-7}	2.17553×10^{-11}	5.698×10^{-10}
0.9	2.42861×10^{-4}	2.49026×10^{-6}	1.0693×10^{-10}	7.5132×10^{-10}
1	1.25411×10^{-3}	6.25466×10^{-6}	5.86238×10^{-10}	2.68463×10^{-9}

Table 2: The absolute errors of the control variable u(t) for problem (1) at different values of N.

t	Absolute errors (N=3)	Absolute errors (N=5)	Absolute errors (N=8)	Absolute errors (N=10)
0	3.77566×10^{-4}	1.88239×10^{-6}	3.27507×10^{-10}	8.1688×10^{-11}
0.1	1.09654×10^{-4}	7.0819×10^{-7}	6.36999×10^{-11}	2.70723×10^{-11}
0.2	1.42091×10^{-4}	3.9987×10^{-7}	1.83641×10^{-11}	1.54725×10^{-11}
0.3	8.61549×10^{-6}	4.98381×10^{-7}	5.90761×10^{-11}	9.72683×10^{-12}
0.4	1.13021×10^{-4}	2.49281×10^{-7}	8.13922×10^{-11}	2.06191×10^{-11}
0.5	1.3787×10^{-4}	5.81012×10^{-7}	3.49132×10^{-12}	2.08388×10^{-11}
0.6	5.73272×10^{-5}	4.96686×10^{-8}	7.88284×10^{-11}	1.56014×10^{-11}
0.7	7.57957×10^{-5}	5.92684×10^{-7}	6.40287×10^{-11}	3.26667×10^{-11}
0.8	1.60967×10^{-4}	2.40909×10^{-7}	1.19774×10^{-11}	8.78833×10^{-11}
0.9	6.26788×10^{-5}	7.611×10^{-7}	5.92166×10^{-11}	1.10133×10^{-11}
1	3.77513×10^{-4}	1.88239×10^{-6}	3.25376×10^{-10}	1.70672×10^{-10}

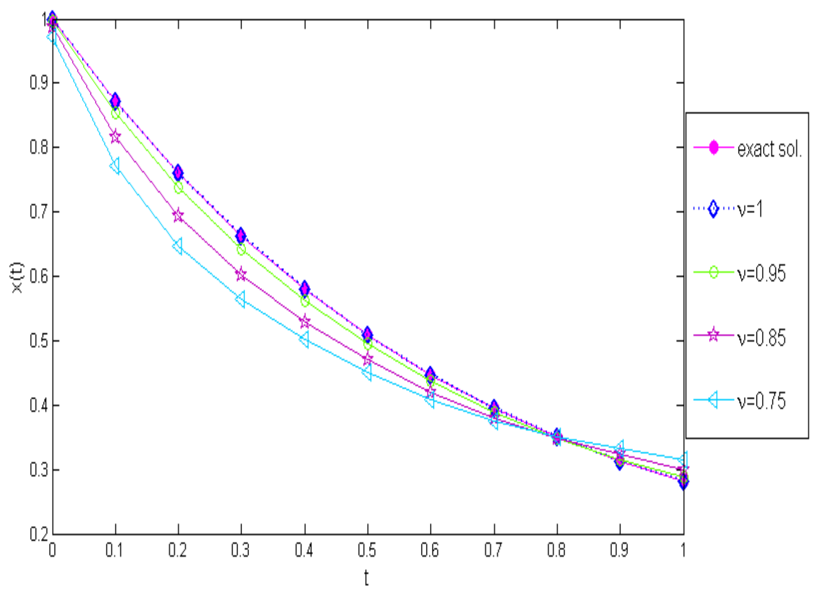

Fig. 1: The behavior of $x(t)$ for $N=5$ and $\nu=0.75,0.85,0.95,1,$ with the exact solution for problem (1)

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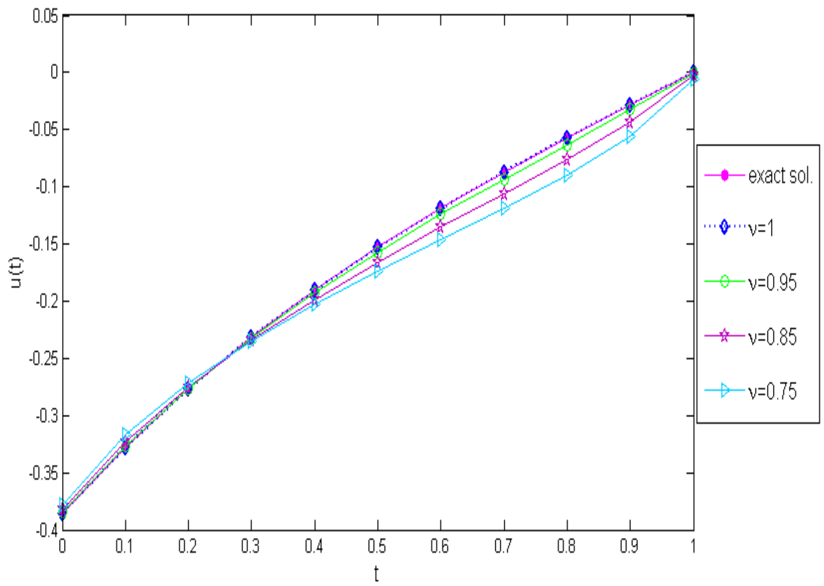

Fig. 2: The behavior of $u(t)$ for $N = 5$ and $\nu = 0.75, 0.85, 0.95, 1,$ with the exact solution for problem (1)

By applying our proposed technique to problem (1), the resultant numerical results for the state and the control variables are displayed through Figures 1 and 2, respectively at $\nu = 0.75, 0.85, 0.95, 1$ with the exact solutions for N=5. We noted that the obtained solutions cover the classical results when the value of the fractional order tends to unity. In addition, as in Tables 1 and 2, the absolute errors of the state variable x(t) and the control variable u(t) for problem (1) are calculated at different choices of N. It's observed that the efficiency of our proposed method increases by increasing N.

Problem (2)

Consider the following FOCP [9,34]

$$Min.J = \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))dt,$$

under the constraints

$$\begin{aligned} &D^\nu x(t) = tx(t) + u(t), \quad 0 \le \nu \le 1, \\ &x(0) = 1. \end{aligned}$$

Table 3: Approximate values of J at different values of ν and N=8 for problem (2)

ν	Present method	Method in [17]	Method in [35]
1	0.484268	0.48426	0.48427
0.99	0.483463	0.48346	0.48347
0.90	0.475883	0.47588	0.47605
0.80	0.466978	0.46697	0.46722

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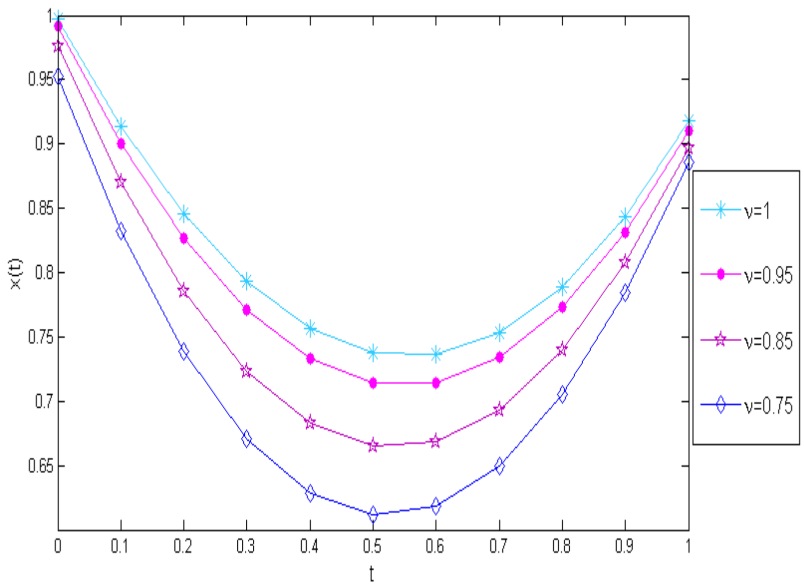

Fig. 3: The behavior of $x(t)$ **for** $N = 3$ **and** $\nu = 0.75, 0.85, 0.95, 1$ **for problem (2)**

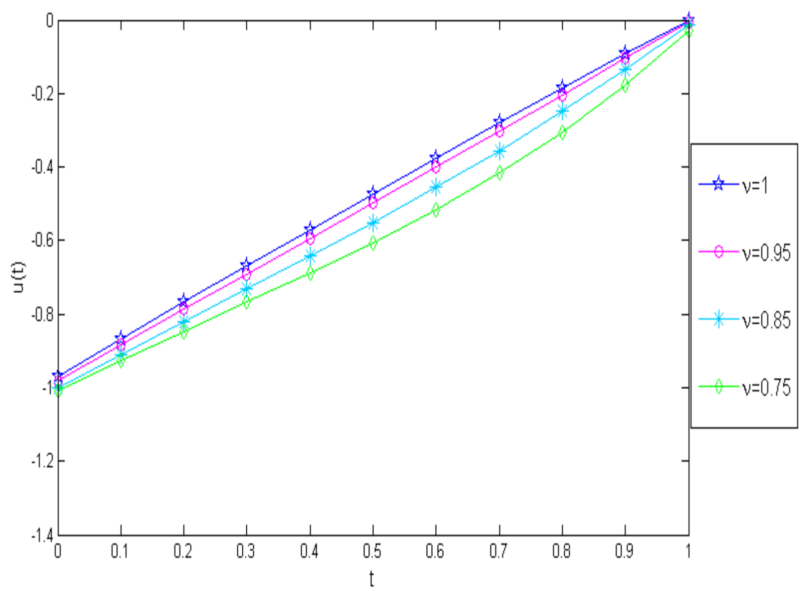

Fig. 4: The behavior of $u(t)$ **for** $N = 3$ **and** $\nu = 0.75, 0.85, 0.95, 1$ **for problem (2)**

In Figures 3 and 4, the approximated results of the variables $x(t)$ and $u(t)$ of problem (2) are plotted for different values of ν. In Table 3, comparisons of our obtained results for the minimum values of J of problem (2) with different values of ν at N=8 compared with results in [17] and [35] are tabulated. Obviously, our estimated results coincides with the results in [17] and [35].

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Problem (3)
Consider the following FOCP [36]

$$Min.J = \int_0^1 \left(\left(x(t) - t^2\right)^2 + \left(u(t) + t^4 - \frac{20t^{\frac{9}{10}}}{9\Gamma(\frac{9}{10})}\right)^2 \right) dt,$$

subject to,

$$D^{\nu}x(t) = t^2 x(t) + u(t), \quad 1 \leq \nu \leq 2,$$
$$x(0) = \dot{x}(0) = 0.$$

Table 4: Approximate values of J at various choices of N and $\nu = 1.1$ for problem (3)

N	Present method	Method in [36]
4	2.23277×10^{-6}	4.76932×10^{-6}
5	8.24619×10^{-7}	1.47243×10^{-6}
6	3.56358×10^{-7}	5.37825×10^{-7}
8	9.08978×10^{-8}	1.06099×10^{-7}
9	5.12433×10^{-8}	5.44304×10^{-8}

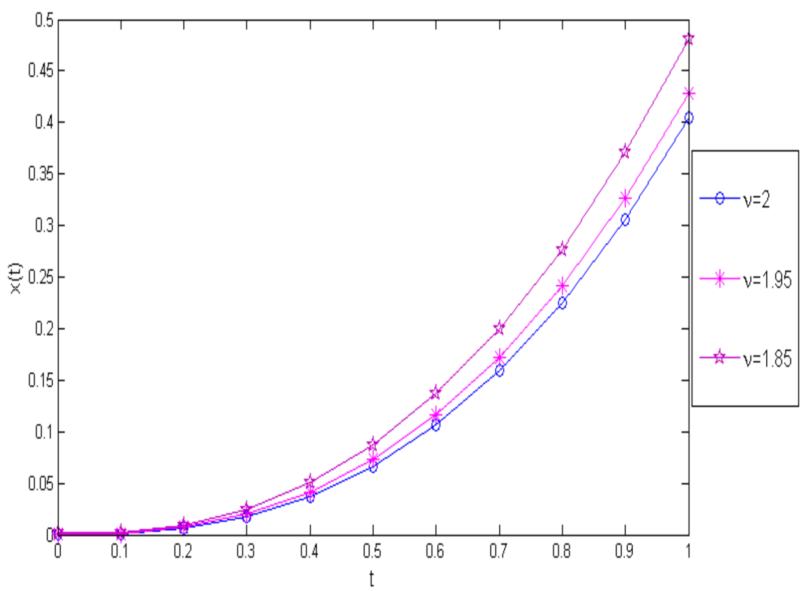

Fig. 5: The behavior of $x(t)$ for $N = 3$ and $\nu = 1.85, 1.95, 2$ for problem (3)

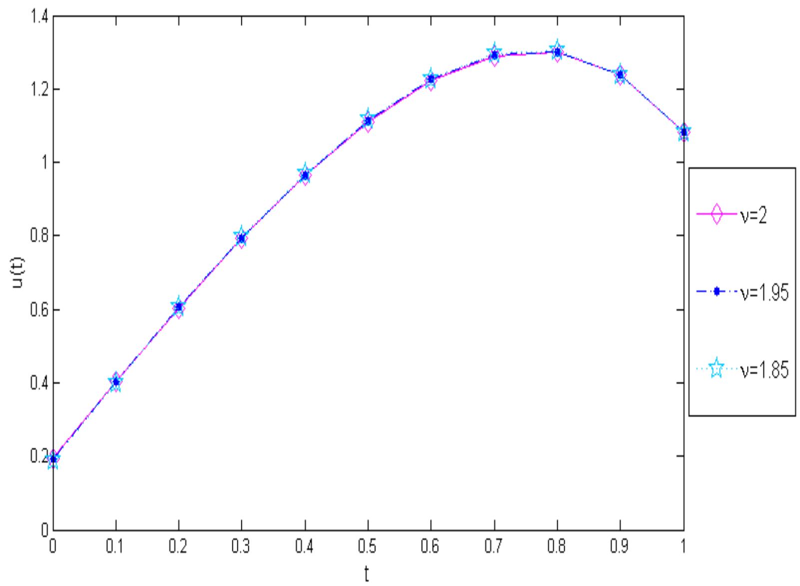

Fig. 6: The behavior of $u(t)$ for $N = 3$ and $\nu = 1.85, 1.95, 2$ for problem (3)

Figures 5 and 6 depict the behavior $x(t)$ and $u(t)$ of problem (3) at $N = 3$ and $\nu = 1.85, 1.95$ and 2. Table 4 lists the obtained results for the minimum values of J of problem (3) and the results in [36] for various choices of N. It is noted that the obtained results by using the suggested technique have high accuracy in comparison with [36].

Problem (4)
Consider the following FOCS [37]

$$Min.J = \frac{1}{2}\int_0^1 (x_1^2(t) + x_2^2(t) + u^2(t))dt,$$

subject to

$$\begin{aligned} D^\nu x_1(t) &= -x_1(t) + x_2(t) + u(t), \\ D^\nu x_2(t) &= -2x_2(t), \\ x_1(0) &= x_2(0) = 1. \end{aligned}$$

This problem has exact solution at $\nu = 1$ as

$$\begin{aligned} x_1(t) &= 0.018352e^{\sqrt{2}t} + 2.48165e^{-\sqrt{2}t} - \frac{3e^{-2t}}{2}, \\ x_2(t) &= e^{-2t}, \\ u(t) &= 0.044305e^{\sqrt{2}t} - 1.0279322e^{-\sqrt{2}t} + \frac{e^{-2t}}{2}. \end{aligned}$$

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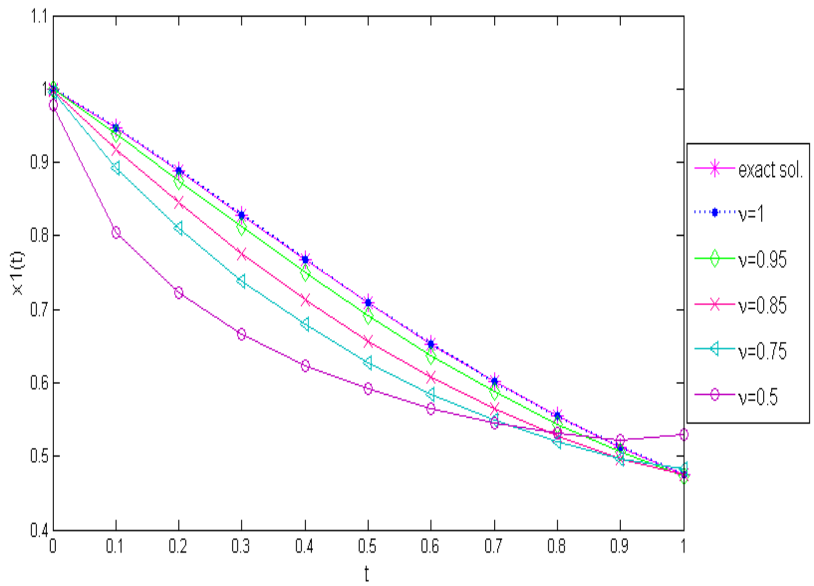

Fig. 7: The behavior of $x_1(t)$ for $N = 8$ and $\nu = 0.5, 0.75, 0.85, 0.95, 1,$ with the exact solution for problem (4)

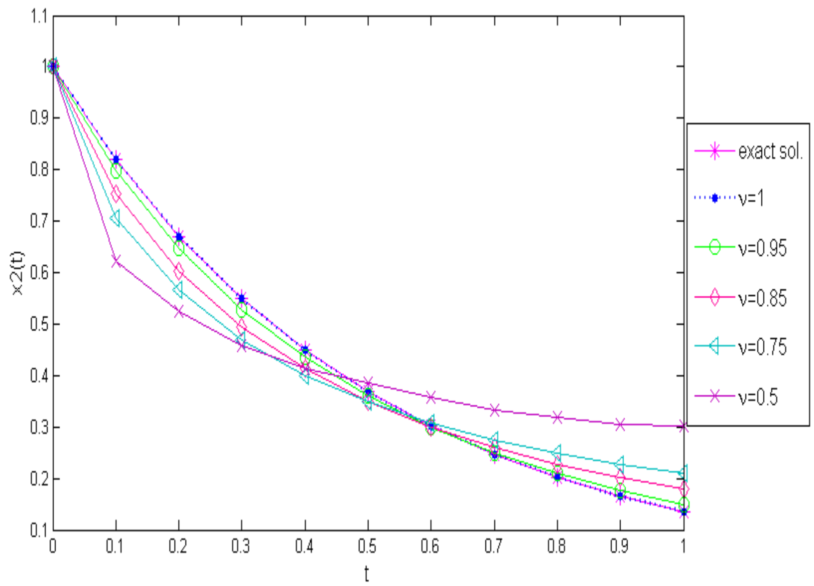

Fig. 8: The behavior of $x_2(t)$ for $N = 8$ and $\nu = 0.5, 0.75, 0.85, 0.95, 1,$ with the exact solution for problem (4)

Figures 7- 9 illustrate the behavior of state variables $x_1(t), x_2(t)$ and control variable u(t), respectively for $N = 8$ and $\nu = 0.5, 0.75, 0.85, 0.95$ and 1 with the exact solutions. At Tables 5- 7, the absolute errors of $x_1(t), x_2(t)$ and u(t) for problem (4) are calculated at different values of N. This problem was treated in Ref. [37] by a another technique. Our estimated results, shown in Figures 7- 9 agree with the results established in [37]. But we obtained good estimates by using at last 8 numbers of the SGP, whereas a number of approximations starting in 8 and increasing up to 128 are used in Ref. [37]. So we can deduce that our numerical technique takes less computational steps or power than that of Ref. [37].

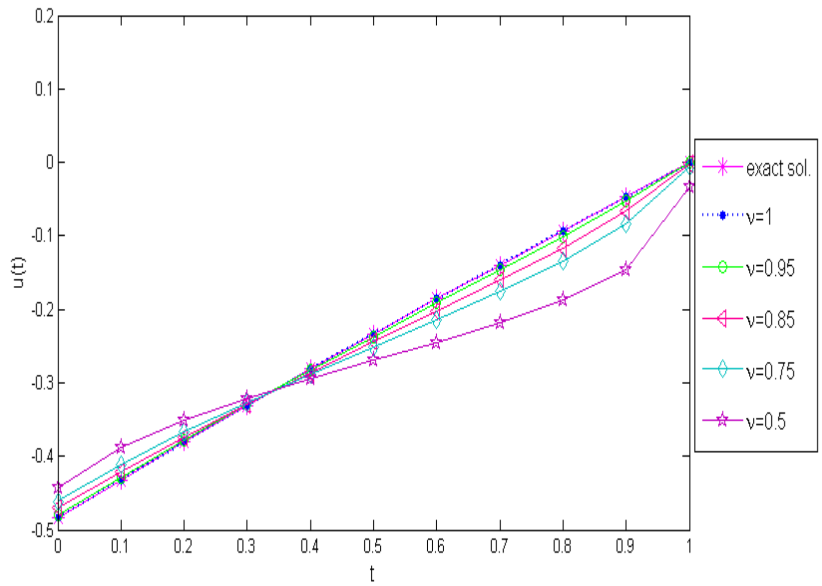

Fig. 9: The behavior of $u(t)$ for $N = 8$ and $\nu = 0.5, 0.75, 0.85, 0.95, 1,$ with the exact solution for problem (4)

Table 5: The absolute errors of $x_1(t)$ for problem (4) at different values of N

t	Absolute errors (N=3)	Absolute errors (N=5)	Absolute errors (N=7)	Absolute errors (N=8)
0	2.54299×10^{-3}	4.00336×10^{-5}	2.41907×10^{-7}	1.49311×10^{-8}
0.1	4.59984×10^{-4}	1.21716×10^{-5}	1.65758×10^{-8}	3.89009×10^{-9}
0.2	4.25396×10^{-4}	1.31491×10^{-5}	6.98381×10^{-8}	3.84311×10^{-10}
0.3	6.09729×10^{-4}	1.66176×10^{-5}	3.52928×10^{-8}	6.76104×10^{-10}
0.4	1.46752×10^{-3}	1.8063×10^{-6}	3.37502×10^{-8}	5.84229×10^{-9}
0.5	1.63318×10^{-3}	4.72386×10^{-6}	1.02771×10^{-7}	1.79443×10^{-9}
0.6	1.09888×10^{-3}	6.54949×10^{-6}	2.12302×10^{-8}	1.22336×10^{-9}
0.7	2.39567×10^{-4}	2.00603×10^{-5}	2.23335×10^{-8}	5.37209×10^{-9}
0.8	2.85639×10^{-4}	1.2638×10^{-5}	9.47197×10^{-8}	2.56206×10^{-9}
0.9	3.88259×10^{-4}	8.68881×10^{-6}	2.68914×10^{-8}	3.33488×10^{-10}
1	3.26993×10^{-3}	4.7211×10^{-5}	2.78275×10^{-7}	1.28264×10^{-8}

Table 6: The absolute errors of $x_2(t)$ for problem (4) at different values of N

t	Absolute errors (N=3)	Absolute errors (N=5)	Absolute errors (N=7)	Absolute errors (N=8)
0	2.77556×10^{-17}	1.11022×10^{-16}	2.60209×10^{-18}	1.01481×10^{-16}
0.1	4.13717×10^{-3}	4.42703×10^{-5}	1.52144×10^{-7}	6.80051×10^{-9}
0.2	3.9441×10^{-3}	1.72479×10^{-5}	9.02464×10^{-8}	8.14885×10^{-9}
0.3	2.20723×10^{-3}	1.0479×10^{-5}	1.54371×10^{-7}	7.91399×10^{-9}
0.4	6.333×10^{-4}	2.1207×10^{-5}	8.58013×10^{-8}	2.19109×10^{-9}
0.5	4.50871×10^{-5}	2.49134×10^{-5}	1.84447×10^{-8}	4.34762×10^{-9}
0.6	5.41637×10^{-4}	1.23822×10^{-5}	6.97336×10^{-8}	5.89944×10^{-9}
0.7	1.62945×10^{-3}	2.28276×10^{-6}	9.38397×10^{-8}	4.64195×10^{-10}
0.8	2.3299×10^{-3}	2.57998×10^{-6}	5.67684×10^{-9}	1.98716×10^{-9}
0.9	1.26715×10^{-3}	2.09563×10^{-5}	3.89227×10^{-8}	3.69174×10^{-9}
1	3.25981×10^{-3}	3.20173×10^{-5}	1.62211×10^{-7}	1.24681×10^{-8}

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Table 7: The absolute errors of $u(t)$ for problem (4) at different values of N

t	Absolute errors (N=3)	Absolute errors (N=5)	Absolute errors (N=7)	Absolute errors (N=8)
0	1.22508×10^{-3}	1.70068×10^{-5}	9.90901×10^{-8}	5.79697×10^{-9}
0.1	1.60038×10^{-4}	1.19007×10^{-6}	2.01279×10^{-8}	2.11329×10^{-9}
0.2	7.66304×10^{-5}	6.57099×10^{-6}	3.47713×10^{-8}	8.09481×10^{-10}
0.3	3.4207×10^{-4}	7.21889×10^{-6}	3.4856×10^{-9}	3.08876×10^{-10}
0.4	5.76103×10^{-4}	1.72718×10^{-6}	1.64181×10^{-8}	2.19585×10^{-9}
0.5	5.97729×10^{-4}	1.09956×10^{-6}	3.67555×10^{-8}	7.44108×10^{-10}
0.6	3.84806×10^{-4}	2.03656×10^{-6}	6.8909×10^{-9}	3.63189×10^{-10}
0.7	4.41158×10^{-5}	6.04497×10^{-6}	1.06384×10^{-8}	1.55692×10^{-9}
0.9	7.02178×10^{-5}	4.65966×10^{-6}	1.10424×10^{-10}	5.89212×10^{-10}
1	8.5567×10^{-4}	1.33851×10^{-5}	8.0739×10^{-8}	4.72284×10^{-9}

7 Conclusion

We derived a new numerical mechanism to find approximate solutions of the FOCPs, based on the SGOM of the RL fractional integral. The SGOM of fractional integration reduces the FOCP into an equivalent integral problem. The properties of the SGPs together with the RRM are used to transform the equivalent functional integral equation problem to an algebraic system, which is easily solved. The applicability, accuracy, and rapidity by using few terms of the SGPs of the suggested mechanism are demonstrated by numerical applications, including first- and second-dimensional FOCPs.

References

[1] P. J. Torvik and R. L. Bagley, On the appearance of the fractional derivative in the behavior of real materials, *J. Appl. Mech.* **51**, 294–298 (1984).
[2] V. Lakshmikantham and A. S. Vatsala, Basic theory of fractional differential equations, *Nonlin. Anal.Theor. Meth.Appl.* **69**, 2677–2682 (2008).
[3] N. H. Sweilam, M. M. Khader and A. M. Mahdy, On the numerical solution for the linear fractional Klein-Gordon equation using Legendre pseudospectral method, *I. J. Math. Comp. Appl. Res.* **2**, 1–10 (2012).
[4] A. C. Sparavigna and P. Milligan, Using fractional differentiation in astronomy, arXiv preprint arXiv:0910.4243, (2009).
[5] M. Zamani, M. Karimi-Ghartemani and N. Sadati, FOPID controller design for robust performance using particle swarm optimization, *Frac. Calc. Appl. Anal.* **10**, 169–187 (2007).
[6] G. W. Bohannan, Analog fractional order controller in temperature and motor control applications, *J. Vibr. Contr.* **14**, 1487–1498 (2008).
[7] I. S. Jesus and J. T. Machado, Fractional control of heat diffusion systems, *Nonlinear Dyn.* **54**, 263–282 (2008).
[8] J. I. Surez, B. M. Vinagre and Y. Chen, A fractional adaptation scheme for lateral control of an AGV, *J. Vibr.Contr.* **14**, 1499–1511 (2008).
[9] O. P. Agrawal, A general formulation and solution scheme for fractional optimal control problems, *Nonlinear Dyn.* **38**, 323–337 (2004).
[10] O. P. Agrawal, Fractional optimal control of a distributed system using eigenfunctions, *J. Comput.Nonlin. Dyn.* **3**, 021204 (2008).
[11] O. P. Agrawal, A quadratic numerical scheme for fractional optimal control problems, *J. Dyn. Sys. Measur. Cont.* **130**, 011010 (2008).
[12] C. Tricaud and Y. Q. Chen, An approximate method for numerically solving fractional order optimal control problems of general form, *Comput. Math. Appl.* **59**, 1644–1655 (2010).
[13] E. Tohidi and H. S. Nik, A Bessel collocation method for solving fractional optimal control problems, *Appl. Math. Mod.* **39**, 455–465 (2015).
[14] A. Debbouche, J. J. Nieto and D. F. Torres, Optimal solutions to relaxation in multiple control problems of Sobolev type with nonlocal nonlinear fractional differential equations, *J. Opt. Theor. Appl.* **174**, 7–31 (2017).
[15] D. I. Xiaoli and J. J. Nieto, Controllability and optimality of linear time-invariant neutral control systems with different fractional orders, *Acta Math. Sci.* **35**, 1003–1013 (2015).
[16] A. H. Bhrawy, E. H. Doha, D. Baleanu and S. S. Ezz-Eldien, A spectral tau algorithm based on Jacobi operational matrix for numerical solution of time fractional diffusion-wave equations, *J. Comp. Phys.* **293**, 142–156 (2015).
[17] E. H. Doha, A. H. Bhrawy, D. Baleanu, S. S. Ezz-Eldien and R. M. Hafez, An efficient numerical scheme based on the shifted orthonormal Jacobi polynomials for solving fractional optimal control problems, *Adv. Differ. Equ.* **2015**, 15 (2015).

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[18] G. H. Erjaee, M. H. Akrami and M. H. Atabakzadeh, The operational matrix of fractional integration for shifted Legendre polynomials, *Iranian J. Sci. Tech. Sci.* **237**, 439–444 (2013).
[19] E. H. Doha, A. H. Bhrawy and S. S. Ezz-Eldien, An efficient Legendre spectral tau matrix formulation for solving fractional subdiffusion and reaction subdiffusion equations, *J. Comput. Nonlin. Dyn.* **10**, 021019 (2015).
[20] A. H. Bhrawy, D. Baleanu and L. M. Assas, Efficient generalized Laguerre-spectral methods for solving multi-term fractional differential equations on the half line. *J. Vibr. Contr.* **20**, 973–985 (2014).
[21] A. Nemati, S. Yousefi, F. Soltanian and J. S. Ardabili, An efficient numerical solution of fractional optimal control problems by using the Ritz method and Bernstein operational matrix, *Asian J. Contr.* **18**, 2272–2282 (2016).
[22] M. Behroozifar and N. Habibi, A numerical approach for solving a class of fractional optimal control problems via operational matrix Bernoulli polynomials, *J. Vibr. Contr.* 1077546316688608 (2017).
[23] K. T. Elgindy, High order numerical solution of second order one dimensional hyperbolic telegraph equation using a shifted Gegenbauer pseudospectral method, *Num. Meth. Part. Differ. Equ.* **32**, 307–349 (2016).
[24] K. T. Elgindy and K. A. Smith-Miles, Optimal Gegenbauer quadrature over arbitrary integration nodes, *J. Comput. Appl. Math.* **242**, 82–106 (2013).
[25] W. M. Abd-Elhameed, Y. Youssri and E. H. Doha, New solutions for singular Lane-Emden equations arising in astrophysics based on shifted ultraspherical operational matrices of derivatives, *Comput. Meth. Differ. Equs.* **2**, 171–185 (2014).
[26] Y. H. Youssri, W. M. Abd-Elhameed and E. H. Doha, Ultraspherical wavelets method for solving LaneEmden type equations, *Rom. Journ. Phys.* **60(9)**, 1298–1314 (2015).
[27] E. H. Doha, W. M. Abd-Elhameed and Y. H. Youssri, New ultraspherical wavelets collocation method for solving 2nth-order initial and boundary value problems, *J. Egypt Math. Soc.* **24**, 319–327 (2016).
[28] W. M. Abd-Elhameed and Y. H. Youssri, New ultraspherical wavelets spectral solutions for fractional Riccati differential equations, *Abstr. Appl. Anal.* Article ID 626275, 8 pages (2014).
[29] M. M. Izadkhah and J. SaberiNadjafi, Gegenbauer spectral method for time- fractional convection diffusion equations with variable coefficients. *Math. Meth. Appl. Sci.* **38**, 3183–3194 (2015).
[30] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, (1993).
[31] E. Kreyszig, *Introductory functional analysis with applications*, New York, Wiley, (1989).
[32] T. J. Rivlin, *An introduction to the approximation of functions*, Courier Corporation, (2003).
[33] C. Phang, N. F. Ismail, A. Isah and J. R. Loh, A new efficient numerical scheme for solving fractional optimal control problems via a Genocchi operational matrix of integration, *J. Vibr. Contr.* doi:1077546317698909, (2017).
[34] A. Lotfi, M. Dehghan and S. A. Yousefi, A numerical technique for solving fractional optimal control problems, *Comput. Math. Appl.* **62**, 1055–1067 (2011).
[35] T. Akbarian and M. Keyanpour, A new approach to the numerical solution of fractional order optimal control problems, *Appl. Appl. Math.* **8**, 523–534 (2013).
[36] A. H. Bhrawy, E. H. Doha, D. Baleanu, S. S. Ezz-Eldien and M. A. Abdelkawy, An accurate numerical technique for solving fractional optimal control problems, *Adv. Differ. Equ.* **15**, 23 (2015).
[37] O. Defterli, A numerical scheme for two-dimensional optimal control problems with memory effect, *Comput. Math. Appl.* **59**, 1630–1636 (2010).

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