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Interactive Approach for Multi-level Multi-objective Fractional Programming Problem under hybrid uncertainty

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Abstract: In this paper, an interactive approach for solving multi-level multi-objective fractional programming (ML-MOFP) problem under hybrid uncertainty is developed. The proposed interactive approach makes an extension work of Shi and Xia [22]. In the current model the left-hand- and right-hand-side variables in the constraints are influenced by hybrid uncertainty (i.e. both fuzziness and randomness); represented by fuzzy random variables (FRVs). In the first phase, we make the best use of the chance-constrained programming approach and the α-cut approach to obtain the equivalent deterministic model of the ML-MOFP problem with FRVs. Then, the linear model of the crisp ML-MOFP problem is formulated. In the second phase, the interactive approach simplifies the ML-MOLP model by changing it into isolated multi-objective decision-making (MODM) problems, to avoid non-convexity. Also, each separate MODM problem of the linear model is solved by the ε-constraint method and the concept of satisfactoriness. Finally, illustrative example and comparison with the existing techniques are provided to indicate the efficiency of the interactive approach.

Keywords: Multi-level programming; Multi-objective programming; Fractional programming; Fuzzy chance-constrained programming; Fuzzy sets; ε-constraint method.

1 Introduction

Hierarchical decision structures are prevalent in government systems, competitive economic organizations, supply chains, agriculture, biofuel production, and so on [5]. The area of multi-level mathematical programming (MLMP) provides the art and science of making such decisions. Several mathematical models for such problems have been exhibited [1,5,17, 19].

The fundamental idea of MLMP methodology is that the first-level decision maker (FLDM) decides his objectives and choices, hence asks each inferior level of the association for their solutions, which obtained individually. The lower level decision makers' choices are then presented and altered by the FLDM in light of the general advantage for the association [1,5].

A significant amount of effort have been devoted to solve MLMP and many efficient algorithms have been proposed [1,2,5,6,17]. Shi and Xia [22] introduced interactive bi-level decision-making problems. Interactive fuzzy programming has been extended by Sakawa et al. [24] to thoroughly consider in solving MLMP problems under fuzziness.

The balance space approach was modified to solve MLMP problems by Abo-Sinna et al. [1]. Baky [5] presented fuzzy goal programming (FGP) methodology to tackle ML-MOLP problems. Interactive fuzzy random bi-level programming via fractile criterion optimization has been presented by Sakawa et al. [25]. Osman et al. [17] proposed an interactive methodology for tri-level MODM problems. Chen and Chen [7] utilized a fuzzy variable for relative satisfactions among leader- and -follower to solve the decentralized bi-level programming problem (BLPP). Arora and Gupta [2] exhibited an interactive FGP methodology for BLPP with the merits of dynamic programming.

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Also, a fuzzy programming approach for bi-level stochastic programming was studied by Modak et al. [16]. A fuzzy BLPP via nearest interval approximation technique and KKT optimality conditions has been introduced in [20].

Fractional programming problems originate from the fact that programming paradigm could better fit the genuine problems if the optimization of a proportion among the physical and economic quantities is considered [9]. In the course of recent decades, such problem has been one of the powerful planning tools. It is routinely used in engineering applications, business and different disciplines [8,11,26]. Recently, Lachhwani [14] introduced FGP methodology exhibited in [5,6] with some alterations for ML-MOFP problems.

Interactive FGP approach using Jacobian matrix for decentralized fractional BLPP has been introduced by Toksari and Bilim [26]. An interactive algorithm to a certain type of bi-level integer multi-objective fractional programming problem was studied by Emam [9].

Amid the previous two decades, the larger part of research on the multi-level programming issues have been focused on the deterministic form in which the coefficients and decision variables in the objective function and the constraints are thought to be fresh esteems. Be that as it may, in all actuality, it is normally hard to know accurately the estimations of the coefficients because of the presence of loose or indeterminate data while building up multi-level models [15,20,21].

There are two noteworthy sorts of approaches for handling vulnerabilities existing in decision making issues: fuzzy mathematical programming (FMP) and stochastic mathematical programming (SMP). FMP is compelling in managing choice issues under the fuzzy goals and in taking care of questionable coefficients of target capacity and requirements caused by imprecision and unclearness [6,21]. SMP is an augmentation of numerical programming to choice issues whose coefficients (input information) are not positively referred to but rather could be spoken to as possibilities or probabilities [10,11,16].

In true enhancement issues, the kind of uncertainty that gets huge consideration is "haphazardness" related with different right-hand side parameters in the requirements [10,12]. At the point when some right-hand side parameters are of stochastic elements and can be spoken to as likelihood conveyance, the chance-constrained programming (CCP) strategy can be utilized. Information once in a while can't be measured/gathered accurately. This uncertainty may happen in stochastic or non-stochastic (i.e. fuzzy) sense or both stochastic and fuzzy faculties together. During the time spent inferring models of CCP, consider that the conceivable estimations of the arbitrary parameters under the event of occasions as fuzzy numbers [16].

In this investigation, randomness and fuzziness are considered all the while as FRVs. The idea of FRVs was first presented by Kwakernaak [13]. Buckly [3,28] characterized fuzzy probabilities utilizing fuzzy numbers as parameters in probability density function and probability function. FGP strategy for tackling CCP issues including FRVs is as of late concentrated in [4,16]. Parametric ML-MOFP problem with fuzziness in the constraints has been exhibited by Osman et al. [18]. The proposed interactive approach makes an extension work of Shi and Xia [22].

Moreover, interactive mechanism for solving multi-level MODM problems simplifies these problems by changing them into isolated MODM problems at the different levels. In this way, the trouble related with non-convex numerical programming to get a compromise solution was avoided. likewise, the algorithm raised the satisfactoriness concept as only for the FLDM predilection [17,22].

The point of this paper is to build up an interactive approach for solving ML-MOFP problem with FRVs in the constraints. These FRVs represents the uncertainty in decision-making problems. In order to do so, the problem is first changed over into interval esteemed programming problem based on CCP procedure and α-cut of fuzzy sets. Then the ML-MOFP problem is changed over into its equivalent deterministic form using fuzzy partial order relation.

Then, the linear model of the deterministic ML-MOFP problem is formulated by extending the work of M. Chakraborty and S. Gupta [8]. Moreover, the interactive approach simplifies the linear model by converting it into isolated MODM problems. In addition to that, each separate MODM problem of the linear model is solved by the ε-constraint method and the concept of satisfactoriness.

Finally, An algorithm to clarify the developed interactive methodology for the ML-MOFP problem with FRVs is exhibited.

The rest of this paper is composed as takes after. Section 2 introduces some basic definitions and preliminary results. In Section 3, formulation of the ML-MOFP problem with FRVs is exhibited. Its proportionate deterministic model is formulated in Section 4. Section 5 develops the linear model of the problem. The interactive models for solving ML-MOFP Problem with FRVs are introduced in Section 6. An interactive algorithm for ML-MOFP Problems with FRVs is proposed in Section 7. A numerical illustration and correlation with the current techniques are given in in Section 8. Closing comments are given toward the end.

2 Preliminaries

In this area, some essential ideas and preparatory outcomes utilized as a part of this paper are quickly presented.

Definition 1 *Let R^1 be the set of all real numbers. Then a real fuzzy number $\tilde{a}$ is defined by its membership function $\mu_{\tilde{a}}(x)$ that satisfies:*

1. *A continuous mapping from R^1 to the closed internal $[0,\ 1]$.*
2. *$\mu_{\tilde{a}}(x) = 0$ for all $x \in (-\infty, a]$.*
3. *Strictly increasing and continuous on $[a,b]$.*
4. *$\mu_{\tilde{a}}(x) = 1$ for all $x \in [b,c]$.*
5. *Strictly decreasing and continuous on $[c,d]$.*
6. *$\mu_{\tilde{a}}(x) = 0$ for all $x \in [d, +\infty)$ [29].*

Definition 2 *A fuzzy number $\tilde{a}$ is said to be an $\mathscr{L}\mathscr{R}$-fuzzy number if*

$$\mu_{\tilde{a}}(x) = \begin{cases} \mathscr{L}\left(\frac{a-x}{\gamma^a}\right) & x \le a, \quad \gamma^a > 0, \\ \mathscr{R}\left(\frac{x-a}{\beta^a}\right) & x \ge a, \quad \beta^a > 0, \end{cases} \tag{1}$$

where a is the mean value of $\tilde{a}$ and γ^a and β^a are positive numbers expressing the left and right spreads of $\tilde{a}$ and reference functions $\mathscr{L}$, $\mathscr{R} : [0,1] \to [0,1]$ with $\mathscr{L}(1) = \mathscr{R}(1) = 0$ and $\mathscr{L}(0) = \mathscr{R}(0) = 1$ are non-increasing, continuous functions [29].

Utilizing its mean esteem and left and right spreads, and shape functions, such an $\mathscr{L}\mathscr{R}$-fuzzy number is emblematically composed as $\tilde{a} = (a, \gamma^a, \beta^a)_{LR}$

Definition 3 *The α-level set of the fuzzy parameter $\tilde{a}$, is defined as an ordinary set $L_\alpha(\tilde{a})$ for which the degree of its membership function exceeds the level set $\alpha \in [0,1]$, where [6,29]:*

$$L_\alpha(\tilde{a}) = \{a \in R^m \mid \mu_{\tilde{\mathbf{a}}}(x) \ge \ \alpha\} = \left\{a \in \left[\tilde{a}^L_\alpha,\ a^U_\alpha\right] \,\middle|\, \mu_{\tilde{\mathbf{a}}}(x) \ge \alpha,\ \right\},$$

where $\tilde{a}^L_\alpha = a - \gamma^a \mathscr{L}^{-1}(\alpha)$ and $\tilde{a}^U_\alpha = a + \beta^a \mathscr{R}^{-1}(\alpha)$.

For two $\mathscr{L}\mathscr{R}$-fuzzy numbers $\tilde{a} = (a, \gamma^a, \beta^a)_{LR}$ and $\tilde{b} = \left(b, \gamma^b, \beta^b\right)_{LR}$ the formula for the extended addition becomes [29]:

1. $(a, \gamma^a, \beta^a)_{LR} + \left(b, \gamma^b, \beta^b\right)_{LR} = \left(a+b, \gamma^a + \gamma^b, \beta^a + \beta^b\right)_{LR}$,
2. $(a, \gamma^a, \beta^a)_{LR} - \left(b, \gamma^b, \beta^b\right)_{LR} = \left(a-b, \gamma^a + \beta^b, \beta^a + \gamma^b\right)_{LR}$
3. $(a, \gamma^a, \beta^a)_{LR} \times \left(b, \gamma^b, \beta^b\right)_{LR} \cong \left(ab, a\gamma^b + b\gamma^a, a\beta^b + b\beta^a\right)_{LR} \quad if \ \ a > 0,\ \ b > 0,$
4. $\lambda(a, \gamma^a, \beta^a)_{LR} = \begin{cases} (\lambda a, \lambda\gamma^a, \lambda\beta^a)_{LR} & if \ \ \lambda \ge 0, \\ (\lambda a, -\lambda\beta^a, -\lambda\gamma^a)_{RL} & if \ \ \lambda < 0, \end{cases} \qquad \lambda$ is a scalar

All through this paper, we should take the ordering between two fuzzy numbers,$\tilde{a}$ and $\tilde{b}$. According to the following definition.

Definition 4 *Let $\tilde{a}_\alpha = \left[\tilde{a}^L_\alpha, \tilde{a}^U_\alpha\right]$ and $\tilde{b}_\alpha = \left[\tilde{b}^L_\alpha, \tilde{b}^U_\alpha\right]$ be two intervals. The order relations $\preccurlyeq_{LR}$ and $\prec_{LR}$ between $\tilde{a}_\alpha$ and $\tilde{b}_\alpha$ are defined as [20,27]:*

1. *$\tilde{a}_\alpha \preccurlyeq_{LR} \tilde{b}_\alpha$ if and only if $\tilde{a}^L_\alpha \le \tilde{b}^L_\alpha$ and $\tilde{a}^U_\alpha \le \tilde{b}^U_\alpha$,*
2. *$\tilde{a}_\alpha \prec_{LR} \tilde{b}_\alpha$ if and only if $\tilde{a}_\alpha \preccurlyeq_{LR} \tilde{b}_\alpha$ and $\tilde{a}_\alpha \ne \tilde{b}_\alpha$,*

Definition 5 *A fuzzy random variable is a random variable whose parameter is fuzzy number. Let $\tilde{X}$ be continuous random variable with fuzzy parameter $\widetilde{\theta}$ and P as fuzzy probability, then $\tilde{X}$ is said to be continuous fuzzy random variable with probability density function $f\left(x; \widetilde{\theta}\right)$ with the property [3,4]*

$$\int_{-\infty}^{\infty} f(x;\theta)dx = 1;\ \theta \in \widetilde{\theta}[\alpha]. \tag{2}$$

3 Problem Formulation

Consider the hierarchical system be made out of a p-level decision maker (DM). Let the DM at the i^{th}-level denoted by DM_i controls over the decision variable $\boldsymbol{x_i} = (x_{i1}, x_{i2}, \cdots, x_{in_i}) \in R^{n_i},\ i = 1,2,\cdots,p$. where $\boldsymbol{x} = (\boldsymbol{x_1}, \boldsymbol{x_2}, \cdots, \boldsymbol{x_p}) \in R^n$ and $n = \sum_{i=1}^{p} n_i$ and furthermore assumed that

$$F_i(\boldsymbol{x_1}, \boldsymbol{x_2}, \cdots, \boldsymbol{x_p}) \equiv F_i(\boldsymbol{x}) :\ R^{n_1} \times R^{n_2} \times \cdots \times R^{n_p} \to R^{m_i}, \quad i=1,2,\cdots,p, \tag{3}$$

are the vector of fractional objective functions for DM_i, $i = 1, 2, \cdots, p$. numerically, the ML-MOFP problem with FRVs in the constraints may be formulated as follows [1,5,7,19];

1^{st} ***Level***

$$\underbrace{max}_{\boldsymbol{x_1}} \; F_1(\boldsymbol{x}) = \underbrace{max}_{\boldsymbol{x_1}} \left(f_{11}(\boldsymbol{x}),\; f_{12}(\boldsymbol{x}),\; \cdots,\; f_{1k_1}(\boldsymbol{x}) \right), \tag{4}$$

where $\boldsymbol{x_2}, \boldsymbol{x_3}, \cdots, \boldsymbol{x_p}$ solves

2^{nd} ***Level***

$$\underbrace{max}_{\boldsymbol{x_2}} \; F_2(\boldsymbol{x}) = \underbrace{max}_{\boldsymbol{x_2}} \left(f_{21}(\boldsymbol{x}),\; f_{22}(\boldsymbol{x}),\; \cdots,\; f_{2k_2}(\boldsymbol{x}) \right), \tag{5}$$

$\vdots$

where $\boldsymbol{x_p}$ solves

p^{th} ***Level***

$$\underbrace{max}_{\boldsymbol{x_p}} \; F_p(\boldsymbol{x}) = \underbrace{max}_{\boldsymbol{x_p}} \left(f_{p1}(\boldsymbol{x}),\; f_{p2}(\boldsymbol{x}),\; \cdots,\; f_{pk_p}(\boldsymbol{x}) \right), \tag{6}$$

subject to

$$\text{Prob}\left[\sum_{j=1}^{n} \tilde{a}_{ij} x_j \le \tilde{b}_i\right] \ge 1 - p_i, \qquad i = 1, 2, \cdots, r_0, \tag{7}$$

$$\text{Prob}\left[\sum_{j=1}^{n} \tilde{e}_{ij} x_j \le \tilde{b}_i\right] \ge 1 - p_i, \qquad i = r_0 + 1, r_0 + 2, \cdots, r_1, \tag{8}$$

$$\text{Prob}\left[\sum_{j=1}^{n} \tilde{e}_{ij} x_j \le \tilde{b}_i\right] \ge 1 - p_i, \quad i = r_1 + 1, r_1 + 2, \cdots, m, \tag{9}$$

$$x_j \ge 0, \quad j = 1, 2, \cdots, n. \tag{10}$$

where

$$f_{ij}(\boldsymbol{x}) = \frac{N_{ij}(\boldsymbol{x})}{D_{ij}(\boldsymbol{x})} = \frac{\boldsymbol{c}_1^{ij}\boldsymbol{x_1} + \boldsymbol{c}_2^{ij}\boldsymbol{x_2} + \cdots + \boldsymbol{c}_p^{ij}\boldsymbol{x_p} + \alpha^{ij}}{\boldsymbol{d}_1^{ij}\boldsymbol{x_1} + \boldsymbol{d}_2^{ij}\boldsymbol{x_2} + \cdots + \boldsymbol{d}_p^{ij}\boldsymbol{x_p} + \beta^{ij}}, \quad i = 1, 2, \cdots, p, \quad j = 1, 2, \cdots, k_i. \tag{11}$$

Also, $\boldsymbol{c}_k^{ij}$ and $\boldsymbol{d}_k^{ij}$are n_j-dimensional row vector for the coefficient of the j^{th} decision vector of the i^{th} objective function; α^{ij} and β^{ij} are scalars.$\tilde{a}_{ij}$ and $\tilde{b}_i, (i = 1, 2, \cdots, r_0)$ are independent normally distributed FRVs. Also, $\tilde{b}_i, (i = r_0 + 1, r_0 + 2, \cdots, r_1)$ is exponentially distributed FRVs.

Moreover, $\tilde{b}_i, (i = r_1 + 1, r_1 + 2, \cdots, m)$ follow Weibull distributed FRVs. While $\tilde{e}_{ij}$, $(i = r_0 + 1, r_0 + 2, \cdots, m)$, represents the fuzzy coefficients of the j^{th} decision variable in the i^{th} stochastic constraints; p_i, $0 \le p_i = 1$, is the tolerance measures which represent the admissible risk of constrain violation. The i^{th} imperative is happy with no less than a likelihood of $1 - p_i$.

Every one of the parameters are communicated as fuzzy numbers described by any type of membership functions, contingent upon DM's inclination. It is standard to expect that $D_{ij}(\boldsymbol{x}) > 0$ for all estimations of decision variables.

4 The Equivalent Deterministic Model

In this section, the ML-MOFP problem with FRVs in the constraints is transformed into the deterministic model.

Case 1: Fuzzy normal distribution

In this case, it is accepted that the random variable ξ has a normal distribution, i.e. $\xi \sim N(m, \delta^2)$, where m and δ^2 denotes the mean value and variance, respectively; because of subjective and objective impacting components, m and δ^2 are likewise considered as indeterminate and will be portrayed by fuzzy probabilistic distribution, i.e. $\tilde{m}$ and $\widetilde{\delta}^2$.

Therefore, the probability density function of the fuzzy normal distribution [15,16]:

$$f\left(x;\tilde{m},\tilde{\delta}^2\right) = \frac{1}{\sqrt{2\pi}\,\tilde{\delta}} e^{\left(-\frac{(x-\tilde{m})^2}{2\tilde{\delta}^2}\right)} \tag{12}$$

Let the coefficients $\tilde{a}_{ij}$ and $\tilde{b}_i, (i = 1,2,\cdots,r_0)$ be independent normally distributed FRVs which is presented in the i^{th} constraint.

The i^{th} CCP follows as:

$$\text{Prob}\left[\sum_{j=1}^{n} \tilde{a}_{ij}x_j \le \tilde{b}_i\right] \ge 1-p_i, \qquad (i = 1,2,\cdots,r_0), \tag{13}$$

Thus the normally distributed FRVs $\tilde{a}_{ij}$ and $\tilde{b}_i$ are expressed as $N(\tilde{m}_{\tilde{a}_{ij}},\tilde{\delta}^2_{\tilde{a}_{ij}})$ and $N(\tilde{m}_{\tilde{b}_i},\tilde{\delta}^2_{\tilde{b}_i})$ respectively; the fuzzy mean $\tilde{m}$ and fuzzy variance $\tilde{\delta}^2$ are thought to be $\mathscr{L}\mathscr{R}$-fuzzy numbers.

Thus based on the stochastic CCP to handle constraint (13), let $\tilde{u}_i = \sum_{j=1}^{n} \tilde{a}_{ij}x_j - \tilde{b}_i$, such that $\tilde{u}_i$ can be expressed as a fuzzy normal distribution $N(\tilde{m}_{\tilde{u}_i},\tilde{\delta}^2_{\tilde{u}_i})$ [10 ,15]. Moreover, $(\tilde{u}_i - \tilde{m}_{\tilde{u}_i})/\tilde{\delta}_{\tilde{u}_i}$, follow standard normal distribution. In the interim, the requirement (13) could be changed into a deterministic nonlinear disparity as takes after:

$$\text{Prob}\left[\sum_{j=1}^{n} \tilde{a}_{ij}x_j \le \tilde{b}_i\right] \Longleftrightarrow \text{Prob}\left[\tilde{u}_i \le 0\right] \Longleftrightarrow \text{Prob}\left[\frac{(\tilde{u}_i - \tilde{m}_{\tilde{u}_i})}{\sqrt{\tilde{\delta}^2_{\tilde{u}_i}}} \le \frac{-\tilde{m}_{\tilde{u}_i}}{\sqrt{\tilde{\delta}^2_{\tilde{u}_i}}}\right] \ge 1-p_i, \tag{14}$$

$$\varphi^{-1}(1-p_i) \le \left[\frac{-\tilde{m}_{\tilde{u}_i}}{\sqrt{\tilde{\delta}^2_{\tilde{u}_i}}}\right] \qquad i = 1,2,\cdots,r_0, \tag{15}$$

$$\sum_{j=1}^{n} \tilde{m}_{\tilde{a}_{ij}}x_j + \varphi^{-1}(1-p_i)\sqrt{\sum_{j=1}^{n} \tilde{\delta}^2_{\tilde{a}_{ij}}x_j^2 + \tilde{\delta}^2_{\tilde{b}_i}} \le \tilde{m}_{\tilde{b}_i} \qquad i = 1,2,\cdots,r_0, \tag{16}$$

Thus, based on level set properties and the partial order relations [27], presented in section 2 then the constraint (16) can be transformed to the following crisp equivalents:

$$\sum_{j=1}^{n} \left(\tilde{m}_{\tilde{a}_{ij}}\right)^U_\alpha x_j + \varphi^{-1}(1-p_i)\sqrt{\sum_{j=1}^{n}\left(\tilde{\delta}^2_{\tilde{a}_{ij}}\right)^U_\alpha x_j^2 + \left(\tilde{\delta}^2_{\tilde{b}_i}\right)^U_\alpha} \le \left(\tilde{m}_{\tilde{b}_i}\right)^U_\alpha \qquad i = 1,2,\cdots,r_0, \tag{17}$$

$$\sum_{j=1}^{n} \left(\tilde{m}_{\tilde{a}_{ij}}\right)^L_\alpha x_j + \varphi^{-1}(1-p_i)\sqrt{\sum_{j=1}^{n}\left(\tilde{\delta}^2_{\tilde{a}_{ij}}\right)^L_\alpha x_j^2 + \left(\tilde{\delta}^2_{\tilde{b}_i}\right)^L_\alpha} \le \left(\tilde{m}_{\tilde{b}_i}\right)^L_\alpha \qquad i = 1,2,\cdots,r_0, \tag{18}$$

where the upper and lower α-cuts of the fuzzy means and fuzzy variances for $\tilde{a}_{ij}$ and $\tilde{b}_i$ follows as:

$$\left(\tilde{m}_{\tilde{a}_{ij}}\right)^U_\alpha = m_{\tilde{a}_{ij}} + \beta^m_{\tilde{a}_{ij}}\mathscr{R}^{-1}(\alpha), \qquad \text{and} \qquad \left(\tilde{m}_{\tilde{a}_{ij}}\right)^L_\alpha = m_{\tilde{a}_{ij}} - \gamma^m_{\tilde{a}_{ij}}\mathscr{L}^{-1}(\alpha), \tag{19}$$

$$\left(\tilde{\delta}^2_{\tilde{a}_{ij}}\right)^U_\alpha = \delta_{\tilde{a}_{ij}} + \beta^\delta_{\tilde{a}_{ij}}\mathscr{R}^{-1}(\alpha), \qquad \text{and} \qquad \left(\tilde{\delta}^2_{\tilde{a}_{ij}}\right)^L_\alpha = \delta_{\tilde{a}_{ij}} - \gamma^\delta_{\tilde{a}_{ij}}\mathscr{L}^{-1}(\alpha), \tag{20}$$

$$\left(\tilde{m}_{\tilde{b}_i}\right)^U_\alpha = m_{\tilde{b}_i} + \beta^m_{\tilde{b}_i}\mathscr{R}^{-1}(\alpha), \qquad \text{and} \qquad \left(\tilde{m}_{\tilde{b}_i}\right)^L_\alpha = m_{\tilde{b}_i} - \gamma^m_{\tilde{b}_i}\mathscr{L}^{-1}(\alpha), \tag{21}$$

$$\left(\tilde{\delta}^2_{\tilde{b}_i}\right)^U_\alpha = \delta_{\tilde{b}_i} + \beta^\delta_{\tilde{b}_i}\mathscr{R}^{-1}(\alpha), \qquad \text{and} \qquad \left(\tilde{\delta}^2_{\tilde{b}_i}\right)^L_\alpha = \delta_{\tilde{b}_i} - \gamma^\delta_{\tilde{b}_i}\mathscr{L}^{-1}(\alpha), \tag{22}$$

Case 2: Fuzzy exponential distribution

In this case, it is assumed that $\tilde{b}_i$, $(i = r_0+1, r_0+2, \cdots, r_1)$, in (8), is exponentially distributed FRVs. Thus $\tilde{b}_i$ has probability density function follows as [4,28]:

$$f\left(b_i; \widetilde{\theta}_i\right) = \widetilde{\theta}_i e^{-b_i\ \widetilde{\theta}_i} \qquad i = r_0+1, r_0+2, \cdots, r_1, \tag{23}$$

To convert the constrain (8) to its equivalent deterministic form let $\tilde{y}_i = \sum_{j=1}^{n} \tilde{e}_{ij} x_j, \quad i = r_0+1, \cdots, r_1$, then Prob $\left[\sum_{j=1}^{n} \tilde{e}_{ij} x_j \le \tilde{b}_i\right]$ = Prob $\left[\tilde{y}_i \le \tilde{b}_i\right]$ [3,4]. So, based on α-cut properties and the partial order relations presented in section 2 then the constraint (8) expressed as:

$$\text{Prob}\left[\tilde{y}_i \le \tilde{b}_i\right](\alpha) = \left\{\int_{y_i}^{\infty} \theta_i e^{-b_i\ \theta_i}\, db_i : \theta_i \in \widetilde{\theta}_i(\alpha), y_i \in \tilde{y}_i(\alpha)\right\} \ge 1 - p_i, \tag{24}$$

$$\text{Prob}\left[\tilde{y}_i \le \tilde{b}_i\right](\alpha) = \left\{e^{-y_i\ \theta_i} : \theta_i \in \widetilde{\theta}_i(\alpha), y_i \in \tilde{y}_i(\alpha)\right\} \ge\ 1 - p_i, \tag{25}$$

Since, the coefficients $\tilde{e}_{ij}$, and the parameter $\widetilde{\theta}_i$ of the FRV $\tilde{b}_i$ are considered as an $\mathscr{L}\mathscr{R}$-fuzzy number. Hence the equivalent crisp constraints of the probabilistic constraints (8) follows as:

$$\sum_{j=1}^{n} \left(e_{ij} + \beta_{ij}^{e}\, \mathscr{R}^{-1}(\alpha)\right) x_j \left(\theta_i + \beta_i^{\theta}\ \mathscr{R}^{-1}(\alpha)\right) \le -ln(1-p_i)\ , \quad i = r_0+1, r_0+2, \cdots, r_1, \tag{26}$$

$$\sum_{j=1}^{n} \left(e_{ij} - \gamma_{ij}^{e}\, \mathscr{L}^{-1}(\alpha)\right) x_j \left(\theta_i - \gamma_{ij}^{\theta}\ \mathscr{L}^{-1}(\alpha)\right) \le -ln(1-p_i)\ , \quad i = r_0+1, r_0+2, \cdots, r_1, \tag{27}$$

Case 3: Fuzzy Weibull distribution

In this case, it is assumed that $\tilde{b}_i$, $(i = r_1+1, r_1+2, \cdots, m)$, in (9), represents Weibull distributed FRVs, thus its probability density function is composed as [12]:

$$f\left(b_i; \widetilde{\mu}_i, \widetilde{\rho}_i, c_i\right) = \frac{c_i}{\widetilde{\rho}_i}\left(\frac{b_i - \widetilde{\mu}_i}{\widetilde{\rho}_i}\right)^{c_i - 1} e^{-\left(\frac{b_i - \widetilde{\mu}_i}{\widetilde{\rho}_i}\right)^{c_i}} \qquad i = r_1+1, r_1+2, \cdots, m, \tag{28}$$

It is also known that $\tilde{b}_i$ has three parameters $\widetilde{\mu}_i, \widetilde{\rho}_i$ *and* c_i where $\widetilde{\rho}_i$ *and* c_i are the scale parameters and $\widetilde{\mu}_i$ is the location parameter. To convert the constrain (9) to its equivalent deterministic form let $\tilde{y}_i = \sum_{j=1}^{n} \tilde{e}_{ij} x_j, \quad i = r_1+1, \cdots, m$, then Prob $\left[\sum_{j=1}^{n} \tilde{e}_{ij} x_j \le \tilde{b}_i\right]$ = Prob $\left[\tilde{y}_i \le \tilde{b}_i\right]$ [12]. So, based on α-cut properties and the partial order relations presented in section 2 then the constraint (9) follows as:

$$\text{Prob}\left[\tilde{y}_i \le \tilde{b}_i\right](\alpha) = \left\{\int_{y_i}^{\infty} \frac{c_i}{\rho_i}\left(\frac{b_i - \mu_i}{\rho_i}\right)^{c_i - 1} e^{-\left(\frac{b_i - \mu_i}{\rho_i}\right)^{c_i}} db_i \;\middle|\; \begin{array}{c} \mu_i \in \widetilde{\mu}_i(\alpha), \rho_i \in \widetilde{\rho}_i(\alpha), \\ y_i \in \tilde{y}_i(\alpha) \end{array}\right\} \ge 1 - p_i, \tag{29}$$

$$\text{Prob}\left[\tilde{y}_i \le \tilde{b}_i\right](\alpha) = \left\{e^{-\left(\frac{y_i - \mu_i}{\rho_i}\right)^{c_i}} \;\middle|\; \begin{array}{c} \mu_i \in \widetilde{\mu}_i(\alpha), \rho_i \in \widetilde{\rho}_i(\alpha), \\ y_i \in \tilde{y}_i(\alpha) \end{array}\right\} \ge 1 - p_i, \tag{30}$$

Since, the coefficients $\tilde{e}_{ij}$, and the parameter $\widetilde{\mu}_i$, $\widetilde{\rho}_i$ of the FRV $\tilde{b}_i$ are considered as an $\mathscr{L}\mathscr{R}$-fuzzy number. Hence the equivalent crisp constraints of the probabilistic constraints (9) expressed as:

$$\sum_{j=1}^{n} \left[e_{ij} + \beta_{ij}^{e} \mathscr{R}^{-1}(\alpha)\right] x_j \le \left[\mu_i + \beta_i^{\mu} \mathscr{R}^{-1}(\alpha)\right] + \left[\rho_i + \beta_i^{\rho} \mathscr{R}^{-1}(\alpha)\right] \left[-ln(1-p_i)\right]^{\frac{1}{c_i}},\ i = r_1+1, \cdots, m, \tag{31}$$

$$\sum_{j=1}^{n} \left[e_{ij} - \gamma_{ij}^{e} \mathscr{L}^{-1}(\alpha)\right] x_j \le \left[\mu_i - \gamma_i^{\mu} \mathscr{L}^{-1}(\alpha)\right] + \left[\rho_i - \gamma_i^{\rho} \mathscr{L}^{-1}(\alpha)\right] \left[-ln(1-p_i)\right]^{\frac{1}{c_i}},\ i = r_1+1, \cdots, m, \tag{32}$$

Hence, for a coveted estimation of α, the ML-MOFP problem with FRVs in the constraints can be changed into the α-(ML-MOFP) model as takes after:
1^{st} ***Level***

$$\underbrace{max}_{\boldsymbol{x_1}} \; F_1(\boldsymbol{x}) = \underbrace{max}_{\boldsymbol{x_1}} \left(f_{11}(\boldsymbol{x}),\; f_{12}(\boldsymbol{x}),\; \cdots, f_{1k_1}(\boldsymbol{x})\right), \tag{33}$$

where $\boldsymbol{x_2}, \boldsymbol{x_3}, \cdots, \boldsymbol{x_p}$ solves
2^{nd} ***Level***

$$\underbrace{max}_{\boldsymbol{x_2}} \; F_2(\boldsymbol{x}) = \underbrace{max}_{\boldsymbol{x_2}} \left(f_{21}(\boldsymbol{x}),\; f_{22}(\boldsymbol{x}),\; \cdots, f_{2k_2}(\boldsymbol{x})\right), \tag{34}$$

$\vdots$

where $\boldsymbol{x_p}$ solves
p^{th} ***Level***

$$\underbrace{max}_{\boldsymbol{x_p}} \; F_t(\boldsymbol{x}) = \underbrace{max}_{\boldsymbol{x_p}} \left(f_{p1}(\boldsymbol{x}),\; f_{p2}(\boldsymbol{x}),\; \cdots, f_{pk_p}(\boldsymbol{x})\right), \tag{35}$$

subject to

$$\sum_{j=1}^{n} \left(\tilde{m}_{\tilde{a}_{ij}}\right)_{\alpha}^{U} x_j + \varphi^{-1}(1-p_i) \sqrt{\sum_{j=1}^{n} \left(\widetilde{\delta}^2_{\tilde{a}_{ij}}\right)_{\alpha}^{U} x_j^2 + \left(\widetilde{\delta}^2_{\tilde{b}_i}\right)_{\alpha}^{U}} \leq \left(\tilde{m}_{\tilde{b}_i}\right)_{\alpha}^{U} \qquad i = 1, 2, \cdots, r_0, \tag{36}$$

$$\sum_{j=1}^{n} \left(\tilde{m}_{\tilde{a}_{ij}}\right)_{\alpha}^{L} x_j + \varphi^{-1}(1-p_i) \sqrt{\sum_{j=1}^{n} \left(\widetilde{\delta}^2_{\tilde{a}_{ij}}\right)_{\alpha}^{L} x_j^2 + \left(\widetilde{\delta}^2_{\tilde{b}_i}\right)_{\alpha}^{L}} \leq \left(\tilde{m}_{\tilde{b}_i}\right)_{\alpha}^{L} \qquad i = 1, 2, \cdots, r_0, \tag{37}$$

$$\sum_{j=1}^{n} \left(e_{ij} + \beta_{ij}^{e} \mathscr{R}^{-1}(\alpha)\right) x_j \left(\theta_i + \beta_i^{\theta} \mathscr{R}^{-1}(\alpha)\right) \leq -ln(1-p_i), \qquad i = r_0+1, r_0+2, \cdots, r_1, \tag{38}$$

$$\sum_{j=1}^{n} \left(e_{ij} - \gamma_{ij}^{e} \mathscr{L}^{-1}(\alpha)\right) x_j \left(\theta_i - \gamma_{ij}^{e} \mathscr{L}^{-1}(\alpha)\right) \leq -ln(1-p_i), \qquad i = r_0+1, r_0+2, \cdots, r_1, \tag{39}$$

$$\sum_{j=1}^{n} \left[e_{ij} + \beta_{ij}^{e} \mathscr{R}^{-1}(\alpha)\right] x_j \leq \left[\mu_i + \beta_i^{\mu} \mathscr{R}^{-1}(\alpha)\right] + \left[\rho_i + \beta_i^{\rho} \mathscr{R}^{-1}(\alpha)\right] \left[-ln(1-p_i)\right]^{\frac{1}{c_i}}, \; i = r_1+1, \cdots, m, \tag{40}$$

$$\sum_{j=1}^{n} \left[e_{ij} - \gamma_{ij}^{e} \mathscr{L}^{-1}(\alpha)\right] x_j \leq \left[\mu_i - \gamma_i^{\mu} \mathscr{L}^{-1}(\alpha)\right] + \left[\rho_i - \gamma_i^{\rho} \mathscr{L}^{-1}(\alpha)\right] \left[-ln(1-p_i)\right]^{\frac{1}{c_i}}, \quad i = r_1+1, \cdots, m, \tag{41}$$

$$x_j \geq 0, \quad j = 1, 2, \cdots, n. \tag{42}$$

5 Linear Model Development of The α-(ML-MOFP) Problem

An equivalent linear multi-objective programming problem to multi-objective fractional programming (MOFP) problem has been presented [8]. Now, we make further extensions on the article of M. Chakraborty and S. Gupta [8], to develop a methodology for obtaining the linear model of the α-(ML-MOFP) problem. Since the MOFP problem for the i^{th}-level decision maker may be written as:

$$\underbrace{max}_{\boldsymbol{x_i}} F_i(\boldsymbol{x}) = \underbrace{max}_{\boldsymbol{x_i}} \left(f_{i1}(\boldsymbol{x}),\; f_{i2}(\boldsymbol{x}),\; \cdots, f_{ik_i}(\boldsymbol{x})\right), \tag{43}$$

subject to

$$\sum_{j=1}^{n} \left(\tilde{m}_{\tilde{a}_{ij}}\right)_{\alpha}^{U} x_j + \varphi^{-1}(1-p_i) \sqrt{\sum_{j=1}^{n} \left(\widetilde{\delta}^2_{\tilde{a}_{ij}}\right)_{\alpha}^{U} x_j^2 + \left(\widetilde{\delta}^2_{\tilde{b}_i}\right)_{\alpha}^{U}} \leq \left(\tilde{m}_{\tilde{b}_i}\right)_{\alpha}^{U} \qquad i = 1, 2, \cdots, r_0, \tag{44}$$

$$\sum_{j=1}^{n} \left(\tilde{m}_{\tilde{a}_{ij}}\right)_{\alpha}^{L} x_j + \varphi^{-1}(1-p_i) \sqrt{\sum_{j=1}^{n} \left(\widetilde{\delta}^2_{\tilde{a}_{ij}}\right)_{\alpha}^{L} x_j^2 + \left(\widetilde{\delta}^2_{\tilde{b}_i}\right)_{\alpha}^{L}} \leq \left(\tilde{m}_{\tilde{b}_i}\right)_{\alpha}^{L} \qquad i = 1, 2, \cdots, r_0, \tag{45}$$

$$\sum_{j=1}^{n}\left(e_{ij}+\beta_{ij}^{e}\mathscr{R}^{-1}(\alpha)\right)x_j\left(\theta_i+\beta_i^{\theta}\mathscr{R}^{-1}(\alpha)\right)\le -ln(1-p_i), \qquad i=r_0+1,r_0+2,\cdots,r_1, \tag{46}$$

$$\sum_{j=1}^{n}\left(e_{ij}-\gamma_{ij}^{e}\mathscr{L}^{-1}(\alpha)\right)x_j\left(\theta_i-\gamma_i^{\theta}\mathscr{L}^{-1}(\alpha)\right)\le -ln(1-p_i), \qquad i=r_0+1,r_0+2,\cdots,r_1, \tag{47}$$

$$\sum_{j=1}^{n}\left[e_{ij}+\beta_{ij}^{e}\mathscr{R}^{-1}(\alpha)\right]x_j\le\left[\mu_i+\beta_i^{\mu}\mathscr{R}^{-1}(\alpha)\right]+\left[\rho_i+\beta_i^{\rho}\mathscr{R}^{-1}(\alpha)\right]\left[-ln(1-p_i)\right]^{\frac{1}{c_i}} \quad i=r_1+1,\cdots,m, \tag{48}$$

$$\sum_{j=1}^{n}\left[e_{ij}-\gamma_{ij}^{e}\mathscr{L}^{-1}(\alpha)\right]x_j\le\left[\mu_i-\gamma_i^{\mu}\mathscr{L}^{-1}(\alpha)\right]+\left[\rho_i-\gamma_i^{\rho}\mathscr{L}^{-1}(\alpha)\right]\left[-ln(1-p_i)\right]^{\frac{1}{c_i}}, \quad i=r_1+1,\cdots,m, \tag{49}$$

$$x_j\ge 0, \quad j=1,2,\cdots,n. \tag{50}$$

where

$$f_{ij}(\boldsymbol{x})=\frac{N_{ij}(\boldsymbol{x})}{D_{ij}(\boldsymbol{x})}=\frac{\boldsymbol{c}^{ij}\ \boldsymbol{x}\ +\alpha^{ij}}{\boldsymbol{d}^{ij}\ \boldsymbol{x}\ +\beta^{ij}}, \qquad (i=1,2,\cdots,p),\ (j=1,2,\cdots,k_i), \tag{51}$$

Assuming that, the set of constraints (44)-(50) denoted by G_α, I be the index set such that $I=\{j:N_{ij}(\boldsymbol{x})\ge 0\ \forall\, \boldsymbol{x}\in G_\alpha\}$ so also $I^c=\{j:N_{ij}(\boldsymbol{x})<0\ \forall\, \boldsymbol{x}\in G_\alpha\}$, where $I\cup I^c=\{1,2,\cdots,k_i\}$. If $N_{ij}(\boldsymbol{x})$ is concave, $D_{ij}(\boldsymbol{x})$ is concave and positive on G_α and $N_{ij}(\boldsymbol{x})$ is negative for all $\boldsymbol{x}\in G_\alpha$, so [8]:

$$\underbrace{max}_{\boldsymbol{x}\in G_\alpha}=\frac{N_{ij}(\boldsymbol{x})}{D_{ij}(\boldsymbol{x})} \iff \underbrace{min}_{\boldsymbol{x}\in G_\alpha}=\frac{-N_{ij}(\boldsymbol{x})}{D_{ij}(\boldsymbol{x})} \iff \underbrace{max}_{\boldsymbol{x}\in G_\alpha}=\frac{D_{ij}(\boldsymbol{x})}{-N_{ij}(\boldsymbol{x})}, \tag{52}$$

where G_α, is nonempty and bounded. For simplicity, assume that $1/\boldsymbol{d}^{ij}\boldsymbol{x}+\beta^{ij}$ and $-1/\boldsymbol{c}^{ij}\boldsymbol{x}+\alpha^{ij}$ is t for $j\in I$ and $j\in I^c$, respectively, i.e.

$$\bigcap_{i\in I}\frac{1}{\boldsymbol{d}^{ij}\ \boldsymbol{x}+\beta^{ij}}=t \qquad \text{and} \qquad \bigcap_{i\in I^c}\frac{-1}{\boldsymbol{c}^{ij}\ \boldsymbol{x}+\alpha^{ij}}=t, \tag{53}$$

which is amounting to:

$$\frac{1}{\boldsymbol{d}^{ij}\ \boldsymbol{x}\ +\beta^{ij}}\ge t \quad \text{for } j\in I \quad \text{and} \quad \frac{-1}{\boldsymbol{c}^{ij}\ \boldsymbol{x}\ +\alpha^{ij}}\ge t \quad \text{for } j\in I^c, \tag{54}$$

Based on the transformation $\boldsymbol{y}=t\boldsymbol{x}\ (t>0)$, $\boldsymbol{y}\in R^n$, $t\in R$, and the above inequalities in equation (54) therefore, the linear model of the i^{th} level decision maker problem is defined as takes after:

$$\underbrace{max}_{\boldsymbol{y_i}}\left(t\left(N_{ij}(\boldsymbol{y}/t)\right), \quad \text{for } j\in I;\ t\left(D_{ij}(\boldsymbol{y}/t)\right), \quad \text{for } j\in I^c\right), \tag{55}$$

subject to

$$t\left(D_{ij}(\boldsymbol{y}/t)\right)\le 1, \qquad \text{for } j\in I \qquad (i=1,2,\cdots,p),\ (j=1,2,\cdots,k_i), \tag{56}$$

$$-t\left(N_{ij}(\boldsymbol{y}/t)\right)=1, \qquad \text{for } j\in I^C \qquad (i=1,2,\cdots,p),\ (j=1,2,\cdots,k_i), \tag{57}$$

$$\sum_{j=1}^{n}\left(\tilde{m}_{\tilde{a}_{ij}}\right)_{\alpha}^{U}y_j+\varphi^{-1}(1-p_i)\sqrt{\sum_{j=1}^{n}\left(\widetilde{\delta}^2_{\tilde{a}_{ij}}\right)_{\alpha}^{U}y_j^2+\left(\widetilde{\delta}^2_{\tilde{b}_i}\right)_{\alpha}^{U}t^2}\le\left(\tilde{m}_{\tilde{b}_i}\right)_{\alpha}^{U}t \qquad i=1,2,\cdots,r_0, \tag{58}$$

$$\sum_{j=1}^{n}\left(\tilde{m}_{\tilde{a}_{ij}}\right)_{\alpha}^{L}y_j+\varphi^{-1}(1-p_i)\sqrt{\sum_{j=1}^{n}\left(\widetilde{\delta}^2_{\tilde{a}_{ij}}\right)_{\alpha}^{L}y_j^2+\left(\widetilde{\delta}^2_{\tilde{b}_i}\right)_{\alpha}^{L}t^2}\le\left(\tilde{m}_{\tilde{b}_i}\right)_{\alpha}^{L}t \qquad i=1,2,\cdots,r_0, \tag{59}$$

$$\sum_{j=1}^{n}\left(e_{ij}+\beta_{ij}^{e}\mathscr{R}^{-1}(\alpha)\right)y_j\left(\theta_i+\beta_i^{\theta}\mathscr{R}^{-1}(\alpha)\right)\le\left((-ln(1-p_i))\right)t, \qquad i=r_0+1,r_0+2,\cdots,r_1, \tag{60}$$

$$\sum_{j=1}^{n}\left(e_{ij}-\gamma_{ij}^{e}\mathscr{L}^{-1}(\alpha)\right)y_j\left(\theta_i-\gamma_i^{\theta}\mathscr{L}^{-1}(\alpha)\right)\leq((-ln(1-p_i))t,\qquad i=r_0+1,r_0+2,\cdots,r_1, \tag{61}$$

$$\sum_{j=1}^{n}\left[e_{ij}+\beta_{ij}^{e}\mathscr{R}^{-1}(\alpha)\right]y_j\leq\left[\left[\mu_i+\beta_i^{\mu}\mathscr{R}^{-1}(\alpha)\right]+\left[\rho_i+\beta_i^{\rho}\mathscr{R}^{-1}(\alpha)\right]\left[-ln(1-p_i)\right]^{\frac{1}{c_i}}\right]t, i=r_1+1,\cdots,m \tag{62}$$

$$\sum_{j=1}^{n}\left[e_{ij}-\gamma_{ij}^{e}\mathscr{L}^{-1}(\alpha)\right]y_j\leq\left[\left[\mu_i-\gamma_i^{\mu}\mathscr{L}^{-1}(\alpha)\right]+\left[\rho_i-\gamma_i^{\rho}\mathscr{L}^{-1}(\alpha)\right]\left[-ln(1-p_i)\right]^{\frac{1}{c_i}}\right]t,\ i=r_1+1,\cdots,m \tag{63}$$

$$y_j\geq 0,\quad t>0,\qquad j=1,2,\cdots,n. \tag{64}$$

Following the above discussion thus, the α-(ML-MOLP) model of the α-(ML-MOFP) problem is formulated as follows:

1^{st} ***Level***

$$\underbrace{max}_{\boldsymbol{y_1}}F_1(\boldsymbol{y},t)=\underbrace{max}_{\boldsymbol{y_1}}\left(t\left(N_{1j}(\boldsymbol{y}/t)\right),\text{ for } j\in I;\, t\left(D_{1j}(\boldsymbol{y}/t)\right),\text{ for } j\in I^c\right),\ (j=1,\cdots,k_1), \tag{65}$$

where $\boldsymbol{y_2},\boldsymbol{y_3},\cdots,\boldsymbol{y_p}$ solves

2^{nd} ***Level***

$$\underbrace{max}_{\boldsymbol{y_2}}F_2(\boldsymbol{y},t)=\underbrace{max}_{\boldsymbol{y_2}}\left(t\left(N_{2j}(\boldsymbol{y}/t)\right),\text{ for } j\in I;\, t\left(D_{2j}(\boldsymbol{y}/t)\right),\text{ for } j\in I^c\right),\ (j=1,\cdots,k_2), \tag{66}$$

$\vdots$

where $\boldsymbol{y_p}$ solves

p^{th} ***Level***

$$\underbrace{max}_{\boldsymbol{y_p}}F_p(\boldsymbol{y},t)=\underbrace{max}_{\boldsymbol{y_p}}\left(t\left(N_{pj}(\boldsymbol{y}/t)\right),\text{ for } j\in I; t\left(D_{pj}(\boldsymbol{y}/t)\right),\text{ for } j\in I^c\right),\ (j=1,\cdots,k_p), \tag{67}$$

subject to

$$t\left(D_{ij}(\boldsymbol{y}/t)\right)\leq 1,\qquad \text{for } j\in I\qquad (i=1,2,\cdots,p),\ (j=1,2,\cdots,k_i), \tag{68}$$

$$-t\left(N_{ij}(\boldsymbol{y}/t)\right)=1,\qquad \text{for } j\in I^C\qquad (i=1,2,\cdots,p),\ (j=1,2,\cdots,k_i), \tag{69}$$

$$\sum_{j=1}^{n}\left(\tilde{m}_{\tilde{a}_{ij}}\right)_{\alpha}^{U}y_j+\varphi^{-1}(1-p_i)\sqrt{\sum_{j=1}^{n}\left(\widetilde{\delta}_{\tilde{a}_{ij}}^2\right)_{\alpha}^{U}y_j^2+\left(\widetilde{\delta}_{\tilde{b}_i}^2\right)_{\alpha}^{U}t^2}\leq\left(\tilde{m}_{\tilde{b}_i}\right)_{\alpha}^{U}t\qquad i=1,2,\cdots,r_0, \tag{70}$$

$$\sum_{j=1}^{n}\left(\tilde{m}_{\tilde{a}_{ij}}\right)_{\alpha}^{L}y_j+\varphi^{-1}(1-p_i)\sqrt{\sum_{j=1}^{n}\left(\widetilde{\delta}_{\tilde{a}_{ij}}^2\right)_{\alpha}^{L}y_j^2+\left(\widetilde{\delta}_{\tilde{b}_i}^2\right)_{\alpha}^{L}t^2}\leq\left(\tilde{m}_{\tilde{b}_i}\right)_{\alpha}^{L}t\qquad i=1,2,\cdots,r_0, \tag{71}$$

$$\sum_{j=1}^{n}\left(e_{ij}+\beta_{ij}^{e}\mathscr{R}^{-1}(\alpha)\right)y_j\left(\theta_i+\beta_i^{\theta}\mathscr{R}^{-1}(\alpha)\right)\leq(-ln(1-p_i))t,\qquad i=r_0+1,r_0+2,\cdots,r_1, \tag{72}$$

$$\sum_{j=1}^{n}\left(e_{ij}-\gamma_{ij}^{e}\mathscr{L}^{-1}(\alpha)\right)y_j\left(\theta_i-\gamma_i^{e}\mathscr{L}^{-1}(\alpha)\right)\leq(-ln(1-p_i))t,\qquad i=r_0+1,r_0+2,\cdots,r_1, \tag{73}$$

$$\sum_{j=1}^{n}\left[e_{ij}+\beta_{ij}^{e}\mathscr{R}^{-1}(\alpha)\right]y_j\leq\left[\left[\mu_i+\beta_i^{\mu}\mathscr{R}^{-1}(\alpha)\right]+\left[\rho_i+\beta_i^{\rho}\mathscr{R}^{-1}(\alpha)\right]\left[-ln(1-p_i)\right]^{\frac{1}{c_i}}\right]t,\ i=r_1+1,\cdots,m \tag{74}$$

$$\sum_{j=1}^{n}\left[e_{ij}-\gamma_{ij}^{e}\mathscr{L}^{-1}(\alpha)\right]y_j\leq\left[\left[\mu_i-\gamma_i^{\mu}\mathscr{L}^{-1}(\alpha)\right]+\left[\rho_i-\gamma_i^{\rho}\mathscr{L}^{-1}(\alpha)\right]\left[-ln(1-p_i)\right]^{\frac{1}{c_i}}\right]t,\ i=r_1+1,\cdots,m \tag{75}$$

$$y_j\geq 0,\quad t>0,\qquad j=1,2,\cdots,n. \tag{76}$$

where the system of constraints, in equations (68)-(76), at an α-level denoted by $\boldsymbol{S}_{\alpha}$, which form a nonempty convex set.

6 An Interactive Models for α-(ML-MOFP) Problem

To obtain the α-Pareto optimal solution (preferred solution) of the α-(ML-MOFP) problem with FRVs in the constraints firstly, the crisp model, equations (33)-(42), is developed based on CCP technique and the α-cut of fuzzy sets. Consequently, the α-(ML-MOLP) model is formulated as presented in the previous section equations (65)-(76).

In the interactive mechanism, after obtaining the preferred or satisfactory solutions by the ε-constraint method and the concept of satisfactoriness, the FLDM gives the favored arrangements that are satisfactory in rank request referring to the satisfactoriness of the preferred solutions to the SLDM.

Then, the SLDM uses the ε-constraint method to achieve the solution that progressively accesses the favored solution of the FLDM [22]. Afterwards, the acquired solutions are delivered to the PLDM who will look up the solution by the ε-constraint method and the concept of satisfactoriness to attain the solution that is closest to the favored solution of the top levels.

At long last, the top level determine the favored solution of the α-(ML-MOLP) problem as indicated by their satisfactoriness. Then, the corresponding preferred solution to the α-(ML-MOFP) problem is obtained.

6.1 The First Level Decision Maker (FLDM) Problem

The first level decision-making problem of the α-(ML-MOLP) model follows as:

$$\underbrace{max}_{y_1}\left(t\left(N_{1j}(\boldsymbol{y}/t)\right),\quad \text{for } j\in I;\ t\left(D_{1j}(\boldsymbol{y}/t)\right),\quad \text{for } j\in I^c\right),\qquad (j=1,2,\cdots,k_1),\tag{77}$$

subject to

$$(y_1,y_2,\cdots,y_p,t)\in \boldsymbol{S}_\alpha.\tag{78}$$

To obtain the α-Pareto optimal solution of the FLDM; we change the MODM problem, model (77)-(78), by the ε-constraint method into the accompanying single-objective decision-making (SODM) problem:

$$max\left(t\left(N_{1j}(\boldsymbol{y}/t)\right),\quad \text{for } j\in I;\ t\left(D_{1j}(\boldsymbol{y}/t)\right),\quad \text{for } j\in I^c\right),\qquad (j=\ell),\tag{79}$$

subject to

$$t\left(D_{1j}(\boldsymbol{y}/t)\right)\geq \delta_{1j},\qquad \text{for } j\in I\qquad (j=1,2,\cdots,k_1),\quad (j\neq\ell),\tag{80}$$

$$-t\left(N_{1j}(\boldsymbol{y}/t)\right)=\delta_{1j},\qquad \text{for } j\in I^C\qquad (j=1,2,\cdots,k_1),\quad (j\neq\ell),\tag{81}$$

$$(y_1,y_2,\cdots,y_p,t)\in \boldsymbol{S}_\alpha.\tag{82}$$

So the solution of the first level is obtained by executing algorithm **I**, as $\left(y_1^*,y_2^*,\cdots,y_p^*,t\right)=\left(y_1^F,y_2^F,\cdots,y_p^F,t\right)$.

6.2 The Second Level Decision Maker (SLDM) Problem

Secondly, following the concept of MLMP problems, the first level decision variable y_1^F should be included in the SLDM problem; hence, the problem of SLDM can be expressed as:

$$\underbrace{max}_{y_2}\left(t\left(N_{2j}(\boldsymbol{y}/t)\right),\quad \text{for } j\in I;\ t\left(D_{2j}(\boldsymbol{y}/t)\right),\quad \text{for } j\in I^c\right),\qquad (j=1,2,\cdots,k_2),\tag{83}$$

subject to

$$\left(y_1^F,y_2,\cdots,y_p,t\right)\in \boldsymbol{S_\alpha}.\tag{84}$$

The ε-constraint method is utilized to formulate the SODM problem as follows:

$$max\left(t\left(N_{2j}(\boldsymbol{y}/t)\right),\quad \text{for } j\in I;\ t\left(D_{2j}(\boldsymbol{y}/t)\right),\quad \text{for } j\in I^c\right),\qquad (j=\ell),\tag{85}$$

subject to

$$t\left(D_{2j}(\boldsymbol{y}/t)\right)\geq \delta_{2j},\qquad \text{for } j\in I\qquad (j=1,2,\cdots,k_2),\quad (j\neq\ell),\tag{86}$$

$$-t\left(N_{2j}\left(\boldsymbol{y}/t\right)\right)=\delta_{2j},\quad \text{for } j\in I^{C}\qquad (j=1,2,\cdots,k_2),\qquad (j\neq\ell),\tag{87}$$

$$\left(y_1^F,y_2,\cdots,y_p,t\right)\in \boldsymbol{S_\alpha}.\tag{88}$$

Our basic thought on solving model (85)-(88) is to obtain the second level non-inferior solution $\left(y_1^F,y_2^S,\cdots,y_p^S,t\right)$ that is closest to the FLDM solution $\left(y_1^F,y_2^F,\cdots,y_p^F,t\right)$by following algorithm **I**.

Therefore, we will test whether $\left(y_1^F,y_2^S,\cdots,y_p^S,t\right)$ is a preferred solution to the FLDM or it might be changed according to the accompanying test: If

$$\frac{\left\|F_1\left(y_1^F,y_2^F,\cdots,y_p^F,t\right)-F_1\left(y_1^F,y_2^S,\cdots,y_p^S,t\right)\right\|_2}{\left\|F_1\left(y_1^F,y_2^S,\cdots,y_p^S,t\right)\right\|_2}<\sigma^F\tag{89}$$

Then,$\left(y_1^F,y_2^S,\cdots,y_p^S,t\right)$ is a favored solution to the FLDM, where σ^F is a small positive constant given by the FLDM.

6.3 The Pth Level Decision Maker (Pth LDM) Problem

Consequently, as indicated by the concept of MLMP problems the decision variables of top levels $\left(y_1^F,y_2^S,\cdots,y_{(p-1)}^{(p-1)}\right)$ should be given to the Pth LDM problem; hence, the problem of Pth LDM can be defined as:

$$\underbrace{max}_{\boldsymbol{y_p}}\left(t\left(N_{pj}(\boldsymbol{y}/t)\right),\quad \text{for } j\in I;\ t\left(D_{pj}(\boldsymbol{y}/t)\right),\quad \text{for } j\in I^c\right),\qquad (j=1,2,\cdots,k_p),\tag{90}$$

subject to

$$\left(y_1^F,y_2^S,\cdots y_{(P-1)}^{(P-1)},y_p,t\right)\in \boldsymbol{S_\alpha}.\tag{91}$$

Based on the ε-constraint method the SODM problem of the Pth LDM follows as:

$$max\left(t\left(N_{pj}(\boldsymbol{y}/t)\right),\quad \text{for } j\in I;\ t\left(D_{pj}(\boldsymbol{y}/t)\right),\quad \text{for } j\in I^c\right),\qquad (j=\ell),\tag{92}$$

subject to

$$t\left(D_{pj}\left(\boldsymbol{y}/t\right)\right)\geq\delta_{pj},\qquad \text{for } j\in I\qquad (j=1,2,\cdots,k_p),\qquad (j\neq\ell),\tag{93}$$

$$-t\left(N_{pj}(\boldsymbol{y}/t)\right)=\delta_{pj},\quad \text{for } j\in I^{C}\qquad (j=1,2,\cdots,k_p),\qquad (j\neq\ell),\tag{94}$$

$$\left(y_1^F,y_2^S,\cdots,y_{(P-1)}^{(P-1)},y_p,t\right)\in \boldsymbol{S_\alpha}.\tag{95}$$

The aim of solving model (92)-(95) is to find the Pth LDM non-inferior solution closest to the preferred solutions of the top levels $\left(y_1^F,y_2^S,\cdots y_{(P-1)}^{(P-1)},y_p^p,t\right)$, by following algorithm **I**.

Now, we will test whether $\left(y_1^F,y_2^S,\cdots,y_p^p,t\right)$ is a favored solution to the $(\mathrm{P}-1)^{th}$ LDM or it might be changed according to the accompanying test: If

$$\frac{\left\|F_{(p-1)}\left(y_1^F,y_2^S,\cdots,y_p^{(p-1)},t\right)-F_{(p-1)}\left(y_1^F,y_2^S,\cdots,y_p^p,t\right)\right\|_2}{\left\|F_{(p-1)}\left(y_1^F,y_2^S,\cdots,y_p^p,t\right)\right\|_2}<\sigma^{(p-1)}\tag{96}$$

Then, $\left(y_1^F,y_2^S,\cdots,y_p^p,t\right)$ is a preferred solution to the Pth LDM, which means $\left(x_1^F,x_2^S,\cdots,x_p^p\right)$ is the corresponding preferred solution of the α-(ML-MOFP) problem. Where $\sigma^{(p-1)}$ is a small positive constant given by the $(\mathrm{p}-1)^{th}$ LDM.

For the i^{th} LDM problem δ_{ij}, b_{ij} and a_{ij} are defined as:

$$\delta_{ij}=\left(b_{ij}-a_{ij}\right)s_i+a_{ij},\qquad (i=1,2,\cdots,p),\quad (j=1,2,\cdots,k_p),\tag{97}$$

$$b_{ij}=\underbrace{max}_{(y,t)\in\boldsymbol{S_\alpha}}\left(t\left(N_{ij}(\boldsymbol{y}/t)\right),\quad \text{for } j\in I;\ t\left(D_{ij}(\boldsymbol{y}/t)\right),\quad \text{for } j\in I^c\right),\tag{98}$$

$$a_{ij} = \underbrace{min}_{(y,t)\in \boldsymbol{S_\alpha}} \left(t\left(N_{ij}(\boldsymbol{y}/t)\right), \quad \text{for } j\in I;\ t\left(D_{ij}(\boldsymbol{y}/t)\right), \quad \text{for } j\in I^c\right), \tag{99}$$

where s_i is the satisfactoriness given by the i^{th} level decision maker.

The preferred solution of the i^{th} LDM problem is obtained by the following algorithm:

Algorithm I:

Step 1.	Set the satisfactoriness s_{iv}, $(i = 1,2,\cdots,p)$, $v = 1,2,\cdots$. Let $s_i = s_{i0}$ toward the start, and let $s_i = s_{i1}, s_{i2}, s_{i3}, \cdots, (i = 1,2,\cdots,p)$ respectively.
Step 2.	Set up the ε-constraint problem $P(\varepsilon_{-\ell}(s_{iv}))$, if $P(\varepsilon_{-\ell}(s_{iv}))$ has no solution or has an ideal solution with $t(N_{ij}(\boldsymbol{y}/t)) < \delta_{\ell j}$, for $j\in I$; $t(D_{ij}(\boldsymbol{y}/t)) < \delta_{\ell j}$, for $j\in I^c$, then go to step 1, to adjust $s = s_{i(v+1)} < s_{iv}$. Otherwise, go to step 3.
Step 3.	If the DM is happy with $\left(\frac{y_1^*}{t}, \frac{y_2^*}{t}, \cdots, \frac{y_p^*}{t}\right) = \left(x_1^*, x_2^*, \cdots, x_p^*\right)$, then it is the favored solution of the i^{th}LDM, go to step 5. Otherwise, go to step 4.
Step 4.	Adjust satisfactoriness, let $s_{i(v+1)} > s_{iv}$ and go to step 2.
Step 5.	Stop.

7 Interactive Algorithm for the ML-MOFP Problem with FRVs in the Constraints

Following the discussion in the previous sections, the proposed interactive algorithm will be developed for fathoming the ML-MOFP problems with FRVs as follows:

Step 1.	Formulate the deterministic ML-MOFP model, equations (33)-(42), at the specified value of α.
Step 2.	Compute the individual best and worst values of each objective function in the α-(ML-MOFP) model.
Step 3.	Formulate the linear model of the α-(ML-MOFP) problem, equations (65)-(76).
Step 4.	Calculate the individual best and worst values for each objective function in the linear model (65)-(76).
Step 5.	Set $r = 0$.
Step 6.	Execute the steps presented in Algorithm **I** to acquire an arrangement of favored solutions for the FLDM problem equations (79)-(82). The FLDM puts these solutions in order as indicated by the accompanying configuration : Preferred solution $\left(y_1^r,\cdots,y_p^r\right),\cdots,$ $\left(y_1^{r+n},\cdots,y_p^{r+n}\right) = \left(x_1^r,\cdots,x_p^r\right),\cdots,$ $\left(x_1^{r+n},\cdots,x_p^{r+n}\right)$. Preferred ranking $\left(y_1^r,\cdots,y_p^r\right) \succ \left(y_1^{r+1},\cdots,y_p^{r+1}\right) \succ \cdots \succ \left(y_1^{r+n},\cdots,y_p^{r+n}\right) = \left(x_1^r,\cdots,x_p^r\right) \succ \left(x_1^{r+1},\cdots,x_p^{r+1}\right) \succ\cdots\succ \left(x_1^{r+n},\cdots,x_p^{r+n}\right)$.
Step 7.	Given $y_1^F = y_1^r$, to the SLD. Solve the SLDM problem, equations (85)-(88), following Algorithm **I** and obtain $\left(y_2^s, y_3^s, \cdots, y_p^s, t\right) = \left(y_2^*, y_3^*, \cdots, y_p^*, t\right)$.
Step 8.	If $\frac{\left\Vert F_1\left(y_1^F, y_2^F,\cdots y_p^F,t\right) - F_1\left(y_1^F, y_2^S,\cdots y_p^S,t\right)\right\Vert_2}{\left\Vert F_1\left(y_1^F, y_2^S,\cdots y_p^S,t\right)\right\Vert_2} < \sigma^F$, then go to Step 9. Otherwise go to Step 12.
Step 9.	Given $\left(y_1^F, y_2^S, \cdots, y_{(P-1)}^{(P-1)}, t\right)$ to the PthLDM problem, solve the PthLDM problem, equations (91)-(94), following Algorithm **I** to obtain $\left(y_1^F, y_2^S, \cdots, y_{(P-1)}^{(P-1)}, y_P^p, t\right)$.
Step 10.	If $\frac{\left\Vert F_{(p-1)}\left(y_1^F, y_2^S,\cdots y_p^{(p-1)},t\right) - F_{(p-1)}\left(y_1^F, y_2^S,\cdots y_p^p,t\right)\right\Vert_2}{\left\Vert F_{(p-1)}\left(y_1^F, y_2^S,\cdots y_p^p,t\right)\right\Vert_2} < \sigma^{(p-1)}$, then go to Step 11. Otherwise, go to Step 12.
Step 11.	If the FLDM is happy with $\left(y_1^F, y_2^S, \cdots, y_p^P, t\right)$ and $F_1\left(y_1^F, y_2^S, \cdots, y_p^P, t\right)$, then $\left(\frac{y_1^F}{t}, \frac{y_2^S}{t}, \cdots, \frac{y_p^P}{t}\right) = \left(x_1^F, x_2^S, \cdots, x_p^P\right)$ is the preferred solution of the α-(ML-MOFP) problem, go to Step 13. Otherwise go to Step 12.
Step 12.	Let $r = r+1$, and go to Step 7.
Step 13.	Stop.

8 Numerical Example

To illustrate the proposed interactive approach, consider the following ML-MOFP problem with FRVs in the constraints.

1^{st} ***Level***

$$\underbrace{max}_{\boldsymbol{x_1}} \left(f_{11} = \frac{1.25x_1 + 4.5x_2 - 0.75x_3 - 0.75}{1.5x_1 + 2.5x_2 + 0.75x_3 + 1.5}, \quad f_{12} = \frac{2.5x_1 - 0.75x_2 - 2.5x_3 + 4.5}{1.5x_1 - 1.25x_2 + 0.75x_3 + 4} \right),$$

where $\boldsymbol{x_2}, \boldsymbol{x_3}$ solves
2^{nd} ***Level***

$$\underbrace{max}_{\boldsymbol{x_2}} \left(f_{21} = \frac{-2.5x_1 + 2.5x_2 - 1.5x_3}{0.75x_1 + 0.75x_2 + 0.75x_3 + 2.5}, \quad f_{22} = \frac{7.5x_1 + 2.5x_2 - 0.75x_3 - 0.75}{4x_1 + 1.5x_2 + 0.75x_3 + 0.75} \right),$$

where $\boldsymbol{x_3}$ solves 3^{rd} ***Level***

$$\underbrace{max}_{\boldsymbol{x_3}} \left(f_{31} = \frac{-0.75x_1 + 0.75x_2 - 0.75x_3 + 4.5}{0.75x_1 + 9x_3 + 5.5}, \quad f_{32} = \frac{-1.5x_1 + 0.75x_2 - 0.75x_3 + 3.5}{-1.25x_1 + 0.75x_2 + 0.75x_3 + 8} \right).$$

subject to

$$\text{Prob}\left[\tilde{a}_{11}x_1 + \tilde{a}_{12}x_2 + \tilde{a}_{13}x_3 \leq \tilde{b}_1\right] \geq 0.9,$$
$$\text{Prob}\left[\tilde{e}_{21}x_1 + \tilde{e}_{22}x_2 + \tilde{e}_{23}x_3 \leq \tilde{b}_2\right] \geq 0.95,$$
$$\text{Prob}\left[\tilde{e}_{31}x_1 + \tilde{e}_{32}x_2 + \tilde{e}_{33}x_3 \leq \tilde{b}_3\right] \geq 0.9,$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$$

where $\tilde{a}_{11}$, $\tilde{a}_{12}$, $\tilde{a}_{13}$ and $\tilde{b}_1$ follow normally distributed independent FRVs with fuzzy means and fuzzy variances as follows: $\tilde{m}_{\tilde{a}_{11}} = (5,2,2)_{LR}$, $\widetilde{\delta}^2_{\tilde{a}_{11}} = (3,1,2)_{LR}$, $\tilde{m}_{\tilde{a}_{12}} = (6,1,2)_{LR}$, $\widetilde{\delta}^2_{\tilde{a}_{12}} = (4,2,2)_{LR}$, $\tilde{m}_{\tilde{a}_{13}} = (8,1,2)_{LR}$, $\widetilde{\delta}^2_{\tilde{a}_{13}} = (5,2,2)_{LR}$, $\tilde{m}_{\tilde{b}_1} = (8,1,2)_{LR}$, $\widetilde{\delta}^2_{\tilde{b}_1} = (4,1,3)_{LR}$. $\tilde{b}_2$ is exponentially distributed FRV with parameter $\widetilde{\theta}_2 = (0.03,0.02,0.02)_{LR}$. Also, $\tilde{b}_3$ represents Weibull distributed independent FRV having parameters $\widetilde{\rho} = (2,1,2)_{LR}$, $\widetilde{\mu} = (20,2,4)_{LR}$ and $c = 2$. On the other hand, the coefficients of the probabilistic constraints $\tilde{e}_{21}$, $\tilde{e}_{22}$, $\tilde{e}_{23}$, $\tilde{e}_{31}$, $\tilde{e}_{32}$, and $\tilde{e}_{33}$ are thought to be LR-fuzzy numbers: $\tilde{e}_{21} = (3,1,2)_{LR}$, $\tilde{e}_{22} = (4,1,2)_{LR}$, $\tilde{e}_{23} = (1,1,2)_{LR}$, $\tilde{e}_{31} = (1,1,2)_{LR}$, $\tilde{e}_{32} = (2,1,2)_{LR}$, and $\tilde{e}_{33} = (7,2,3)_{LR}$.

Following the proposed interactive algorithm, the solution of the ML-MOFP problem with FRVs in the constraints follows as:

Based on the chance-constrained approach and α-level properties, expect that an α-level of 0.5 is is acknowledged by the three level DMs. Then, the deterministic model of the ML-MOFP problem with FRVs, is gotten as takes after:
1^{st} ***Level***

$$\underbrace{max}_{\boldsymbol{x_1}} \left(f_{11} = \frac{1.25x_1 + 4.5x_2 - 0.75x_3 - 0.75}{1.5x_1 + 2.5x_2 + 0.75x_3 + 1.5}, \quad f_{12} = \frac{2.5x_1 - 0.75x_2 - 2.5x_3 + 4.5}{1.5x_1 - 1.25x_2 + 0.75x_3 + 4} \right),$$

where $\boldsymbol{x_2}, \boldsymbol{x_3}$ solves
2^{nd} ***Level***

$$\underbrace{max}_{\boldsymbol{x_2}} \left(f_{21} = \frac{-2.5x_1 + 2.5x_2 - 1.5x_3}{0.75x_1 + 0.75x_2 + 0.75x_3 + 2.5}, \quad f_{22} = \frac{7.5x_1 + 2.5x_2 - 0.75x_3 - 0.75}{4x_1 + 1.5x_2 + 0.75x_3 + 0.75} \right),$$

where $\boldsymbol{x_3}$ solves
3^{rd} ***Level***

$$\underbrace{max}_{\boldsymbol{x_3}} \left(f_{31} = \frac{-0.75x_1 + 0.75x_2 - 0.75x_3 + 4.5}{0.75x_1 + 9x_3 + 5.5}, \quad f_{32} = \frac{-1.5x_1 + 0.75x_2 - 0.75x_3 - 3.5}{-1.25x_1 + 0.75x_2 + 0.75x_3 + 8} \right),$$

subject to

$$4x_1 + 5.5x_2 + 7.5x_3 + 1.28\sqrt{2.5x_1^2 + 3x_2^2 + 4x_3^2 + 3.5} \leq 7.5,$$
$$6x_1 + 7x_2 + 9x_3 + 1.28\sqrt{4x_1^2 + 5x_2^2 + 6x_3^2 + 5.5} \leq 9,$$
$$0.05x_1 + 0.07x_2 + 0.01x_3 \leq -\ln 0.95\ ,$$
$$0.16x_1 + 0.2x_2 + 0.08x_3 \leq -\ln 0.95\ ,$$
$$0.5x_1 + 1.5x_2 + 6x_3 \leq 19 + 1.5\sqrt{-\ln 0.9}\,,$$
$$2x_1 + 3x_2 + 8.5x_3 \leq 22 + 3\sqrt{-\ln 0.9}\,,$$
$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$$

Table 1: Individual best, worst values for the fractional model.

	$f_{11}(\boldsymbol{x})$	$f_{12}(\boldsymbol{x})$	$f_{21}(\boldsymbol{x})$	$f_{22}(\boldsymbol{x})$	$f_{31}(\boldsymbol{x})$	$f_{32}(\boldsymbol{x})$
$max\ \ f_{ij}(\boldsymbol{x})$	0.186	1.183	0.237	0.81	0.853	-0.4
$min\ \ f_{ij}(\boldsymbol{x})$	-0.616	0.67	-0.319	-1	0.369	-0.523

The individual best and worst values for the fractional model exhibited in Table 1.

Definition of the liner model for the crisp ML-MOFP problem:

1^{st} ***Level***

$$\underbrace{max}_{y_1} = \begin{pmatrix} f_{11}(y,t) = 1.25y_1 + 4.5y_2 - 0.75y_3 - 0.75t, \\ f_{12}(y,t) = 2.5y_1 - 0.75y_2 - 2.5y_3 + 4.5t \end{pmatrix},$$

where $\boldsymbol{y_2}, \boldsymbol{y_3}$ solves

2^{nd} ***Level***

$$\underbrace{max}_{y_2} = \begin{pmatrix} f_{21}(y,t) = -2.5y_1 + 2.5y_2 - 1.5y_3, \\ f_{22}(y,t) = 7.5y_1 + 2.5y_2 - 0.75y_3 - 0.75t \end{pmatrix},$$

where $\boldsymbol{y_3}$ solves

3^{rd} ***Level***

$$\underbrace{max}_{y_3} = \begin{pmatrix} f_{31}(y,t) = -0.75y_1 + 0.75y_2 - 0.75y_3 + 4.5t, \\ f_{32}(y,t) = -1.25y_1 + 0.75y_2 + 0.75y_3 + 8t \end{pmatrix},$$

subject to

$$4y_1 + 5.5y_2 + 7.5y_3 + 1.28\sqrt{2.5y_1^2 + 3y_2^2 + 4y_3^2 + 3.5t^2} - 7.5t \le 0,$$

$$6y_1 + 7y_2 + 9y_3 + 1.28\sqrt{4y_1^2 + 5y_2^2 + 6y_3^2 + 5.5t^2} - 9t \le 0,$$

$$0.05y_1 + 0.07y_2 + 0.01y_3 + (\ln 0.95\ )\,t \le 0,$$

$$0.16y_1 + 0.2y_2 + 0.08y_3 + (\ln 0.95\ )\,t \le 0,$$

$$0.5y_1 + 1.5y_2 + 6y_3 - \left(19 + 1.5\sqrt{-\ln 0.9}\right) t \le 0,$$

$$2y_1 + 3y_2 + 8.5y_3 - \left(22 + 3\sqrt{-\ln 0.9}\right) t \le 0,$$

$$1.5y_1 + 2.5y_2 + 0.75y_3 + 1.5t \le 1,$$

$$1.5y_1 - 1.25y_2 + 0.75y_3 + 4t \le 1,$$

$$0.75y_1 + 0.75y_2 + 0.75y_3 + 2.5t \le 1,$$

$$4y_1 + 1.5y_2 + 0.75y_3 + 0.75t \le 1,$$

$$0.75y_1 + 9y_3 + 5.5t \le 1,$$

$$1.5y_1 - 0.75y_2 + 0.75y_3 + 3.5t \le 1,$$

$$y_1 \ge 0, y_2 \ge 0, y_3 \ge 0, t > 0,$$

The individual best and worst values for the linear model exhibited in Table 2.

Table 2: Individual best, worst values for the linear model.

	$f_{11}(\boldsymbol{y},t)$	$f_{12}(\boldsymbol{y},t)$	$f_{21}(\boldsymbol{y},t)$	$f_{22}(\boldsymbol{y},t)$	$f_{31}(\boldsymbol{y},t)$	$f_{32}(\boldsymbol{y},t)$
$b_{ij} = max(f_{ij}(\boldsymbol{y},t))$	0.072	0.923	0.116	0.286	0.853	1.489
$a_{ij}{=}min(f_{ij}(\boldsymbol{y},t))$	-0.136	0	-0.139	-0.136	0	0

Formulate and solve the FLDM single-objective decision-making problem (79)-(82):

$$max\ f_{11}(y,t) = 1.25y_1 + 4.5y_2 - 0.75y_3 - 0.75t,$$

subject to

$$2.5y_1 - 0.75y_2 - 2.5y_3 + 4.5t \geq 0.738,$$
$$4y_1 + 5.5y_2 + 7.5y_3 + 1.28\sqrt{2.5y_1^2 + 3y_2^2 + 4y_3^2 + 3.5t^2} - 7.5t \leq 0,$$
$$6y_1 + 7y_2 + 9y_3 + 1.28\sqrt{4y_1^2 + 5y_2^2 + 6y_3^2 + 5.5t^2} - 9t \leq 0,$$
$$0.05y_1 + 0.07y_2 + 0.01y_3 + (\ln 0.95\ )t \leq 0,$$
$$0.16y_1 + 0.2y_2 + 0.08y_3 + (\ln 0.95\ )t \leq 0,$$
$$0.5y_1 + 1.5y_2 + 6y_3 - \left(19 + 1.5\sqrt{-\ln 0.9}\right)t \leq 0,$$
$$2y_1 + 3y_2 + 8.5y_3 - \left(22 + 3\sqrt{-\ln 0.9}\right)t \leq 0,$$
$$1.5y_1 + 2.5y_2 + 0.75y_3 + 1.5t \leq 1,$$
$$1.5y_1 - 1.25y_2 + 0.75y_3 + 4t \leq 1,$$
$$0.75y_1 + 0.75y_2 + 0.75y_3 + 2.5t \leq 1,$$
$$4y_1 + 1.5y_2 + 0.75y_3 + 0.75t \leq 1,$$
$$0.75y_1 + 9y_3 + 5.5t \leq 1,$$
$$1.5y_1 - 0.75y_2 + 0.75y_3 + 3.5t \leq 1,$$
$$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, t > 0,$$

where $\delta_{12} = (b_{12} - a_{12})s_1 + a_{12} = 0.738$, so the solution of the FLDM is $\left(y_1^F, y_2^F, y_3^F, t\right) = (0,\ \ 0.04636,\ \ 0,\ 0.1818)$ and $s_1 = 0.8$, $\sigma^F = 0.05$ are given by the FLDM.

Secondly, formulate the SLDM single-objective decision-making problem (85)-(88):

$$max\ f_{21}(y,t) = -2.5y_1 + 2.5y_2 - 1.5y_3,$$

subject to

$$7.5y_1 + 2.5y_2 - 0.75y_3 - 0.75t \geq -0.031,$$
$$4y_1 + 5.5y_2 + 7.5y_3 + 1.28\sqrt{2.5y_1^2 + 3y_2^2 + 4y_3^2 + 3.5t^2} - 7.5t \leq 0,$$
$$6y_1 + 7y_2 + 9y_3 + 1.28\sqrt{4y_1^2 + 5y_2^2 + 6y_3^2 + 5.5t^2} - 9t \leq 0,$$
$$0.05y_1 + 0.07y_2 + 0.01y_3 + (\ln 0.95\ )t \leq 0,$$
$$0.16y_1 + 0.2y_2 + 0.08y_3 + (\ln 0.95\ )t \leq 0,$$
$$0.5y_1 + 1.5y_2 + 6y_3 - \left(19 + 1.5\sqrt{-\ln 0.9}\right)t \leq 0,$$
$$2y_1 + 3y_2 + 8.5y_3 - \left(22 + 3\sqrt{-\ln 0.9}\right)t \leq 0,$$
$$1.5y_1 + 2.5y_2 + 0.75y_3 + 1.5t \leq 1,$$
$$1.5y_1 - 1.25y_2 + 0.75y_3 + 4t \leq 1,$$
$$0.75y_1 + 0.75y_2 + 0.75y_3 + 2.5t \leq 1,$$
$$4y_1 + 1.5y_2 + 0.75y_3 + 0.75t \leq 1,$$
$$0.75y_1 + 9y_3 + 5.5t \leq 1,$$
$$1.5y_1 - 0.75y_2 + 0.75y_3 + 3.5t \leq 1,$$
$$y_1 = 0,$$
$$y_2 \geq 0, y_3 \geq 0, t > 0,$$

where $\delta_{22}=(b_{22}-a_{22})s_2+a_{22}=-0.031$, so the SLDM solution is $\left(y_1^F,y_2^S,y_3^S,t\right)=(0,\ \ 0.04636,\ \ 0,\ 0.1818)$ and $s_2=0.25$, $\sigma^S=0.05$ are given by the SLDM.

Now, the FLDM test function, equation (89), will be utilized to decide whether the solution $(0,\ \ 0.04636,\ \ 0,\ 0.1818)$ is acceptable or not:

$$\frac{\|F_1(0,0.04636,0,0.1818)-F_1(0,0.04636,0,0.1818)\|_2}{\|F_1(0,0.04636,0,0.1818)\|_2}=\frac{\|(0.0723,\ 0.7833)-(0.0723,0.7833)\|_2}{\|(0.0723,\ 0.7833)\|_2}=0<0.005$$

Finally, formulate the TLDM single-objective decision-making problem (92)-(95):

$$max\ f_{31}(y,t)=-0.75y_1+0.75y_2-0.75y_3+4.5t,$$

subject to

$$-1.25y_1+0.75y_2+0.75y_3+8t\geq 1.489,$$

$$4y_1+5.5y_2+7.5y_3+1.28\sqrt{2.5y_1^2+3y_2^2+4y_3^2+3.5t^2}-7.5t\leq 0,$$

$$6y_1+7y_2+9y_3+1.28\sqrt{4y_1^2+5y_2^2+6y_3^2+5.5t^2}-9t\leq 0,$$

$$0.05y_1+0.07y_2+0.01y_3+(\ln 0.95\ )t\leq 0,$$

$$0.16y_1+0.2y_2+0.08y_3+(\ln 0.95\ )t\leq 0,$$

$$0.5y_1+1.5y_2+6y_3-\left(19+1.5\sqrt{-\ln 0.9}\right)t\leq 0,$$

$$2y_1+3y_2+8.5y_3-\left(22+3\sqrt{-\ln 0.9}\right)t\leq 0,$$

$$1.5y_1+2.5y_2+0.75y_3+1.5t\leq 1,$$

$$1.5y_1-1.25y_2+0.75y_3+4t\leq 1,$$

$$0.75y_1+0.75y_2+0.75y_3+2.5t\leq 1,$$

$$4y_1+1.5y_2+0.75y_3+0.75t\leq 1,$$

$$0.75y_1+9y_3+5.5t\leq 1,$$

$$1.5y_1-0.75y_2+0.75y_3+3.5t\leq 1,$$

$$y_1=0,$$

$$y_2=0.04636,$$

$$y_3\geq 0,t>0,$$

where $\delta_{32}=(b_{32}-a_{32})s_3+a_{32}=1.489$, so the TLDM solution is $\left(y_1^F,y_2^S,y_3^T,t\right)=(0,\ \ 0.04636,\ 0,\ 0.1818)$ and $s_3=1$, is given by the TLDM.

Now, the SLDM test function, equation (96), will be utilized to decide whether the solution $(0,\ 0.04636,\ 0,\ 0.1818)$ is acceptable or not:

$$\frac{\|F_2(0,0.04636,0,0.1818)-F_2(0,0.04636,0,0.1818)\|_2}{\|F_2(0,0.04636,0,0.1818)\|_2}=\frac{\|(0.1159,\ -0.0205)-(0.1159,0.0205)\|_2}{\|(0.1159,\ 0.0205)\|_2}=0<0.005$$

So $\left(y_1^F,y_2^S,y_3^T,t\right)=(0,\ 0.04636,\ 0,\ 0.1818)$ is the preferred solution, which means that $(x_1,x_2,x_3)=(0,\ 0.255,\ 0)$ is the corresponding preferred solution to the ML-MOFP problem.

The comparison between the proposed interactive approach and the strategy for Lachhwani [14] is given in Table 3. The outcomes demonstrate that the preferred solution of the proposed interactive approach and the method of Lachhwani 14 are close to one another.

Table 3: Comparison between the proposed interactive approach and Lachhwani [14].

Interactive approach		Lachhwani [14]		Ideal objective vector
$f_{11} = 0.1859$	$\mu_{11} = 0.999$	$f_{11} = 0.182$	$\mu_{11} = 0.995$	$f_{11} = 0.186$
$f_{12} = 1.17$	$\mu_{12} = 0.975$	$f_{12} = 1.17$	$\mu_{12} = 0.975$	$f_{12} = 1.183$
$f_{21} = 0.2368$	$\mu_{21} = 0.999$	$f_{21} = 0.235$	$\mu_{21} = 0.996$	$f_{21} = 0.237$
$f_{22} = -0.099$	$\mu_{22} = 0.498$	$f_{22} = -0.1$	$\mu_{22} = 0.497$	$f_{22} = 0.81$
$f_{31} = 0.853$	$\mu_{31} = 1$	$f_{31} = 0.853$	$\mu_{31} = 1$	$f_{31} = 0.853$
$f_{32} = -0.4$	$\mu_{32} = 1$	$f_{32} = -0.4$	$\mu_{32} = 1$	$f_{32} = -0.4$

9 Conclusions

In this paper, an effective and powerful interactive approach is presented to solve ML-MOFP problem with FRVs in the constraints. Applying the CCP technique and α-level concept, a numerical crisp model of the ML-MOFP is formulated. Moreover, the linear model of the problem is developed by extending the work of M. Chakraborty and S. Gupta [8]. Then, the interactive approach simplifies the ML-MOLP model by changing it into isolated MODM problems, thus the complex MLMP problem is simplified and the non-convex mathematical programming difficulty has been avoided. Hence, the ε-constraint method and the concept of satisfactoriness is utilized to solve each isolated MODM problem of the ML-MOLP model. A procedure has been suggested for solving the ML-MOFP problem with FRVs in the constraints.

Several open points for research in the area of ML-MOFP, from our point of view, to be studied in the future. Some of these focuses are given in the following:

1. The interactive algorithm is needed for dealing with ML-MOFP in a rough environment.
2. The interactive algorithm is needed for tackling integer ML-MOFP with fuzzy parameters.

References

[1] M. A. Abo-Sinna, I.A. Baky, Interactive balance space approach for solving multi-level multi-objective programming problems, Information Sciences, 177 (2007) 3397- 3410.

[2] S.R. Arora, R. Gupta, interactive fuzzy goal programming approach for bilevel programming problem, European Journal of Operational Research 194, (2) (2009) 368-376.

[3] J. J. Buckley, E. Eslami, uncertain probabilities-II, Soft Computing, 8 (2004) 193-199.

[4] A. Biswas, A. K. De, on solving multi-objective quadratic programming problems in a probabilistic fuzzy environment, Fourth International Conference on Soft Computing for Problem Solving, Advances in Intelligent Systems and Computing 335, DOI 10.1007/978-81-322-2217-0_44.

[5] Ibrahim A. Baky, Solving multi-level multi-objective linear programming problems through fuzzy goal programming approach, Applied Mathematical Modelling, 34 (9) (2010) 2377-2387.

[6] Ibrahim A. Baky, M.H. Eid, M.A. Elsayed, Bi-level multi-objective programming problem with fuzzy demands: a fuzzy goal programming algorithm, Journal of Operational Research Society of India, 51 (2) (2014) 280-296.

[7] L.H. Chen and H.H. Chen, A two-phase fuzzy approach for solving multi-level decision-making problems, Knowledge-Based Systems 76 (2015) 189-199.

[8] M. Chakraborty and S. Gupta, Fuzzy mathematical programming for multi-objective linear fractional programming problem, Fuzzy Sets and Systems, 125 (2002) 335-342.

[9] O. E. Emam, Interactive approach to bi-level integer multi-objective fractional programming problem, Applied Mathematics and Computation, 233(2013) 17-24.

[10] S. Hulsurkar, M. P. Biswal, S. B. Sinha, fuzzy programming approach to multi-objective stochastic linear programming problems, Fuzzy Sets and Systems, 88 (1997) 173-181.

[11] M. G. Iskander, a possibility programming approach for stochastic fuzzy multi-objective linear fractional programs, Computer and Mathematics with Applications, 48 (2004) 1603-1609.

[12] S. Javaid, S. I. Ansari, Z. Anwar, multi-objective stochastic linear programming problem when b_i's follow Weibull distribution OPSEARCH 50 (2) (2013) 250-259.

[13] H. Kwakernaak, Fuzzy random variables – I. definitions and theorems, Information Sciences 15 (1978) 1–29.

[14] K. Lachhwani, Modified FGP approach for multi-level multi objective linear fractional programming problems, Applied Mathematics and Computation 266 (2015) 1038-1049.

[15] F. Liu, Z. Wen, Y. Xu, a dual-uncertainty-based chance-constrained model for municipal solid waste management, Applied Mathematical Modelling, 37 (2013) 9147-9159.

[16] N. Modak, A. Biswas, a fuzzy programming approach for bi-level stochastic programming, Second International Conference on Soft Computing for Problem Solving, December 28-30, 2012, Advances in Intelligent Systems and Computing 236, DOI 10.1007/978-81-322-1602-5_14.

[17] M. S. Osman, M. A. Abo-Sinna, A. H. Amer and O. E. Emam, (2003) Interactive decision making for three-level multi-objective non-linear programming problems, in:The 38th Annual conference on Statistics, Computer Sciences and Operation Research, (2003) 95–110.
[18] M. S. Osman, O. E. Emam, M. A. Elsayed, on parametric multi-level multi-objective fractional programming problems with fuzziness in the constraints, British Journal of Mathematics & Computer Science, 18 (5) (2016) 1-19.
[19] S. Pramanik and T.K. Roy, Fuzzy goal programming approach to multi-level programming problems, European Journal of Operational Research, 176 (2) (2007)1151-1166.
[20] A. Ren, Y. Wang, An approach for solving a fuzzy bi-level programming problem through nearest interval approximation approach and KKT optimality conditions, Soft Computing, DOI 10.1007/s 00500-016-2144-8.
[21] A. Ren, Y. Wang, X. Xue, Interactive programming approach for solving the fully fuzzy bi-level linear programming problem, Knowledge-Based Systems, 99 (2016) 103-111.
[22] X. Shi and H. Xia, Interactive bi-level multi-objective decision making, Journal of the Operational Research Society, 48 (9) (1997) 943-949.
[23] M. Sakawa, I. Nishizaki and Y. Uemura, A decentralized two-level transportation problem in a housing material manufacturer: interactive fuzzy programming approach, European Journal of Operational Research,141(2002) 167-185.
[24] M. Sakawa, I. Nishizaki and Y. Uemura, Interactive fuzzy programming for multi-level linear programming problems with fuzzy parameters, Fuzzy Sets and Systems, 109 (1) (2000) 03- 19.
[25] M. Sakawa, I. Nishizaki and Y. Uemura, interactive fuzzy random two-level linear programming through fractile criterion optimization, Mathematical and Computer Modelling, 54 (2011) 3153-3163.
[26] M. D. Toksari and Y. Bilim, Interactive fuzzy goal programming based on jacobian matrix to solve decentralized bi-level multi-objective fractional programming problems, International Journal of Fuzzy System, 17 (4) (2015) 499-508.
[27] G. Zhang, J. Lu, T. Dillon, decentralized multi-objective bi-level decision making with fuzzy demands, Knowledge-Based Systems 20 (2007) 495–507.
[28] J. J. Buckley, Fuzzy probability and statistics, Springer-Verlag Berlin Heidelberg 2006.
[29] M. Sakawa, Fuzzy sets and interactive multi-objective optimization, Plenum Press, New York, NY, USA, 1993.

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