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Generation of Distribution Functions: A Survey

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Abstract: Generation of distributions, by composition of cumulative distribution functions (CDFs), compounding or mixing, leads the population distribution to be more flexible to analyzing data. On the other hand, heavy-tailed skewed distributions can be generated by compounding. Different generating methods are surveyed. Such methods include generation by composition, compounding and mixing (countable or finite). Relationships of some of the generated distributions to other distributions, or functions are presented.

Keywords: Generation, composition, compounding, mixture of distributions

1 Introduction

New distribution functions can be generated by using different generating methods. For example, composition of CDF H with another CDF G, generates a new CDF that performs better than the base line CDF G, in the setting. In this article, the following generating methods will be surveyed in Sections 2-4, followed by some concluding remarks in Section 5.

2. Generation by Composition
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2 Generation by Composition

2.1 Composition of a CDF with another CDF on the support $(0,1)$

Suppose that $H(.)$ and $G(.)$ are two absolutely continuous CDFs whose corresponding PDFs are $h(.)$ and $g(.)$, respectively. Suppose also that the composition of $H(.)$ and $G(.)$ yields a CDF given by

$$F(x) = H[G(x)] = \int_0^{G(x)} h(y)\mathrm{d}y, \tag{1}$$

with PDF $f(.)$. In this composition, $H(.)$ is assumed to have support the unit interval $(0,1)$, while $G(.)$ is an arbitrary CDF, defined on the whole real line. Two choices for the PDF $h(.)$ are known in literature:

–when $h(.)$ is the beta PDF,
–when $h(.)$ is the Kumaraswamy (Kw) PDF.

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2.1.1 The beta-G family

In this case, $h(y)$ is chosen to be the beta PDF, given by

$$h(y) = \frac{1}{B(a,b)} y^{a-1}(1-y)^{b-1}, \quad 0 < y < 1.$$

Therefore, Equation (1) becomes

$$F(x) = H[G(x)] = \frac{1}{B(a,b)} \int_0^{G(x)} y^{a-1}(1-y)^{b-1} \quad \mathrm{d}y, \tag{2}$$

where $B(a,b)$ is the beta function. The CDF, given by (2), is known as the beta-G distribution.

Eugene et al [1] studied the beta-normal distribution. Jones [2] generalized the beta-normal distribution to include an arbitrary G instead of specifying G to be normal.

Remark 1 *$F(x)$, given by (2), is the incomplete beta ratio, denoted by $I_G(a,b)$. So, we can write*

$$F(x) = I_G(a,b), \tag{3}$$

where $I_G(a,b)$ is given by (2).

Remark 2 *An important special case of (2) is the case of exponentiated-distributions which can be obtained by taking $b = 1$, in (2), so that*

$$F(x) = [G(x)]^a. \tag{4}$$

For more details on exponentiated distributions, see AL-Hussaini and Ahsanullah [3]. Nadarajah [4] and Nadarajah et al [5] surveyed the exponentiated Weibull and exponentiated exponential distributions, respectively.

By specifying G, some of the beta-G distributions were obtained and studied by the following researchers (Table 1).

2.1.2 Kumaraswamy-G (Kw-G) Family

Kw [6] suggested the use of a CDF of the form

$$H(y) = 1-(1-y^a)^b, \quad 0 < y < 1, \quad (a,b > 0),$$

as a model in hydrology processes. The corresponding PDF is given by

$$h(y) = aby^{a-1}(1-y^a)^{b-1}.$$

So that, for arbitrary G, composite function (1) becomes

$$F(x) = H[G(x)] = 1-(1-[G(x)]^a)^b, \tag{5}$$

and the corresponding PDF takes the form

$$f(x) = ab[G(x)]^{a-1}(1-[G(x)]^a)^{b-1}. \tag{6}$$

The Kw-G distribution was used by some researchers when G is arbitrary and when G has specific form. A list of such researchers is given in Table 2.

The Kw - Weibull model was used in accelerated life testing by Rezk [27], Rezk et al [28] and AL-Dayan et al [29].

Jones [30] compared the beta distribution with Kw distribution. He summarized pros, cons and equivalences for the two distributions and concluded by saying that: “The Kw is certainly not superior to the beta distribution in any way, but it might be worth consideration from time to time by researchers who wish to utilize one or more of its simple properties”. In addition to its “simple properties”, Cordeiro et al [11] noticed that the Kw(a,b) distribution has a physical interpretation when a and b are positive integers. They explain their remark as follows:

Suppose that a system is made of b independent components and that each component is made up of a independent

Table 1: Beta-G family when G is specified as given and reference.

$G(x)$	Reference
Normal	Eugene et al [1]
Fréchet	Nadarajah and Gupta [7]
Gumble	Nadarajah and Kotz [8], Jonsson [9]
Weibull	Famoye et al [10], Cordeiro et al [11]
Exponential	Nadarajah and Kotz [12]
Gamma	Kong et al [13]
Extreme value type III	Zafar and Aleem [14]
Pareto type I	Akinsete et al [15]
Exponentiated Exponential	Barreto-Souza et al [16]
Laplace	Condeiro and Lemonte [17]
Burr type XII	Paranoíba et al [18]
Exponentiated Weibull	Singla et al [19]
Power	Cordeiro and Brito [20]
Exponentiated logistic	Nassar and El-Masry [21]

Table 2: Kw-G family when G is specified as given and reference.

$G(x)$	Reference
Weibull	Cordeiro et al [11]
Modified Weibull	Mateas-Salvadero et al [22]
Arbitrary	Nadarajah et al [23]
Pareto	Bourguignon et al [24]
Log-logistic	De-Santana et al [25]
Generalized gamma	De-Pascoa et al [26]

sub-components. Suppose that the system fails if any of the b components fail and that each component of all of the a sub-components fail. Let X be the lifetime of the entire system. Cordeiro et al [11] showed that the CDF of X is given by

$$P[X \leq x] = 1 - (1 - [G(x)]^a)^b.$$

So that the Kw-G distribution represents the time distribution of the entire system.

Nadarajah [31] pointed out that the Kw distribution is a special case of McDonald's [32] distribution whose PDF is given by

$$f(x) = \frac{\beta x^{\beta a-1}[1-(\frac{x}{\gamma})^{\beta a}]^{b-1}}{B(a,b)\gamma^{\beta a}}, \quad x > 0, \quad (\alpha, \beta, \gamma > 0).$$

In fact, if $a = 1$ and $\gamma = 1$, then

$$f(x) = b\beta x^{\beta-1}[1-x^{\beta}]^{b-1},$$

which is the PDF of Kw (β, b) distribution. It can be shown that the Kw-G distribution is the same as a beta-exponentiated G (EG) distribution, when the beta distribution has parameters $(1, b)$ and the exponent of the EG distribution is c.

2.1.3 Relation of the beta- family to the hyper-geometric function and other CDFs

(i) Relation to the hyper-geometric function:

By expanding $(1-y)^{b-1}$ in the integral $\int_0^{G(x)} y^{a-1}(1-y)^{b-1}dy$, it follows that:

$$I_G(a,b) = \frac{[G(x)]^a}{aB(a,b)}\, {}_2F_1(a, 1-b, a+1; G(x)),$$

$${}_2F_1(a,b,c;z) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_{jz^j}}{(c)_j\, j!} \text{ and } (d)_j = d(d+1)\ldots(d+j-1).$$

(ii) Relation to the binomial distribution:
By repeated integration by parts,

$$I_G(a,n-a+1)=\sum_{j=a}^{n}\binom{n}{j}[G(x)]^j[1-G(x)]^{n-j}.$$

(iii) Relation to the negative binomial distribution:

$$I_{1-G}(a,n)=\sum_{j=a}^{n}\binom{n+j-1}{j}[G(x)]^n[1-G(x)]^j.$$

(iv) Relation to the χ^2-distribution:
If X_1 and X_2 are two independent random variables (RVs) such that $X_i\sim\chi^2(\nu_i),\ i=1,2$, then

$$Z=\frac{X_1}{X_1+X_2}\sim \mathrm{beta}(\nu_1,\nu_2)\Rightarrow F_Z(z)=P(Z\le z)=I_G(\nu_1/2,\nu_2/2).$$

(v) Relation to the t- distribution:

$$I_G(\nu/2,1/2)=1-A(t),$$

where

$$A(t)=\int_{-t}^{t}\frac{1}{\sqrt{2\pi}}e^{-u^2/2}\mathrm{d}u,\qquad x=\frac{\nu}{\nu+t^2}.$$

(vi) Relation to the f- distribution:

$$I_x(\nu_2/2,\nu_1/2)=Q(x),$$

where

$$Q(t)=\int_{t}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-u^2/2}\mathrm{d}u,\quad x=\frac{\nu_1}{\nu_2+\nu_1 t}.$$

See Abramowitz and Stegun ([33], p. 945).

2.2 Composition of a CDF with a function of another CDF, in the general case

2.2.1 Composition of H with $\eta_1(x)=-\ln\overline{G}(x)$

$$F(x)=H[\eta_1(x)]=\int_{-\infty}^{\eta_1(x)}h(y)\mathrm{d}y=\int_{-\infty}^{-\ln\overline{G}(x)}h(y)\mathrm{d}y, \tag{7}$$

where $\overline{G}(x)=1-G(x)$ and H is a CDF over the whole real line.
If H is gamma$(\delta,1)$ with PDF

$$h(y)=\frac{1}{\Gamma(\delta)}y^{\delta-1}e^{-y},\quad y>0,\quad(\delta>0), \tag{8}$$

a generated F is then given by

$$\begin{aligned}F(x)=H[\eta_1(x)]&=\frac{1}{\Gamma(\delta)}\int_{0}^{-\ln\overline{G}(x)}y^{\delta-1}e^{-y}\,\mathrm{d}y\\&=\frac{\gamma[\delta,-\ln\overline{G}(x)]}{\Gamma(\delta)},\end{aligned} \tag{9}$$

where $\gamma(\delta,z)$ is the incomplete gamma function, given by

$$\gamma(\delta,z)=\int_0^z y^{\delta-1}e^{-y}\,\mathrm{d}y. \tag{10}$$

The corresponding PDF is given by

$$f(x)=\frac{1}{\Gamma(\delta)}[-\ln\overline{G}(x)]^{\delta-1}g(x),\quad x>0. \tag{11}$$

The function g is the PDF corresponding to G.

Abdel-Hamid and Albasuoni [34] applied this technique to obtain a new distribution by composing a log-logistic distribution with a Weibull distribution.

Zografos and Balakrishnan [35], showed that if in (11), $\delta=n$ is a positive integer, then (11) is the PDF of the upper record value from a sequence of independently identically distributed (iid) RVs, drawn from a population with PDF g. They also showed that if a RV X follows a distribution with PDF (11), then $Z=-\ln\overline{G}(x)\sim\text{gamma}(\delta,1)$, given by (8) and if $Z\sim\text{gamma}(\delta,1)$, then $X=G^{-1}(1-e^{-Z})\sim f$, given by (11).

2.2.2 Composition of H with $\eta_2(x)=-\ln G(x)$

$$\begin{aligned}\overline{F}(x)=H[\eta_2(x)]&=\int_{-\infty}^{\eta_2(x)}h(y)\mathrm{d}y\\&=\frac{1}{\Gamma(\delta)}\int_0^{-\ln G(x)}y^{\delta-1}e^{-y}\quad\mathrm{d}y\\&=\frac{\gamma(\delta,-\ln G(x))}{\Gamma(\delta)},\end{aligned} \tag{12}$$

where $\gamma(\delta,z)$ is the incomplete gamma function, given by (11) with $z=-\ln G(x)$.

The corresponding PDF is given by

$$f(x)=\frac{1}{\Gamma(\delta)}[-\ln G(x)]^{\delta-1}g(x),\quad x\in\Re,\quad\delta>0. \tag{13}$$

Ristić and Balakrishnan [36] studied this "dual" case whose population survival function (SF) is given by (12). They showed that if δ is a positive integer, then (12) represents the SF of the lower record from a sequence of iid RVs from a population with PDF $g(x)$. Similarly, they showed that if $X\sim f$, given by (13), then $Z=-\ln G(x)\sim\text{gamma}(\delta,1)$ and if $Z\sim\text{gamma}(\delta,1)$, then $X=G^{-1}(e^{-Z})\sim f$ given by (13).

If $G(x)$ is chosen to be exponentiated exponential distribution (EED), given by $G(x)=[1-e^{-\beta x}]^{\alpha}$, then Equation (13) becomes

$$\overline{F}(x)=\frac{1}{\Gamma(\delta)}\int_0^{-\alpha\ln(1-e^{-\beta x})}y^{\delta-1}e^{-y}\quad\mathrm{d}y. \tag{14}$$

Some properties and inferences, in this case were studied by Ristić and Balakrishnan [36].

(i) Relation of $\overline{F}(x)$ to the confluent hyper-geometric function

It can be shown that

$$\overline{F}(x)=\frac{1}{\Gamma(\delta)}\left[\frac{(-\alpha\ln(1-e^{-\beta x}))^{\delta}}{\delta}\right]M[\delta,1+\delta,ln(1-e^{-\beta x})], \tag{15}$$

where $M[a,b;z]$ is the confluent hyper-geometric function, defined by

$$M[a,b;z]=1+\frac{az}{b}+\frac{(a)_2z^2}{(b)_22!}+\cdots+\frac{(a)_nz^n}{(b)_nn!}+\ldots, \tag{16}$$

where $(d)_j=d(d+1)\ldots(d+j-1),\quad j=1,2,\ldots.$

(ii) Relation of $\overline{F}(x)$ to the Poisson distribution

Assuming that $\delta = k$ is a non-negative integer and $-\alpha\ln(1-e^{-\beta x}) = z$, it follows, from (13) that

$$\begin{aligned}\overline{F}(x) =& \frac{\gamma(k,z)}{\Gamma(k)}\\ =& \frac{1}{\Gamma(k)}\left[(k-1)!\sum_{j=k}^{\infty}\frac{z^j}{j!}e^{-z}\right]\\ =& (1-e^{-\beta x})^{\alpha}\sum_{j=k}^{\infty}\frac{[-\ln(1-e^{-\beta x})]^j}{j!}.\end{aligned} \tag{17}$$

(iii) A simple series for the incomplete gamma integral, suggested by Lau [37], can be used to write the SF as follows

$$\begin{aligned}\overline{F}(x) = \frac{\gamma(k,z)}{\Gamma(\delta)} =& \frac{1}{\Gamma(\delta)}\int_0^z y^{\delta-1}e^{-y}\mathrm{d}y\\ =& A\sum_{j=0}^{\infty}C_j(\delta,z),\end{aligned} \tag{18}$$

where

$$z = -\ln G(x), \quad A = \frac{z^{\delta}e^{-\delta}}{\Gamma(\delta+1)}, C_0 = 1 \quad \text{and} \quad C_j(\delta,z) = \frac{z}{\delta+j}C_{j-1}(\delta,z),\ j = 1,2,\ldots.$$

For proof of (18), see [37].

2.3 Composition of a symmetric PDF with a transformation of scale

Jones [38] generated a PDF f by composing a symmetric PDF g with transformation of scale $t(x)$, in such a way that

$$f(x) = 2g[t(x)], \quad -\infty < x < \infty. \tag{19}$$

Two of such transformations were suggested by Baker [39]:

$$t_1(x) = x - \frac{b}{x}, \quad x > 0, \quad b > 0,$$

and

$$t_2(x) = \frac{1}{a}\ln(e^{ax}-1), \quad x > 0, \quad a > 0.$$

In addition, Jones [38] suggested the following four transformations of scale:

$$t_3(x) = c\left(2\sqrt{\frac{x}{c}}-1\right)I(0<x<c) + xI(x\geq c), \quad x>0, \quad c>0,$$

$$t_4(x) = d\left(1-\frac{x^2}{d^2}\right)I(x<-d) + d\left(\frac{x}{d}+\frac{1}{2}+\sqrt{\frac{x}{d}+\frac{4}{5}}\right)I(x\geq -d), \quad -\infty<x<\infty, \quad d>0,$$

$$t_5(x) = \frac{2x}{1+a}I(x<0) + \frac{2x}{1-a}I(x\geq 0), \quad -1<a<1, \quad -\infty<x<\infty,$$

$$\begin{aligned}t_6(x) =& \frac{2x}{1+a}I(x<-\frac{1}{2}(1+a)) + \frac{1}{a}(1-\sqrt{1-a(4x+a)})I(-\frac{1}{2}(1+a)\leq x<\frac{1}{2}(1-a))\\ &+\frac{2x}{1-a}I(x\geq\frac{1}{2}(1-a)), \quad -\infty<x<\infty, \quad -1<a<1,\end{aligned}$$

where $I(A) = \begin{cases}1, \ x\in A,\\ 0, \ x\notin A\end{cases}$ is the indicator function.

For example, if the symmetric PDF g is chosen to be the standard normal distribution $N(0,1)$, then the generated PDF $f(x)$ is given, using $t_3(x)$, by

$$f(x) = 2g(t_3(x)) = \frac{2}{\sqrt{2\pi}} \begin{cases} \exp\left[-\frac{1}{2}\left[c\left(2\sqrt{\frac{x}{c}}-1\right)\right]^2\right], & 0 < x < c, \\ \exp\left(-\frac{x^2}{2}\right), & x \geq c. \end{cases}$$

It can be shown that the generated function f is a PDF. In fact,

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\mathrm{d}x =& \frac{2}{\sqrt{2\pi}}\int_0^c e^{-[c(2\sqrt{\frac{x}{c}}-1)]^2}\mathrm{d}x + \frac{2}{\sqrt{2\pi}}\int_c^{\infty} e^{-x^2/2}\mathrm{d}x \\ =& I_1 + I_2, \end{aligned}$$

where

$$I_1 = \frac{2}{\sqrt{2\pi}}\int_0^c e^{-[c(2\sqrt{\frac{x}{c}}-1)]^2}\mathrm{d}x,$$

and

$$I_2 = \frac{2}{\sqrt{2\pi}}\int_c^{\infty} e^{-x^2/2}\mathrm{d}x = 2P[X > c],$$

where $X \sim N(0,1)$.

By applying the substitution $z = c(2\sqrt{x/c}-1)$ to I_1, we have $x = c[\frac{1}{2}(\frac{z}{c}+1)]^2$, so $\mathrm{d}x = \frac{1}{2}(\frac{z}{c}+1)\mathrm{d}z, (0,c) \to (-c,c)$. It follows that

$$I_1 = \frac{2}{\sqrt{2\pi}}\int_{-c}^{c} e^{-z^2/2}\frac{1}{2}(\frac{z}{c}+1)\mathrm{d}z = \frac{2}{\sqrt{2\pi}}\int_{-c}^{c} e^{-z^2/2}\mathrm{d}z = P[-c < X < c] = 1 - 2P[X < c].$$

Adding up I_1 and I_2, $\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1$.

3 Generation by Compounding

Suppose that CDFs $G(x|\theta)$ and $H(\theta)$ have PDFs $g(x|\theta)$ and $h(\theta)$, respectively. A generated CDF F is given by

$$F(x) = \int_{\Theta} G(x|\theta)\mathrm{d}H(\theta). \tag{20}$$

Fisher [40] called this generated distribution "compound" distribution. Teicher [41] called it "mixture" of the two distributions G and H. If $G(x|\theta)$ and $H(\theta)$ are absolutely continuous, then the corresponding generated PDF is given by

$$f(x) = \int_{\Theta} g(x|\theta)h(\theta)\mathrm{d}\theta. \tag{21}$$

Table 3 displays some generated PDFs $f(x) = \int_0^{\infty} g(x|\theta)h(\theta)\mathrm{d}\theta$ for given $g(x|\theta)$ and $h(\theta)$.

Remark 3 *If, in the Poisson-gamma case (first in Table 3), $\alpha = r$ and $\beta = (1-p)/p$, where r is a positive integer and $0 < p < 1$, then*

$$f(x) = \binom{r+x-1}{r-1} p^r(1-p)^x, \quad x = 0,1,2,\ldots,$$

which is the probability mass function (PMF) of the negative binomial (r,p) distribution.

Remark 4 *If, in the normal-gamma case (fourth in Table 3), $\alpha = r/2$ and $\beta = 2/r$, where r is a positive integer, then X has the student's t-distribution with r degrees of freedom.*

Remark 5 *Heavy-tailed skewed distributions can be obtained by compounding. For example, in the gamma-gamma case, (third in Table 3), the resulting distribution, is also known as the "generalized Pareto distribution". Both of the compound PDFs have their tails thicker than (conditional) gamma distribution.*

Table 3: Generated PDFs $f(x)=\int_0^\infty g(x|\theta)h(\theta)d\theta$ for given $g(x|\theta)$ and $h(\theta)$

$g(x\|\theta)$	$h(\theta)$	$f(x)$
1. Poisson $e^{-\theta}\frac{\theta^k}{x!}$	Gamma(α,β) $\frac{1}{\Gamma(\alpha)\beta^\alpha}\theta^{\alpha-1}e^{-\frac{\theta}{\beta}}$	Poisson-gamma $\frac{\beta^x\Gamma(x+\alpha)}{\Gamma(\alpha)x!(\beta+1)^{\alpha+x}},\ x=0,1,\ldots$
2. Binomial $\binom{n}{x}\theta^x(1-\theta)^{n-x}$	Beta(α,β) $\frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$	Binomial-beta $\binom{n}{x}\frac{B(\alpha+x,n-x+\beta)}{B(\alpha,\beta)},\ x=0,1,\ldots,n$
3. Gamma$(k,\frac{1}{\theta})$ $\frac{\theta^k}{\Gamma(k)}x^{k-1}e^{-\theta x}$	Gamma(α,β) $\frac{1}{\Gamma(\alpha)\beta^\alpha}\theta^{\alpha-1}e^{-\frac{\theta}{\beta}}$	Gamma-gamma $\frac{\beta^k x^{k-1}}{B(\alpha,k)(1+\beta x)^{k+\alpha}},\ x>0$
4. Normal$(0,\frac{1}{\theta})$ $\sqrt{\frac{\theta}{2\pi}}e^{-\theta x^2/2}$	Gamma(α,β) $\frac{1}{\Gamma(\alpha)\beta^\alpha}\theta^{\alpha-1}e^{-\frac{\theta}{\beta}}$	Normal-gamma $\frac{\Gamma[\alpha+1/2]}{\beta^\alpha\sqrt{2\pi}\Gamma[\alpha]}(\frac{2\beta}{2+\beta x^2})^{\alpha+1/2},\ -\infty<x<\infty$
5. Weibull $(b,\frac{1}{\theta})$ $\theta b x^{b-1}e^{-\theta x^b}$	Gamma(α,β) $\frac{1}{\Gamma(\alpha)\beta^\alpha}\theta^{\alpha-1}e^{-\frac{\theta}{\beta}}$	Weibull-gamma $\alpha b\beta(1+\beta x^b)^{-\alpha-1},\ x>0$ Also known as Burr type XII(α,b,β).
6. $e^{\theta[1-e^{-(\lambda x)^\beta}]}$	Exp(α) $\alpha e^{-\alpha\theta}$	Marshall-Olkin $\overline{G}(x)=\frac{\alpha e^{-(\lambda x)^\beta}}{1-(1-\alpha)e^{-(\lambda x)^\beta}},\ x>0.$

Remark 6 *Relation between Pareto type II and Burr type XII.*
When $k=1$, so that $X|\theta\sim Exp(\theta)$, then the gamma-exponential case leads to

$$h(x)=\alpha\beta(1+\beta x)^{-\alpha-1},\quad x>0,\quad(\alpha,\ \beta>0). \tag{22}$$

This is the PDF of Pareto type II (α,β) distribution (also known as Lomax (α,β) or compound exponential (α,β) distribution).

The CDF corresponding to (6) is given by

$$H(x)=1-(1+\beta x)^{-\alpha},\quad x>0. \tag{23}$$

If $X=Y^c$, where $X\sim$ gamma-gamma $(\alpha,\beta,k=1)$ and $c>0$, then $Y\sim$ Burr type XII (α,β,c). This is so because

$$F_Y(y)=F_X(y^c)=1-(1+\beta y^c)^{-\alpha},\quad y>0. \tag{24}$$

This is the three-parameter Burr type XII (α,β,c), which is useful in modeling thicker tailed distributions.

The three-parameter Burr type XII distribution can be obtained by compounding the Weibull with the gamma distributions, as given in the fifth case of Table 3, see [42].

Following the approach of Marshall and Olkin [43], Abdel-Hamid [44] obtained a new distribution by compounding the half-logistic distribution with the Poisson distribution.

4 Generation by Mixing

4.1 Generation by countable mixtures

If, in (20), the entire mass of the corresponding measure of H is confined to a countable number of points $\theta_1,\theta_2,\ldots$, and the masses at θ_j, $j=1,2,\ldots$, are $H(\theta_j)$, then

$$F(x)=\sum_{j=1}^{\infty}G(x|\theta_j)H(\theta_j)=\sum_{j=1}^{\infty}p_jG_j(x), \tag{25}$$

$p_j\equiv H(\theta_j)\Rightarrow 0\le p_j\le 1$ and $\sum_{j=1}^{\infty}p_j=1$, $G_j(x)\equiv G(x|\theta_j)$.
Similarly, (21) becomes

$$f(x)=\sum_{j=1}^{\infty}g(x|\theta_j)H(\theta_j)=\sum_{j=1}^{\infty}p_jg_j(x), \tag{26}$$

$g_j(x)\equiv g(x|\theta_j)$.

Remark 7 *A related but slightly different concept of compounding is as follows: A RV Y is said to have a compound distribution if $Y = \sum_{i=1}^{N} X_i$, where the number of terms N is random. It is assumed that the RVs $X_1, X_2, \ldots, X_N$ are iid and each is independent of N. The CDF of Y is given by*

$$F_Y(y) = \sum_{n=0}^{\infty} p_n G_n(y),$$

where $p_n = P[N = n], G_n(y)$ is the CDF of the sum of $X_1, X_2, \ldots, X_N$ and $G_0(y)$ is the point mass at $y = 0$, (degenerate case).

Remark 8 *Non-central distributions are obtained as countable mixtures of Poisson mixing proportions and (central) distributions such as the χ^2, f, Fisher and beta PDFs are given as follows:*

(a)Non-central $\chi^2(\nu, \lambda)$ PDF is obtained as a countable mixture of Poisson (λ) mass function and central $\chi^2(\nu + 2j)$ distributions, where ν represents the degrees of freedom and λ is the non centrality parameter.

$$f(x) = \sum_{j=0}^{\infty} \frac{e^{-\lambda} \lambda^j x^{(\nu+2j)/2} e^{-x/2}}{j! \Gamma[(\nu + 2j)/2] 2^{(\nu+2j)/2}}.$$

(b)Non-central $f_{(\nu_1,\nu_2,\lambda)}$ PDF is obtained as a countable mixture of Poisson(λ) mass function and central $f_{[(\nu_1+2j)/2,\nu_2]}$ distributions.

$$f(x) = \sum_{j=0}^{\infty} \frac{e^{-\lambda} \lambda^j [(\nu_1 + 2j)/\nu_2]^{(\nu_1+2j)/2} x^{(\nu_1+2j)/2-1}}{j! B[(\nu_1 + 2j)/2, \nu_2/2][1 + (\nu_1 + 2j)x/\nu_2]^{(\nu_1+\nu_2)/2+j}}.$$

(c)Non-central Fisher (ν_1, ν_2, λ) distribution is obtained as a countable mixture of Poisson(λ) and the central Fisher (ν_1, ν_2) distributions.

$$f(x) = \sum_{j=0}^{\infty} \frac{e^{-\lambda} \lambda^j [(\nu_1 + 2j)/\nu_2]^{(\nu_1+2j)/2} x^{(\nu_1+2j)/2-1}}{j! B[(\nu_1 + 2j)/2, \nu_2/2][(\nu_1 + 2j)e^{2x} + /\nu_2]^{(\nu_1+\nu_2)/2}}.$$

(d)Non-central beta $[(\nu_1 + 2j)/2, \nu_2/2, \lambda]$ distribution is obtained as a countable mixture of Poisson(λ) and the central beta$(\nu_1/2, \nu_2/2)$ distributions.

$$f(x) = \sum_{j=0}^{\infty} \frac{e^{-\lambda} \lambda^j}{j! B[(\nu_1 + 2j)/2, \nu_2/2]} x^{(\nu_1+2j)/2-1} (1 - x)^{(\nu_2)/2-1}.$$

In general, we can "choose" other mixing proportions p_j and other PDFs g_j to generate the PDF of the countable mixture, given by (25).

Remark 9 *It may be noticed that all terms vanish in the above non-central cases, when $\lambda = 0$, except the first term. So that, when $\lambda = 0$, expressions (a)-(d) reduce to:*

$(a')\, f(x) = \dfrac{x^{\nu/2-1} e^{-x/2}}{\Gamma[\nu/2] 2^{\nu/2}}$, *a central $\chi^2(\nu)$ PDF.*

$(b')\, f(x) = \dfrac{[\nu_1/\nu_2]^{\nu_1/2} x^{\nu_1/2-1}}{B[\nu_1/2, \nu_2/2][1 + \nu_1 x/\nu_2]^{(\nu_1+\nu_2)/2}}$, *a central $f_{(\nu_1,\nu_2)}$ PDF.*

$(c')\, f(x) = \dfrac{[\nu_1/\nu_2]^{\nu_1/2} x^{\nu_1/2-1}}{B[\nu_1/2, \nu_2/2][\nu_1 e^{2x} + \nu_2]^{(\nu_1+\nu_2)/2}}$, *a central Fisher$(\nu_1, \nu_2)$ PDF.*

$(d')\, f(x) = \frac{1}{B[\nu_1/2,\nu_2/2]} x^{\nu_1/2-1} (1 - x)^{(\nu_2)/2-1}, 0 < x < 1$, *a central beta$[\nu_1/2, \nu_2/2]$ PDF.*

Remark 10 *A countable mixture of SFs $\overline{F}_r(x) = \sum_{j=r}^{\infty} p_j \overline{G}_j(x)$ where $\overline{G}_j(x) = [\overline{G}(x)]^j, j = r, r+1, \ldots$ has the form*

$$\overline{F}_r(x) = \left[\frac{p\overline{G}(x)}{1 - q\overline{G}(x)} \right]^r, \tag{27}$$

if and only if the mixing proportions $p_j = \binom{j-1}{r-1} p^r q^{r-1}, j = r, r+1, \ldots.$

For proof, see AL-Hussaini and Ghitany [45]. Notice that $\overline{F}_r(x)$ is the exponentiated SF of $\frac{p\overline{G}(x)}{1-q\overline{G}(x)}$.

Remark 11 *Suppose that $Y_{1:n},\dots,Y_{n:n}$ are the order statistics of a random sample of size n from a population whose SF and PDF are given, respectively, by*

$$\left.\begin{aligned}\overline{F}_r(x) &= [\overline{G}(x)]^r,\\ f_r(x) &= r[\overline{G}(x)]^{r-1}g(x),\end{aligned}\right\} \tag{28}$$

where $\overline{G}(x)=\frac{p\overline{H}(x)}{1-q\overline{H}(x)}$ and $g(x)=\frac{ph(x)}{[1-q\overline{H}(x)]^2}$. The PDF of the $s-$th order statistic $Y_{s:n}$ can be given by

$$\zeta(y) = rg(y)\sum_{i=0}^{s-1} a_i[\overline{G}(y)]^{rm_i-1}, \tag{29}$$

where $a_i=(-1)^i n\binom{n-1}{s-1}\binom{s-1}{i}$, $m_i=n-s+i+1$.

Remark 12 *It can be shown that the Marshall-Olkin [43] extended Weibull distribution can be obtained by compounding the SF*

$$\overline{G}(x|\theta) = exp[\theta(1-e^{(\lambda x)^\beta})], \quad x>0, \quad (\beta,\lambda,\theta>0),$$

with the exponential PDF

$$h(\theta) = \alpha e^{-\alpha\theta}, \quad \theta>0, \quad (\alpha>0),$$

as follows

$$\begin{aligned}\overline{F}_r(x) &= \int_0^\infty \overline{G}(x|\theta)h(\theta)d\theta\\ &= \int_0^\infty e^{-\theta\left[\alpha-1+e^{-(\lambda x)^\beta}\right]}d\theta\\ &= \frac{\alpha}{e^{-(\lambda x)^\beta}-(1-\alpha)}\\ &= \frac{\alpha e^{-(\lambda x)^\beta}}{1-\overline{\alpha}e^{-(\lambda x)^\beta}}, \quad \overline{\alpha}=1-\alpha.\end{aligned}$$

For more details, see [46].

4.2 Generation by finite mixtures

If, in (20), the entire mass of the corresponding measure of H is confined to a finite number of points $\theta_1,\theta_2,\dots,\theta_k$ and the masses at θ_j, $j=1,2,\dots,k$ are $H(\theta_j)$, similar expressions of CDF (25) and PDF (26) can be shown to be

$$F(x) = \sum_{j=1}^{k} p_j G_j(x), \tag{30}$$

and

$$f(x) = \sum_{j=1}^{k} p_j g_j(x). \tag{31}$$

A CDF of the form (30) (or PDF of the form (31)) is known as finite mixture of k components mixed with proportions $p_1,\dots,p_k$, $0\le p_j\le 1$ and $\sum_{j=1}^k p_j=1$.
A heterogeneous population consisting of k sub-populations, mixed with proportions p_j, $j=1,\dots,k$ can be represented by a finite mixture whose generated CDF is given by (30).

The study of homogeneous populations with "single" component distribution was the main concern of statisticians along history, although Newcomb [47] and Pearson [48] were two pioneers who approached heterogenous populations with finite mixture of distributions. With the advent of computing facilities the study of heterogeneous populations, which is the case of many real world populations, attracted the interest of several researchers from about the middle of the century. Books have collected and organized the results of research published in several articles, see for example [49, 50, 51, 52, 53, 54, 55].

Titterington et al [50] listed and gave examples of direct applications of finite mixture models to fisheries research, economics, medicine, psycology, paleontology, electrophoresis, sedimentology, botany, agriculture, zoology life-testing and reliability, among others. Indirect applications include outliers, normal mixtures as checks to robustness, Gaussian sums, cluster analysis, latent structural models, modeling prior densities, empirical Bayes method, non-parametric (kernel) density estimation, random generation and approximation of mixture models by non-mixture models.

McLachlan and Peel [53] discussed fitting of finite mixtures through use of the EM algorithm, see Dempster et al [56], the properties of maximum likelihood estimators, the assessment of the number of components to be used in the mixture, applications in areas such as unsupervised pattern recognition, speech recognition and medical imaging.

Several publications on finite mixtures appeared in the past years. We list only a few, as the comprehensive list includes much more publications in the same period of time. Examples are Aitkin and Tunnicliffe-Wilson [57], AL-Hussaini and Ahmad [58,59], Aitkin and Rubin [60], Basford and McLachlan [61,62,63], Bernardo and Girón [64], Dean et al [65], AL-Hussaini and Abdel-Hakim [66,67,68], Evans et al [69], Chen [70], Diebolt and Robert [71], Crawford [72], Escobar and West [73], AL-Hussaini and Osman [74], Celeux et al [75], AL-Hussaini et al [76], AL-Hussaini [77], Woodward and Sain [78], McLachlan et al [79], Besbeas and Morgan [80], AL-Hussaini and Abdel-Hamid [81,82], AL-Hussaini and Ghitany [45], Grün and Leisch [83], Abdel-Hamid and AL-Hussaini [84,85], Al-Jaralla and AL-Hussaini [86], AL-Hussaini and Hussein [87], among others. AL-Hussaini and Sultan [88] reviewed reliability and hazard functions under finite mixture models. Finally, Barakat [89] suggested a new distribution family via a mixture of a baseline CDF and its reverse, after adding and subtracting, respectively, to them the same positive location parameter. He also showed that the suggested family is capable of describing many types of statistical data than many other known families. Moreover, via mixture, Barakat and Khaled [90] suggested a new method for constructing a family of CDFs, which contains all the possible types of CDFs and possess very wide range of the indices of skewness and kurtosis.

5 Conclusion

Generation of new distributions may be needed if the new distributions are more flexible to analyzing data in the sense of having better fit, more shapes of the HRF, etc. These features may render to the more parameters that will be added to the new distributions. In this article, we have surveyed and discussed three methods of generation of new distributions. These methods are:

– Generation by composition. This method includes composition of a CDF with another CDF on the support $(0,1)$, composition of a CDF with a function of another CDF, in the general case, and composition of a symmetric PDF with a transformation of scale.

– Generation by compounding.

– Generation by mixing. This method includes generation by countable mixtures and generation by finite mixtures of k distributions.

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Dedication [To the spirit of late Professor AL-Hussaini]

An earlier version of this article was prepared by Professor Essam K. AL-Hussaini (Alexandria University, Egypt) who passed away in Aug., 2015. He did not catch to accomplish it for publication because of his death. I am one of his students (Alaa H. Abdel-Hamid, Beni-Suef University, Egypt, email: hamid_alh@science.bsu.edu.eg). On behalf of Professor AL-Hussaini, I have prepared this article in its final form for publication, with the help of Professor Haroon M. Barakat (Zagazig University email: hbarakat2@hotmail.com). I have the honor to dedicate this article to his spirit.

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