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Prediction of Generalized Order Statistics Based on Some Accelerated Data Using the Tampered Failure Rate Model

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Abstract: Prediction of generalized order sample can be done with normal bounds having suitable confidence. However, given independent samples from the same absolutely continuous distribution, the prediction based on some accelerated bounds can be achieved. In this regard, this paper deals some various exact non-parametric prediction intervals for the future generalized order statistics under normal operating conditions based on stressed generalized order variables were observed in step stress accelerated life test: The tampered failure rate model. The progressive Type-II right-censoring order sample of failure time data that presented by Xiong [1] are used in numerical computations to illustrate the proposed procedures.

Keywords: Tampered failure rate model; Step-stress acceleration; Prediction intervals; Generalized order statistics; Two-sample prediction.

1 Introduction

Our problem here basically are constructed from three pivotal statistical procedures: Non-parametric prediction method, generalized order statistics $(gOSs)$ and the accelerated life test (ALT). Firstly, the prediction of future events is one of the most important problems in statistics. In absence of prior knowledge, it may be resorted to the non-parametric prediction method to avoid many statistical procedures errors, such as selection of nearest suitable parent distribution, regression and estimation etc.. Non-parametric predictive interval uses the observed ordered data without any information about the sampling distribution F other than being continuous. Such these intervals corresponding coverage probabilities are known exactly and do not depend on F.
Next, the concept of $gOSs$ had introduced by Kamps [2] and then developed by Kamps and Cramer [3]. The use of such concept has been steadily growing through the years because such concept includes important well-known models of ordered random variables that have been treated separately in statistical literature.

Moreover, ALT experiments are conducted at stress levels higher than normal use stress levels. Stress can be applied in different ways such that constant stress and step stress. In constant stress ALT, each unit is subjected to an accelerated stress level and this level remains unchanged during the testing period, so the test might delay for a long time if the considered stress is relatively weak. The step stress is considered to reduce the testing time. In this procedure, the stress subjected to each test unit is not constant, but is changing in a stepwise manner and goes up like stairs.
Under the normal operating conditions, many contexts discussed the non-parametric prediction problem using special assumptions [4]-[14]. None of these papers can be useful wherein the observed units are suffered higher stress level than normal. For this purpose, this paper develops non-parametric prediction intervals for future $gOSs$ under normal conditions based on some accelerated generalized order bounds that observed in ALT. We have to get specific model of ALT be compatible with the non-parametric prediction method. Therefor, the tampered failure rate (TFR) model due to

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Bhattacharrya and Soejoeti [15] is devoted.

TFR model is also called, tampered hazard model and proportional hazard model. This model consider some products are undergone a step stress ALT. They suffer a stress S_1 from beginning time to a prescribed time t. Those that survive at t will face a stronger stress $S_2(>S_1)$ such that, the stress reflected in the hazard rate function. Let $f(x)$ and $\overline{F}(x)$ be the pdf and the survival function of the product lifetime X under the stress S_1. Given the failure rate at the initial stress, the change in the stress effect is assumed to multiply the failure rate function $h(x)=\frac{f(x)}{\overline{F}(x)}$ by unknown positive factor α to current failure rate over the remaining life that is related to the time point t. The accelerated survival function $\overline{F}^*(x)=\overline{F}(x\mid S_2)$ relates with $\overline{F}(x)=\overline{F}(x\mid S_1)$ such it depends only on the product distribution and the loss factor under the stress S_1, and it claims nothing from the product life or it's distribution under the stress S_2.
Let $h^*(x)$ denote the failure rate function of the step-stress life time X^*. The proposed TFR model is

$$h^*(x)=\begin{cases} h(x), & x\le t,\\ \alpha h(x), & x>t. \end{cases} \tag{1}$$

From the relation $\overline{F}(x)=e^{-\int_0^x h(y)dy}$, the survival function and the pdf corresponding to TFR model are respectively expressed as

$$\overline{F}^*(x)=\begin{cases} \overline{F}(x), & x\le t,\\ \overline{F}(t)\left[\frac{\overline{F}(x)}{\overline{F}(t)}\right]^{\alpha}, & x>t. \end{cases} \tag{2}$$

$$f^*(x)=\begin{cases} f(x), & x\le t,\\ \alpha\left[\frac{\overline{F}(x)}{\overline{F}(t)}\right]^{1-\alpha} f(x), & x>t. \end{cases} \tag{3}$$

The rest of this paper proceeds as follows: In Section 2, some preliminaries are given. In Section 3, we derive non-parametric prediction intervals for future single $gOSs$ samples under normal operating conditions, based on some stressed generalized order data that suffered the tampered failure rate acceleration model. In Section 4, outer and inner prediction intervals for future $gOSs$ intervals are derived. Section 5, includes numerical computations. Finally, conclusions are given in Section 6.

2 Preliminaries

Suppose the $gOSs$ $\{X_{1,n,\tilde{m},k}<X_{2,n,\tilde{m},k}<\ldots<X_{n,n,\tilde{m},k}\}$ be observed sample of size $n\in N$ having an absolutely continuous cdf F and pdf f that developed by Kamps and Cramer [3] under assumption $\gamma_i\neq\gamma_j$, $i,j=1,2,\ldots,n-1$ and $i\neq j$. The survival function of $X_{i,n,\tilde{m},k}$ is expressed as

$$\overline{F}_{X_{i,n,\tilde{m},k}}(x)=c_{i-1}\sum_{\nu=1}^{i}\frac{a_\nu(i)}{\gamma_\nu}(1-F(x))^{\gamma_\nu}, \tag{4}$$

where $n\in N$, $\tilde{m}=(m_1,\ldots,m_{n-1})\in R_{n-1}$ and $k\ge 1$, be given constants such that for all $1\le i\le n-1$, $\gamma_i=k+n-i+M_i>0$, where $M_i=\sum_{j=i}^{n-1}m_j$ and $a_i(r)=\prod_{j=1,j\neq i}^{r}\frac{1}{\gamma_j-\gamma_i}$, $1\le r\le n$.
Also, the joint pdf of any two $gOSs$, say $X_{r,n,\tilde{m},k}$ and $X_{s,n,\tilde{m},k}$ such that $1\le r<s\le n$, is expressed as

$$f_{X_{i,j,n,\tilde{m},k}}(u,v)=c_{j-1}\left[\sum_{\lambda=i+1}^{j}a_\lambda^{(i)}(j)\left(\frac{\overline{F}(v)}{\overline{F}(u)}\right)^{\gamma_\lambda}\right]\left[\sum_{\mu=1}^{i}a_\mu(i)\left(\overline{F}(u)\right)^{\gamma_\mu}\right]\frac{f(u)f(v)}{\overline{F}(u)\overline{F}(v)}, \tag{5}$$

such that $u<v$, $a_i^{(r)}(s)=\prod_{j=r+1,j\neq i}^{s}\frac{1}{\gamma_j-\gamma_i}$ and $a_i(s)=a_i^{(0)}(s)$. $\forall 1\le i\le n$,

Next, assume $\{Y_{1,n^*,\tilde{m}^*,k^*},Y_{2,n^*,\tilde{m}^*,k^*},\ldots,Y_{n^*,n^*,\tilde{m}^*,k^*}\}$ be future $gOSs$ sample of size n^* from the same population under assumption $\gamma_i^*\neq\gamma_j^*$, $i,j=1,2,\ldots,n^*-1$ and $i\neq j$. Likewise, each of the coefficients of X-sample will carry an asterisk with $(^*)$ when it refers to Y-sample. We aim basically to make inference about this sample.

Lemma 1. The survival function, the pdf of the i^{th} accelerated gOS $X^*_{i,n,\tilde{m},k}$ for $x > t$ and the joint distribution function of i^{th} and j^{th} accelerated $gOSs$ $X^*_{i,n,\tilde{m},k}$ and $X^*_{j,n,\tilde{m},k}$ for $u,v > t$ due to TFR model in terms of F and f, respectively, can expressed as.

$$\overline{F}_{X^*_{i,n,\tilde{m},k}}(x) = c_{r-1}\sum_{\nu=1}^{r}\frac{a_\nu(r)}{\gamma_\nu}\overline{F}^{(1-\alpha)\gamma_\nu}(t)\overline{F}^{\alpha\gamma_\nu}(x), \tag{6}$$

$$f_{X^*_{i,n,\tilde{m},k}}(x) = \alpha c_{i-1}\sum_{\nu=1}^{i}a_\nu(i)\overline{F}^{(1-\alpha)\gamma_\nu}(t)\overline{F}^{\alpha\gamma_\nu-1}(x)f(x), \tag{7}$$

$$f_{X^*_{i,j,n,\tilde{m},k}}(u,v) = \alpha^2 c_{j-1}\sum_{\lambda=i+1}^{j}a_\lambda^{(i)}(j)\sum_{\mu=1}^{i}a_\mu(i)\overline{F}^{(1-\alpha)\gamma_\mu}(t)\overline{F}^{\alpha(\gamma_\mu-\gamma_\lambda)-1}(u)\overline{F}^{\alpha\gamma_\lambda-1}(v)f(u)f(v). \tag{8}$$

3 Prediction of individual gOS

This Section interested in obtaining two-sided non-parametric prediction intervals for a future $Y_{r,n^*,\tilde{m}^*,k^*}$, under normal conditions based on some $gOSs$, which was observed under some kind of stress of the form $(X^*_{i,n,\tilde{m},k}, X^*_{j,n,\tilde{m},k})$ where i and j are integers such that $1 \le i < j \le n$.
Such that, the purposed coverage probability $P\left(X^*_{i,n,\tilde{m},k} \le Y_{r,n^*,\tilde{m}^*,k^*} \le X^*_{j,n,\tilde{m},k}\right) = 1-\varepsilon$, which does not depend on the particular continuous F. Then we call $(X^*_{i,n,\tilde{m},k}, X^*_{j,n,\tilde{m},k})$ a $(1-\varepsilon)100\%$ prediction interval for $Y_{r,n^*,\tilde{m}^*,k^*}$. The corresponding confidence coefficient for the prediction interval is exact and do not depend on the sampling distribution.

The following lemma give accelerated upper bounds for the purposed prediction intervals.

Lemma 2. Under the preliminaries assumptions, then $(-\infty, X^*_{i,n,\tilde{m},k})$, $1 \le i \le n$, is accelerated upper prediction interval for the future $Y_{r,n^*,\tilde{m}^*,k^*}$, with the coverage probability is given by

$$P\left(X^*_{i,n,\tilde{m},k} > Y_{r,n^*,\tilde{m}^*,k^*}\right) = \sum_{\nu=1}^{i}\frac{c_{i-1}a_\nu(i)}{\gamma_\nu}\sum_{\lambda=1}^{r}\frac{c^*_{r-1}a^*_\lambda(r)}{\alpha\gamma_\nu+\gamma^*_\lambda}\left[\overline{F}(t)\right]^{(1-\alpha)\gamma_\nu}. \tag{9}$$

Proof. Under the assumption that $\{Y_{r,n^*,\tilde{m}^*,k^*}, 1 \le r \le n^*\}$ are continuous r.v.'s, we can write

$$P\left(X^*_{i,n,\tilde{m},k} > Y_{r,n^*,\tilde{m}^*,k^*}\right) = \int_{-\infty}^{\infty}P\left(X^*_{i,n,\tilde{m},k} > y\right)f_{Y_{r,n^*,\tilde{m}^*,k^*}}(y)dy. \tag{10}$$

Upon substituting (6) and $f_{Y_{r,n^*,\tilde{m}^*,k^*}}(y) = dF_{Y_{r,n^*,\tilde{m}^*,k^*}}(y)$, then solving the integration (10) using the transformation $\overline{F}(y) = y$, the proof completed.

The corresponding normal upper prediction interval $(-\infty, X_{i,n,\tilde{m},k})$, $1 \le i \le n$, which was discussed separately in [12], can be expressed by setting $\alpha = 1$ in (9) as

$$P\left(X_{i,n,\tilde{m},k} > Y_{r,n^*,\tilde{m}^*,k^*}\right) = \sum_{\nu=1}^{i}\frac{c_{i-1}a_\nu(i)}{\gamma_\nu}\sum_{\lambda=1}^{r}\frac{c^*_{r-1}a^*_\lambda(r)}{\gamma_\nu+\gamma^*_\lambda}. \tag{11}$$

The coverage probability of the event $X_{i,n,\tilde{m},k} \le Y_{r,n^*,\tilde{m}^*,k^*} \le X^*_{j,n,\tilde{m},k}$, which mean the lower bound was observed in normal operating conditions and the upper bound is suffered stress higher than normal, can be set by subtracting (11) from (9), as

$$P\left(X_{i,n,\tilde{m},k} \le Y_{r,n^*,\tilde{m}^*,k^*} \le X^*_{j,n,\tilde{m},k}\right) = \sum_{\lambda=1}^{r}c^*_{r-1}a^*_\lambda(r)\left(\sum_{\nu=1}^{j}\frac{c_{j-1}a_\nu(j)}{\gamma_\nu(\alpha\gamma_\nu+\gamma^*_\lambda)}\left[\overline{F}(t)\right]^{(1-\alpha)\gamma_\nu} - \sum_{\mu=1}^{i}\frac{c_{i-1}a_\mu(i)}{\gamma_\mu(\gamma_\mu+\gamma^*_\lambda)}\right). \tag{12}$$

The following theorem discuss the augmented coverage probabilities when the two bounds are suffered high stress.

Theorem 1. Under the preliminaries assumptions, then $(X^*_{i,n,m,k}, X^*_{j,n,\tilde{m},k})$, $1 \le i < j \le n$, is stressed two sided prediction interval for the future $Y_{r,n^*,\tilde{m}^*,k^*}$, with the coverage probability is given by

$$P\left(X^*_{i,n,\tilde{m},k} \le Y_{r,n^*,\tilde{m}^*,k^*} \le X^*_{j,n,\tilde{m},k}\right) = \sum_{\lambda=i+1}^{j} \frac{c_{j-1} a_\lambda^{(i)}(j)}{\gamma_\lambda} \sum_{\mu=1}^{i} \frac{c^*_{r-1} a_\mu(i)}{\gamma_\mu - \gamma_\lambda} \sum_{\omega=1}^{r} a^*_\omega(r) \left[\overline{F}(t)\right]^{(1-\alpha)\gamma_\mu} \left(\frac{1}{\alpha\gamma_\lambda + \gamma^*_\omega} - \frac{1}{\alpha\gamma_\mu + \gamma^*_\omega}\right).$$

Proof. Under the assumption that $X^*_{i,n,\tilde{m},k}$ and $X^*_{j,n,\tilde{m},k}$ are continuous r.v.'s, we can write

$$P\left(X^*_{i,n,\tilde{m},k} \le y \le X^*_{j,n,\tilde{m},k}\right) = \int_y^\infty \int_{-\infty}^{y} f_{X^*_{i,j,n,\tilde{m},k}}(u,v)\,du\,dv. \tag{13}$$

Using (8) and the transformations $\overline{F}(*) = *$, equation (13) can take the form

$$P\left(X^*_{i,n,\tilde{m},k} \le y \le X^*_{j,n,\tilde{m},k}\right) = c_{j-1} \sum_{\lambda=i+1}^{j} a_\lambda^{(i)}(j) \sum_{\mu=1}^{i} a_\mu(i)\alpha^2 \left[\overline{F}(t)\right]^{(1-\alpha)\gamma_\mu} I^\alpha_{\lambda,\mu}(\overline{F}(y), \overline{F}(y)). \tag{14}$$

Where $I^\alpha_{\lambda,\mu}(x,y)$, is given by

$$\begin{aligned} I^\alpha_{\lambda,\mu}(x,y) &= \int_x^1 u^{\alpha(\gamma_\mu-\gamma_\lambda)-1}du \int_0^y v^{\alpha\gamma_\lambda-1}dv \\ &= \frac{1}{\alpha^2}\left(\frac{1-x^{\alpha(\gamma_\mu-\gamma_\lambda)}}{\gamma_\mu-\gamma_\lambda}\right)\frac{y^{\alpha\gamma_\lambda}}{\gamma_\lambda}. \end{aligned} \tag{15}$$

Such that $Y_{r,n^*,\tilde{m}^*,k^*}$ is continuous r.v., we can write

$$P\left(X^*_{i,n,\tilde{m},k} \le Y_{r,n^*,\tilde{m}^*,k^*} \le X^*_{j,n,\tilde{m},k}\right) = \int_{-\infty}^{\infty} P\left(X^*_{i,n,\tilde{m},k} \le y \le X^*_{j,n,\tilde{m},k}\right) f_{Y_{r,n^*,\tilde{m}^*,k^*}}(y)dy. \tag{16}$$

By using (14) with (15), $\overline{F}(*) = *$ in (16) and simplifying, the proof completed.

4 Outer and inner prediction of *gOSs* intervals

To describe the outer and inner prediction problem generally, suppose $(Y_r, Y_s))$ is an interval of the unobserved Y-sample and let X be the th random variable from observed X-sequence. We are interested in obtaining at least $100(1-\varepsilon)\%$ prediction intervals for it of the form (X_i, X_j). The interval (X_i, X_j) called, outer prediction intervals if $p(X_i \le Y_r \le Y_s \le X_j) \ge 1-\varepsilon$ and inner prediction intervals if $p(Y_r \le X_i \le X_j \le Y_s) \ge 1-\varepsilon$. Such that the interested coverage probabilities that in focuss are exact and free of the parent distribution. This chapter discusses this problem with the consideration, the two samples are *gOSs*, independent and X-sample are suffered the tampered failure rate acceleration model.

Theorem 2. Under the preliminaries assumptions, the coverage probability of the event $X^*_{i,n,m,k} \le Y_{r,n^*,m^*,k^*} < Y_{r,n^*,m^*,k^*} \le X^*_{j,n,m,k}$, can take the form

$$\begin{aligned} &P\left(X^*_{i,n,\tilde{m},k} \le Y_{r,n^*,\tilde{m}^*,k^*} < Y_{s,n^*,\tilde{m}^*,k^*} \le X^*_{j,n,\tilde{m},k}\right) \\ &= \sum_{\lambda=i+1}^{j} \frac{c_{j-1} a_\lambda^{(i)}(j)}{\gamma_\lambda} \sum_{\mu=1}^{i} \frac{c^*_{s-1} a_\mu(i)}{\gamma_\mu-\gamma_\lambda} \sum_{\nu=r+1}^{s} \frac{a_\nu^{*(r)}(s)}{\alpha\gamma_\lambda+\gamma^*_\nu} \sum_{\omega=1}^{r} a^*_\omega(r) \left[\overline{F}(t)\right]^{(1-\alpha)\gamma_\mu} \left(\frac{1}{\alpha\gamma_\lambda+\gamma^*_\omega} - \frac{1}{\alpha\gamma_\mu+\gamma^*_\omega}\right). \end{aligned} \tag{17}$$

Proof. Using (14) with (15), we can write

$$P\left(X^*_{i,n,\tilde{m},k} \le x < y \le X^*_{j,n,\tilde{m},k}\right) = c_{j-1} \sum_{\lambda=i+1}^{j} \frac{a_\lambda^{(i)}(j)}{\gamma_\lambda} \sum_{\mu=1}^{i} \frac{a_\mu(i)}{\gamma_\mu-\gamma_\lambda} \left[\overline{F}(t)\right]^{(1-\alpha)\gamma_\mu} \left(1-\overline{F}^{\alpha(\gamma_\mu-\gamma_\lambda)}(x)\right)\overline{F}^{\alpha\gamma_\lambda}(y). \tag{18}$$

By assuming that Y_{r,n^*,m^*,k^*} and Y_{s,n^*,m^*,k^*} are continuous r.v.'s, we can write

$$P\left(X^*_{i,n,\tilde{m},k} \leq Y_{r,n^*,\tilde{m}^*,k^*} < Y_{s,n^*,\tilde{m}^*,k^*} \leq X^*_{j,n,\tilde{m},k}\right) = \int_{-\infty}^{\infty}\int_{x}^{\infty} P\left(X^*_{i,n,\tilde{m},k} \leq x < y \leq X^*_{j,n,\tilde{m},k}\right) f_{Y_{r,s,n^*,\tilde{m}^*,k^*}}(x,y)dydx. \quad (19)$$

Substituting (18) and $f_{Y_{r,s,n^*,\tilde{m}^*,k^*}}(y)$ in (19), then using $\overline{F}(*) = *$, equation (19) take the form

$$\begin{aligned} &P\left(X^*_{i,n,\tilde{m},k} \leq Y_{r,n^*,\tilde{m}^*,k^*} < Y_{s,n^*,\tilde{m}^*,k^*} \leq X^*_{j,n,\tilde{m},k}\right) \\ &= c_{j-1}\sum_{\lambda=i+1}^{j} \frac{a_\lambda^{(i)}(j)}{\gamma_\lambda} \sum_{\mu=1}^{i} \frac{a_\mu(i)}{\gamma_\mu - \gamma_\lambda} \left[\overline{F}(t)\right]^{(1-\alpha)\gamma_\mu} c^*_{s-1} \sum_{v=r+1}^{s} a_v^{*(r)}(s) \sum_{\omega=1}^{r} a^*_\omega(s) \int_0^1 \left(1 - x^{\alpha(\gamma_\mu-\gamma_\lambda)}\right) x^{\gamma^*_\omega - \gamma^*_v - 1} \\ &\times \left\{\int_0^x y^{\alpha\gamma_\lambda + \gamma^*_v - 1} dy\right\} dx. \end{aligned} \quad (20)$$

By solving the integration in (20) and simplifying, the proof completed.

Theorem 3. Under the preliminaries assumptions, the coverage probability of the event $Y_{r,n^*,\tilde{m}^*,k^*} \leq X^*_{i,n,\tilde{m},k} < X^*_{j,n,\tilde{m},k} \leq Y_{s,n^*,\tilde{m}^*,k^*}$, can take the form

$$\begin{aligned} &P\left(Y_{r,n^*,\tilde{m}^*,k^*} \leq X^*_{i,n,\tilde{m},k} < X^*_{j,n,\tilde{m},k} \leq Y_{s,n^*,\tilde{m}^*,k^*}\right) \\ &= c_{j-1}\sum_{\lambda=i+1}^{j} a_\lambda^{(i)}(j) \sum_{\mu=1}^{i} \frac{a_\mu(i) c^*_{s-1}}{\gamma_\mu - \gamma_\lambda} \sum_{v=r+1}^{s} a_v^{*(r)}(s) \sum_{\omega=1}^{r} \frac{a^*_\omega(r)}{\alpha\gamma_\mu + \gamma^*_\omega} \left[\overline{F}(t)\right]^{(1-\alpha)\gamma_\mu} \\ &\times \left(\frac{1}{\alpha\gamma_\lambda}\left(\frac{1}{\gamma^*_v} - \frac{1}{\alpha\gamma_\lambda + \gamma^*_v}\right) - \frac{\alpha}{\gamma^*_v(\alpha\gamma_\mu + \gamma^*_v)}\right). \end{aligned} \quad (21)$$

Proof. Under the assumption that $X^*_{i,n,\tilde{m},k}$ and $X^*_{j,n,\tilde{m},k}$ are continuous r.v.'s, we can write

$$P\left(x \leq X^*_{i,n,\tilde{m},k} < X^*_{j,n,\tilde{m},k} \leq y\right) = \int_x^y \int_x^v f_{X^*_{i,j,n,\tilde{m},k}}(u,v)dudv. \quad (22)$$

Substituting (8) in (22) then using $\overline{F}(*) = *$, equation (22) can be written as

$$P\left(x \leq X^*_{i,n,\tilde{m},k} < X^*_{j,n,\tilde{m},k} \leq y\right) = c_{j-1}\sum_{\lambda=i+1}^{j} a_\lambda^{(i)}(j) \sum_{\mu=1}^{i} a_\mu(i)\alpha^2 \left[\overline{F}(t)\right]^{(1-\alpha)\gamma_\mu} J^\alpha_{\lambda,\mu}(\overline{F}(x),\overline{F}(y)). \quad (23)$$

Where $J^\alpha_{\lambda,\mu}(x,y)$, is given by

$$\begin{aligned} J^\alpha_{\lambda,\mu}(x,y) &= \int_y^x \left\{\int_v^x u^{\alpha(\gamma_\mu-\gamma_\lambda)-1} du\right\} v^{\alpha\gamma_\lambda - 1} dv \\ &= \frac{1}{\alpha^2(\gamma_\mu - \gamma_\lambda)}\left(x^{\alpha(\gamma_\mu-\gamma_\lambda)}\left(\frac{x^{\alpha\gamma_\lambda} - y^{\alpha\gamma_\lambda}}{\gamma_\lambda}\right) - \frac{x^{\alpha\gamma_\mu} - y^{\alpha\gamma_\mu}}{\gamma_\mu}\right). \end{aligned} \quad (24)$$

Now, the coverage probability of the event $Y_{r,n^*,\tilde{m}^*,k^*} \leq X^*_{i,n,\tilde{m},k} < X^*_{j,n,\tilde{m},k} \leq Y_{s,n^*,\tilde{m}^*,k^*}$, can be derived from

$$\begin{aligned} &P\left(Y_{r,n^*,\tilde{m}^*,k^*} \leq X^*_{i,n,\tilde{m},k} < X^*_{j,n,\tilde{m},k} \leq Y_{s,n^*,\tilde{m}^*,k^*}\right) \\ &= \int_{-\infty}^{\infty}\int_x^{\infty} P\left(x \leq X^*_{i,n,\tilde{m},k} < X^*_{j,n,\tilde{m},k} \leq y\right) f_{Y_{r,s,n^*,\tilde{m}^*,k^*}}(x,y)dydx. \end{aligned} \quad (25)$$

Substituting (23) with (24) and $f_{Y_{r,s,n^*,\tilde{m}^*,k^*}}(y)$ in (25), then using the transformation $\overline{F}(*) = *$, equation (25) can take the form

$$\begin{aligned} &P\left(Y_{r,n^*,\tilde{m}^*,k^*} \leq X^*_{i,n,\tilde{m},k} < X^*_{j,n,\tilde{m},k} \leq Y_{s,n^*,\tilde{m}^*,k^*}\right) \\ &= c_{j-1}\sum_{\lambda=i+1}^{j} a_\lambda^{(i)}(j) \sum_{\mu=1}^{i} \frac{a_\mu(i)}{\gamma_\mu - \gamma_\lambda} \left[\overline{F}(t)\right]^{(1-\alpha)\gamma_\mu} c^*_{s-1} \sum_{v=r+1}^{s} a_v^{*(r)}(s) \sum_{\omega=1}^{r} a^*_\omega(r) J^\alpha_{\lambda,\mu,v,\omega}. \end{aligned} \quad (26)$$

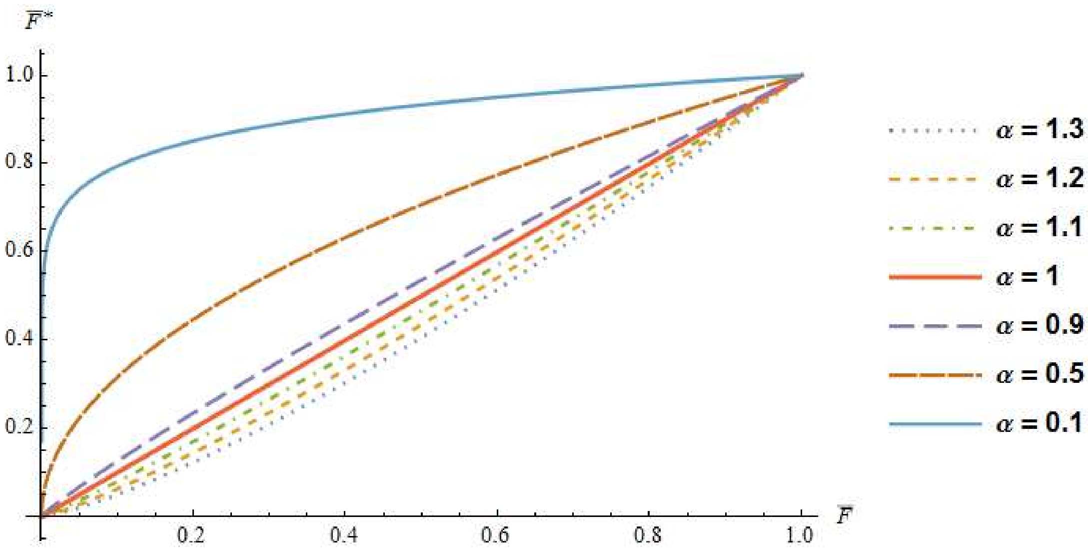

Fig. 1: Plot the relation between $\overline{F}(x)$ and $\overline{F}^*(x), \forall x$ and $t = 0$.

Where $J^{\alpha}_{\lambda,\mu,\nu,\omega}$, is given by

$$J^{\alpha}_{\lambda,\mu,\nu,\omega} = \int_0^1 \int_0^x \left(x^{\alpha(\gamma_\mu - \gamma_\lambda)} \left(\frac{x^{\alpha\gamma_\lambda} - y^{\alpha\gamma_\lambda}}{\gamma_\lambda} \right) - \frac{x^{\alpha\gamma_\mu} - y^{\alpha\gamma_\mu}}{\gamma_\mu} \right) x^{\gamma^*_\omega - \gamma^*_\nu - 1} y^{\gamma^*_\nu - 1} dy dx$$

$$= \left(\frac{1}{\alpha\gamma_\lambda} \left(\frac{1}{\gamma^*_\nu} - \frac{1}{\alpha\gamma_\lambda + \gamma^*_\nu} \right) - \frac{\alpha}{\gamma^*_\nu(\alpha\gamma_\mu + \gamma^*_\nu)} \right) \int_0^1 x^{\alpha\gamma_\mu + \gamma^*_\omega - 1} dx. \quad (27)$$

By using (26) and (27) after completing the integration then simplifying, the proof of **Theorem 3.** completed.

5 Numerical Results

This paper develops non-parametric prediction intervals for *gOSs* under normal operating conditions based on some *gOSs* observations that subjected in step stress *ALT* due to *TFR* model. For high reliable products or material under normal operating conditions in industrial experiments that are tested often requires long periods. It may be resorted to accelerate the lifetime testing whereas the units are subjected to higher stress levels than normal, which results in a shorter lifetime. This lifetime can be accelerated by using very high levels of: temperature, load, force, voltage, vibration, etc. on the lifetime of a product. In this section, to illustrate the inferential procedures developed in the preceding sections, we calculate some prediction coefficients for some choices of i, j, r, s, α and by using $\overline{F}(t) = \frac{x_n - t}{x_n - x_1} \in [0,1]$.

The acceleration factor will typically depend on the nature of the product, the test method, S_l and S_2 ($0 < \alpha < 1$ for slowing down, $\alpha = 1$ give the normal case and $\alpha > 1$ for the acceleration) and possibly also on t but it is plausible to assume that it does not depend on x. Also, because S_2 is the single accelerated stress, we can leave α without specifying a regression structure on the stress variable. Based on equation (2), Figure 1 plots the relation between the normal $\overline{F}(x)$ and the tampered failure rate model of the accelerated $\overline{F}^*(x), \forall x$ for different values of $\alpha = 1.3, 1.2, 1.1, 1, 0.9, 0.5$ and 0.1.

Let us consider the failure time data in Table 1, which presented by Xiong [1]. The data in use represents progressive Type-II right-censoring order statistics sample of size 20, pre-specified number of observed lifetimes 16, low test stress $S_1 = 0.5$, high test stress $S_2 = 1.5$ and the time of stress-change $t = 5$. It consider as a special case of a past $X - gOSs$ (by setting $\gamma_i = 20 - i + 1$, for $1 < i \leq 4$ and $\gamma_i = 16 - i + 1$, for $4 < i \leq 16$).

Table 1:

Stress level	$X_{i:16:20}$					
S_1	2.01	3.60	4.12	4.34		
S_2	5.04	5.94	6.68	7.09	7.17	7.49
	7.60	8.23	8.24	8.25	8.69	12.05

Based on the observed *PCOs* data that given in Table 1, suppose that, we interest in predicting a future complete ordinary order sample *oOSs* of size 15 as special case of the future $Y-gOSs$ (by setting $\gamma_r^*=15-r+1$). For different values of α, which discussed in Table 1, the $P\left(X_{i:16:20}\le Y_{r:15}\le X^*_{j:16:20}\right)$ due to equation (12) are presented in Table 2 and $P\left(X^*_{i:16:20}\le Y_{r:15}\le X^*_{j:16:20}\right)$ due to equation (13) are presented in Table 3. The prediction coefficients are constructed under assumption that $\overline{F}(t) = \frac{12.05-5}{12.05-2.01}$.

Table 2:Values of $P\left(X_{i:16:20}\le Y_{r:15}\le X^*_{j:16:20}\right)$ for some choices of $i, j; r$ and α.

r	i	j	(X_i, X_j^*)	$P\left(X_{i:16:20}\le Y_{r:15}\le X^*_{j:16:20}\right)$						
				$\alpha=1.3$	1.2	1.1	1	0.9	0.5	0.1
1	1	7	$(2.01, 6.68)$	0.5688	0.5662	0.5655	0.5645	0.5629	0.5289	0.3760
2	1	8	$(2.01, 7.09)$	0.8090	0.8089	0.8087	0.8082	0.8073	0.7939	0.7360
3	1	9	$(2.01, 7.17)$	0.9037	0.9052	0.9066	0.9078	0.9089	0.9100	0.8971
4	1	9	$(2.01, 7.17)$	0.8458	0.8502	0.8545	0.8587	0.8628	0.8741	0.8673
5	3	10	$(4.12, 7.49)$	0.8062	0.8153	0.8244	0.8334	0.8420	0.8715	0.8865
6	2	10	$(3.60, 7.49)$	0.8009	0.8181	0.8355	0.8530	0.8724	0.9331	0.9690
7	4	10	$(4.34, 7.49)$	0.6436	0.6697	0.6970	0.7251	0.7539	0.8656	0.9345
8	4	10	$(4.34, 7.49)$	0.5198	0.5530	0.5888	0.6270	0.6675	0.8396	0.9577
9	3	9	$(4.12, 7.17)$	0.9170	0.6363	0.3990	0.4917	0.5589	0.5629	0.5629
10	4	11	$(4.34, 7.60)$	0.2605	0.2881	0.3199	0.3566	0.3990	0.6363	0.9172
11	4	13	$(4.34, 8.24)$	0.4968	0.5479	0.6028	0.6608	0.7206	0.9333	0.9987
12	4	15	$(4.34, 8.69)$	0.5224	0.5796	0.6401	0.7024	0.7649	0.9614	0.9999
13	4	15	$(4.34, 8.69)$	0.5224	0.5796	0.6401	0.7024	0.7649	0.9614	0.9999
14	4	15	$(4.34, 8.69)$	0.3169	0.3704	0.4317	0.5009	0.5772	0.8958	0.9999
15	4	15	$(4.34, 8.69)$	0.1216	0.1527	0.1920	0.2417	0.3038	0.7000	0.9991

Table 3:Values of $P\left(X^*_{i:16:20}\le Y_{r:15}\le X^*_{j:16:20}\right)$ for some choices of $i, j; r$ and α.

r	i	j	(X_i^*, X_j^*)	$P\left(X^*_{i:16:20}\le Y_{r:15}\le X^*_{j:16:20}\right)$						
				$\alpha=1.3$	1.2	1.1	1	0.9	0.5	0.1
1	6	9	$(5.94, 7.17)$	0.0272	0.0118	0.0132	0.0165	0.0219	0.1172	0.3619
2	5	9	$(5.04, 7.17)$	0.0803	0.1045	0.1206	0.1414	0.1699	0.3800	0.6182
3	5	9	$(5.04, 7.17)$	0.2206	0.2391	0.2598	0.2847	0.3148	0.4781	0.6330
4	5	9	$(5.04, 7.17)$	0.3665	0.3849	0.4053	0.4280	0.4528	0.5631	0.6479
5	5	11	$(5.04, 7.60)$	0.5808	0.5950	0.6095	0.6242	0.6389	0.6896	0.7031
6	5	10	$(5.04, 7.49)$	0.6006	0.6196	0.6386	0.6574	0.6756	0.7300	0.7120
7	5	13	$(5.04, 8.24)$	0.8289	0.8387	0.8470	0.8535	0.8578	0.8453	0.7490
8	5	15	$(5.04, 8.69)$	0.9297	0.9326	0.9342	0.9342	0.9324	0.9010	0.7721
9	6	12	$(5.94, 8.23)$	0.6105	0.6449	0.6792	0.7124	0.7434	0.8012	0.5860
10	5	15	$(5.04, 8.69)$	0.9003	0.9196	0.9364	0.9503	0.9609	0.9641	0.8211
11	6	14	$(5.94, 8.25)$	0.6587	0.7048	0.7508	0.7952	0.8364	0.9226	0.6640
12	8	15	$(7.09, 8.69)$	0.6835	0.7275	0.7695	0.8077	0.8396	0.8311	0.2977
13	9	16	$(7.17, 12.05)$	0.7646	0.8018	0.8356	0.8644	0.8860	0.8266	0.1924
14	8	15	$(7.09, 8.69)$	0.3163	0.3694	0.4302	0.4985	0.5735	0.8620	0.4497
15	7	15	$(6.68, 8.69)$	0.1216	0.1526	0.1920	0.2416	0.3036	0.6974	0.7739

6 Conclusions

In many statistics surveys, the observed data often do not fit to known distribution. To improve the inferences about the population and for reducing moral and material costs of sampling, it may be resorted to predict another sample based on this observed data, which was drawn from unknown distribution. The prediction of unobserved statistics arises naturally in several real life situations. In this paper for more modification, some various exact non-parametric prediction intervals for the future generalized order statistics under normal operating conditions, based on accelerated generalized order statistics were observed in simple step stress accelerated life testing due to the tampered failure rate model are constructed. As it was expected, the coverage probabilities are increasing with: decreasing the lower bounds, or increasing the upper bounds, or decreasing the accelerated factor α (for high reliability of the stress-change time t due to the upper prediction interval). The values may be distributed as a skewed bell for different α for specific reliability of t.

The proposed procedure can be extended to construct the prediction coefficient due to multiple step stress the tampered failure rate model. The generality of our work enabled us to compare the values the prediction coefficients in normal operating conditions with the stress cases and choose the best choice corresponding the practical work.

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