Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/040103

An Inverse Problem for the Caputo Fractional Derivative by Means of the Wavelet Transform

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Received: 2 Apr. 2017, Revised: 7 Jul. 2017, Accepted: 17 Jul. 2017
Published online: 1 Jan. 2018

Abstract: In this article, we build an approximate solution to an Inverse Problem that consist in finding a function whose Caputo fractional derivative is given. We decompose and project the data in appropriate wavelet subspaces and, by a Galerkin scheme, we calculate the coefficients of the unknown function in the chosen wavelet basis. Based on properties of the operator and of the basis, the scheme is simple, efficient and the errors introduced by the approximation can be handled and controlled. We illustrate the results with an example.

Keywords: Fractional calculus, Caputo fractional derivative, wavelet transform.

1 Introduction, Motivation and Preliminaries

Fractional calculus is the theory of integrals and derivatives of arbitrary order. Its theoretical foundations go back to the works of mathematicians such as Laplace, Liouville, Fourier, Abel, Riemann and Heaviside. In particular, opposite to classical differentiation, fractional derivatives do not take into account local characteristics of the dynamics but it considers the global evolution of the system. In [1,2,3,4,5,6] properties and formula for fractional operators are studied in detail. In the last century a lot of contributions in this field have been developed since fractional operators have been used to describe dynamics and properties of different materials. There exist various applications of this theory in different areas such as diffusion problems, hydraulics, potential theory, control theory, electrochemistry, viscoelasticity and nanotechnology among others [7,8,9,10,11,12,13]. Moreover, numerous problems in physics, chemistry and engineering are modelled mathematically by systems of fractional differential equations [14,15]. Theoretical results concerning linear and nonlinear fractional differential equations can be found in [16,17,18,19]. Analytical calculus of fractional operators is, in general, very difficult and different numerical approximations have been proposed. Recently approximate solution to Fractional Differential Equations have also been developed [6,20,21,22,23]. In [1] useful mathematical results concerning fractional models are presented. Numerical methods and the application of fractional calculus for modeling processes can be found in [24,25,1,3,26,4,27].

Although the use of fractional operators has increased significantly, its mathematical expressions are sometimes complicate and consequently, finding solutions to equations involving these operators can be a difficult task. In this work we construct an approximate solution to an Inverse Problem involving the Caputo Fractional Derivative. It consists in finding a function whose Caputo Fractional Derivative is given. The Caputo Fractional Derivative introduced by M. Caputo in 1965 is a fractional integral operator with singular kernel. To calculate de solution we adapt an approximation scheme developed in [28] for equations associated to integral operators acting on the Fourier Transform. First we express the equation by means of the Fourier Transform. In that way the singularity of the kernel can be handled. Afterwards we project the data into suitable wavelet subspaces. Finally, by means of a Galerkin scheme, we calculate the coefficients of the unknown function in the chosen wavelet basis. The proposed method is simple. Only the wavelet coefficients of the data and a matrix derived from some normal equations are needed. The error introduced in the approximation can be controlled improving the computation of the elements of the matrix and considering a more accurate truncated projection

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of the data into the wavelet subspaces. Properties of the basis and of the operator warranty that the resulting approximation scheme is efficient and numerically stable and no additional conditions need to be imposed.

This paper is organized as follows. In the next section we present the fractional operator and the Inverse Problem we are interested in. The wavelet basis and the approximation scheme are introduced in Section 3. In Section 4 a solution to the Inverse Problem is proposed. A numerical example is presented in Section 5. Finally, we state some conclusions.

2 An Inverse Problem for the Caputo Fractional Derivative

2.1 The fractional derivative

We denote by $H^1((a,b))$ the Sobolev space $W^{1,2}((a,b))$ of functions $u:(a,b)\to\mathbb{R}$, such that the derivative $u' = D^1 u$ exists in the weak sense and belongs to $L^2((a,b))$.

The Caputo Fractional Derivative (CFD) of order $\alpha > 0$ of a function $f\in H^1((a,b))$, $a<b$ is defined as

$$D_*^\alpha f(t) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau & n-1<\alpha<n\in\mathbb{N}, \\ \dfrac{d^n f(t)}{dt^n} & \alpha = n\in\mathbb{N} \end{cases} \tag{1}$$

where $a\in[-\infty,t]$, [1,2].

This fractional operator has interesting properties such as linearity, the possibility of expressing it in terms of the Riemann-Liouville Derivative, no commutativity and formula for the derivative of the product of functions. For a complete analysis of the properties of this operator we refer to [3,4].

2.2 The inverse problem

In this work we consider $a=-\infty$ and $n=1$. Consequently $\alpha\in(0,1)$. Higher order derivatives can be treated similarly, [1,2,29].

The Inverse Problem (IP) we are interested in consist in finding functions f with weak derivative $f'\in L^1(-\infty,b)\cap L^2(-\infty,b)$ such that

$$D_*^\alpha f(t) = g(t) \tag{2}$$

where $g\in L^2(\mathbb{R})$ is the data and $D_*^\alpha f$ is the CFD of the unknown function f. Thus, we are looking for an approximate inverse of the CFD of order α of a known function g.

In order to find f we apply the approximation scheme proposed in an earlier work [28] for solving integral equations of the type

$$Af(t) = \int_{\mathbb{R}} h(t,\omega)\widehat{f}(\omega)e^{i\omega t}d\omega = g(t), \quad t\in\mathbb{R}. \tag{3}$$

With this aim we rewrite Eq. (2) using the Fourier Transform (in the distributional sense).

Let κ be the casual function, $\kappa(t) = \frac{1}{t^\alpha}$ for $t>0$ then

$$\widehat{\kappa}(\omega) = \Gamma(1-\alpha)(i\omega)^{\alpha-1}. \tag{4}$$

Consequently,

$$D_*^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^t \frac{f'(\tau)}{(t-\tau)^\alpha}d\tau = \frac{1}{2\pi\Gamma(1-\alpha)}\int_{\mathbb{R}} \widehat{f'}\cdot\widehat{\kappa}(\omega)e^{i\omega t}\,d\omega. \tag{5}$$

Note that Eq. (5) is an expression of the fractional derivative that involves an integral operator in terms of the Fourier Transform of the unknown f, with kernel

$$h(\omega) = \frac{1}{2\pi}(i\omega)^\alpha \tag{6}$$

that is a radial function, not depending on t

$$h(t,\omega) = h(\omega), \qquad h(\lambda\omega) = \lambda^\alpha h(\omega), \lambda>0. \tag{7}$$

Now Eq. (2) is expressed as

$$\int_{\mathbb{R}} h(\omega)\widehat{f}(\omega)e^{i\omega t}\,d\omega = g(t) \tag{8}$$

and the approximation scheme proposed in [28] can be applied. In the following section we describe it briefly. We refer to the cited paper for details.

3 The Proposed Solution

3.1 The wavelet basis

We recall that a wavelet is a oscillant function, well localized in the time and the frequency domains, see [30,31].

For special selection of the mother wavelet ψ the family

$$\{\psi_{jk}(t) = 2^{j/2}\,\psi(2^j t - k);\ j,k \in \mathbb{Z}\},$$

is a orthonormal basis of the space $L^2(\mathbb{R})$, associated with a hierarchical structure of the space, the Multiresolution Analysis (MRA). This is a sequence of nested subspaces V_j, the scale-subspaces, such that:

1. $V_j \subset V_{j+1}$
2. $s(t) \in V_j$ if and only if $s(2t) \in V_{j+1}$
3. $s(t) \in V_0$ then $s(t+1) \in V_0$
4. $\cup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and $\cap_{j\in\mathbb{Z}} V_j = \{0\}$
5. There exists a function called a scaling function $\phi \in V_0$, such that the family $\{\phi(t-k), k \in \mathbb{Z}\}$ is an orthonormal basis of V_0.

The wavelet subspace W_j is the orthogonal complement of V_j in V_{j+1}, i. e.,

$$\begin{cases} V_j \perp W_j \\ V_{j+1} = V_j \oplus W_j,\ j \in \mathbb{Z}. \end{cases} \tag{9}$$

The subspace $W_j = span\{\psi_{jk}(t), k \in \mathbb{Z}\}$ contains the detail information needed to go from an approximation at resolution j to an approximation at resolution $j+1$. Consequently

$$L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j.$$

Moreover,

$$\begin{cases} V_n & = \bigoplus_{j<n} W_j \\ L^2(\mathbb{R}) & = \bigoplus_{j\geq n} W_j + V_n, \text{ for any } n \in \mathbb{Z}. \end{cases}$$

The MRA is associated to an efficient method to compute the wavelet coefficients, the Mallat's algorithm [30].

The application in mind suggests the selection of the mother wavelet. Considering that the operator (8) acts on the Fourier transforms, it seems convenient to implement a partition of the frequency domain in quasi-disjoint scale bands:

$$\mathbb{R}_\omega = \bigcup_{j=-\infty}^{\infty} \Omega_j$$

where the two-side bands $\Omega_j \cong \{2^j\pi \leq |\omega| \leq 2^{j+1}\pi\}$ are naturally associated with the wavelet subspaces W_j.

We choose a Meyer wavelet, a band-limited function ψ, having smooth Fourier transform $\widehat{\psi}$. In [32] we define the scale function and the wavelet as:

$$\widehat{\phi}(\omega) = \begin{cases} 1 & |\omega| < \pi - \beta \\ \dfrac{v_\beta(\omega)}{\sqrt{v_\beta^2(\omega) + v_\beta^2(2\beta - \omega)}} & \pi - \beta < |\omega| < \pi + \beta \\ 0 & |\omega| \geq \pi + \beta \end{cases} \tag{10}$$

with

$$v_\beta(\omega) = \begin{cases} \exp\left(-\dfrac{(\frac{\omega-\pi+\beta}{2\beta})}{1-(\frac{\omega-\pi+\beta}{2\beta})^2}\right) & |\omega-\pi+\beta| < 2\beta \\ 0 & |\omega-\pi+\beta| \geq 2\beta \end{cases} \tag{11}$$

and

$$\widehat{\psi}(\omega) = \sqrt{\phi^2(\omega/2) - \phi^2(\omega)}\, e^{-i\omega/2} \tag{12}$$

with parameter $0 < \beta \leq \pi/3$.

We recall that $\psi \in \mathscr{S}$, the Schwartz Class, and the family $\{\psi_{jk}, k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$ associated to a MRA, well localized in both, time and frequency domain. Its spectrum, $|\widehat{\psi}(2^{-j}\omega)|$, is supported on the two-sided band

$$\Omega_j = \left\{\omega : 2^j(\pi-\beta) \leq |\omega| \leq 2^{j+1}(\pi+\beta)\right\} \tag{13}$$

for some $0 < \beta \leq \pi/3$, [32]. Figure 1 show the graph of ψ and $|\widehat{\psi}|$. See [31] for details about the approximation of Sovolev, Besov and other functional spaces, using wavelets in the Schwartz Class.

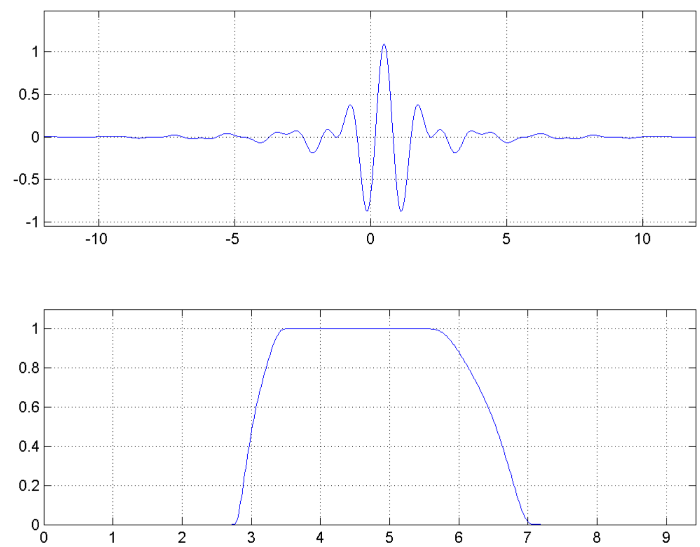

Fig. 1: Mother wavelet for $\beta = \pi/4$ (above) and $|\widehat{\psi}|$ for $\omega \geq 0$ (below)

3.2 The data

We decompose the data $g \in L^2(\mathbb{R})$ as, $g = \sum_{j=J_{min}}^{J_{max}} g_j + r$ where $||r||_2 < \varepsilon\|g\|_2 \cong 0$, $J \in \mathbb{Z}$

$$g_j(t) = \sum_{k\in\mathbb{Z}} c_{jk}\psi_{jk}(t) \in W_j$$

and $c_{jk} = \langle g, \psi_{jk}\rangle$ are the wavelet coefficients. We denote by $\tilde{g}_j$ the truncated projection of the data in W_j,

$$\tilde{g}_j(t) = \sum_{k\in\mathbb{K}_j} c_{jk}\psi_{jk}(t) \tag{14}$$

where $\mathbb{K}_j \subset \mathbb{Z}$, is finite, $|\mathbb{K}_j| = \eta_j < \infty$, and satisfies $\sum_{k\notin\mathbb{K}_j} |\langle g, \psi_{jk}\rangle|^2 < \varepsilon||g_j||^2$ with $\varepsilon \cong 0$.

3.3 The approximation scheme

In this subsection we briefly describe the approximation scheme proposed in [28] to solve an IP described by an integral equation of the type (3). First we look for the images v_{jk} of the wavelet basis defined in Subsection 3.1, i.e., $A\psi_{jk} = v_{jk}$,

$$v_{jk}(t) = \int_{\Omega_j} h(t,\omega)\,\widehat{\psi}_{jk}(\omega)\, e^{i\omega t}\, d\omega. \tag{15}$$

We consider that for each $J_{min} \leq j \leq J_{max}$, $A(W_j) \cong W_j$. If this is not the case, we can proceed in a similar way considering an appropriate union of wavelet subspaces.

Let

$$f(t) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} b_{jk}\psi_{jk}(t), \tag{16}$$

at each level $J_{min} \leq j \leq J_{max}$ we restrict ourselves to $\mathbb{K}_j$ to obtain $\tilde{f}_j = \sum_{k\in\mathbb{K}_j} b_{jk}\psi_{jk}$ (see (14)), and consequently

$$A\tilde{f}_j(t) = \sum_{k\in\mathbb{K}_j} b_{jk}v_{jk}(t). \tag{17}$$

We propose $A\tilde{f}_j(t) \cong \tilde{g}_j(t)$, i.e.,

$$\sum_{k\in\mathbb{K}_j} b_{jk}v_{jk}(t) \cong \sum_{k'\in\mathbb{K}_j} c_{j'k'}\psi_{j'k'}(t) \tag{18}$$

and calculate the vector of coefficients $\mathbf{b}_k^j = \{b_{jk}\}_{k\in\mathbb{K}_j}$ from the normal equations

$$\left\langle \sum_{l\in\mathbb{K}_j} b_{jl}v_{jl}, \psi_{jm} \right\rangle = \sum_{l\in\mathbb{K}_j} b_{jl}\left\langle v_{jl}, \psi_{jm}\right\rangle = c_{jm},\ m\in\mathbb{K}_j. \tag{19}$$

Eq. (19) can be expressed as

$$M^j\mathbf{b}_k^j = \mathbf{c}_k^j,\ k\in\mathbb{K}_j \tag{20}$$

where the matrix $M^j \in \mathbb{R}^{\eta_j}\times\mathbb{R}^{\eta_j}$ contains the inner products

$$M_{lm}^j = \left\langle v_{jl}, \psi_{jm}\right\rangle, \tag{21}$$

and $\mathbf{c}_k^j = \{c_{jk}\}_{k\in\mathbb{K}_j}$ is the vector of coefficients of the data g (see Eq. (14)). Based on the properties of the wavelet basis and the integral operator, on each level j, M^j is invertible and $\mathbf{b}^j$ can be computed.

Finally, for $J_{min} \leq j \leq J_{max}$, we set $\tilde{f}_j = \sum_{k\in\mathbb{K}_j} b_{jk}\psi_{jk}$ and $\tilde{f} = \sum_{j=J_{min}}^{J_{max}} \tilde{f}_j$ is the proposed approximate solution.

Since we work with band limited wavelets, all integrals can be calculated in the frequency domain, that is, in compact subsets. The scheme is numerically stable and efficient. Numerical approximations to compute M^j and v_{jk} can be found in [28].

4 A Solution to the IP for the Caputo Fractional Derivative

In order to apply the approximation scheme proposed in [28] to calculate f in (2), we consider $Af = D_*^\alpha f$ as in (8) and follow the steps described above. Note that in this case the kernel of the integral operator defined in Eq. (6) does not depend on t.

First we calculate the images of the wavelet basis,

$$v_{jk}(t) = D_*^\alpha\psi_{jk}(t) = \int_{\Omega_j} h(\omega)\widehat{\psi}_{jk}(\omega)e^{i\omega t}d\omega, \quad k\in\mathbb{K}_j. \tag{22}$$

In particular, since the kernel defined in (6) satisfies (7), we observe that the family $\{v_{jk},\ J_{min} \leq j \leq J_{max}, k\in\mathbb{K}_j\}$ resembles the wavelet family and dilation and translations formula can be proved.

Lemma 1. *Let* $v(t) = \int_{\mathbb{R}} h(\omega)|\widehat{\psi}(\omega)|e^{-i\omega/2}e^{i\omega t}d\omega$*, then the family* $\{v_{jk},\, J_{min} \le j \le J_{max},\, k \in \mathbb{K}_j\}$ *defined in (22) satisfies the following dilation and translation properties:*

$$v_{jk}(t) = 2^{j(1/2+\alpha)}v(2^jt-k). \tag{23}$$

Proof. Recall that $\widehat{\psi}_{jk}(\omega) = 2^{-j/2}|\widehat{\psi}(2^{-j}\omega)|e^{-i2^{-j}\omega(1/2+k)}$, then from (22) we have

$$\begin{aligned} v_{jk}(t) &= 2^{-j/2}\int_{\mathbb{R}} h(\omega)|\widehat{\psi}(2^{-j}\omega)|e^{-i2^{-j}\omega(1/2+k)}e^{i\omega t}d\omega \\ &= 2^{j(1/2+\alpha)}\int_{\mathbb{R}} h(v)|\widehat{\psi}(v)|e^{-iv(1/2+k)}e^{iv(2^jt)}dv \\ &= 2^{j(1/2+\alpha)}v(2^jt-k). \end{aligned}$$

□

We incorporate these calculations to compute the elements of matrix M^j. We prove that it is a diagonal dominant matrix and consequently the vector $\mathbf{b}^j$ can be easily calculated from (20).

Proposition 1. *M^j is a diagonal dominant matrix.*

Proof. For $A = D^{\alpha}_{*}$, and taking into account (23), the elements $M^j_{lm} = \langle v_{jl}, \psi_{jm}\rangle$ results

$$\langle v_{jl}, \psi_{jm}\rangle = 2^{j(\alpha+1)}\int_{\mathbb{R}} v(2^jt-l)\psi(2^jt-k)\,dt. \tag{24}$$

For $\omega \in \Omega_j$ we can approximate the kernel h by the following expression

$$h(\omega) = a_0 + \sum_{n=1}^{N} a_n\cos(\frac{n}{2^j}\omega) + b_n\sin(\frac{n}{2^j}\omega) \; + \; \varepsilon(\omega)$$

where (a_n, b_n) are the coefficients of the expansion of h and ε is the error that is small for large N. Then

$$v_{jk}(t) \cong \int_{\Omega_j}\left(a_0 + \sum_{n=1}^{N} a_n\cos(\frac{n}{2^j}\omega) + b_n\sin(\frac{n}{2^j}\omega)\right)\,\widehat{\psi}_{jk}(\omega)\,e^{i\omega t}\,d\omega.$$

We observe that, for $1 \le n \le N$

$$\begin{aligned} 2\cos(\tfrac{n}{2^j}\omega)\widehat{\psi}_{jk}(\omega) &= \left(e^{-i\frac{n}{2^j}\omega} + e^{i\frac{n}{2^j}\omega}\right)\widehat{\psi}_{jk}(\omega) = \widehat{\psi}_{j(k+n)}(\omega) + \widehat{\psi}_{j(k-n)}(\omega), \\ 2i\sin(\tfrac{n}{2^j}\omega)\widehat{\psi}_{jk}(\omega) &= \left(-e^{-i\frac{n}{2^j}\omega} + e^{i\frac{n}{2^j}\omega}\right)\widehat{\psi}_{jk}(\omega) = -\widehat{\psi}_{j(k+n)}(\omega) + \widehat{\psi}_{j(k-n)}(\omega) \end{aligned}$$

then

$$v_{jk}(t) \cong 2\pi\sum_{n=-N}^{N}\gamma_n\psi_{j(k-n)}(t).$$

Finally, we can approximate for $0 \le m \le N$

$$< v_{jk}, \psi_{jm} > \cong 2\pi\gamma_{k+m}$$

and the inner products are nulls in another case.

Consequently M^j is a diagonal dominant matrix. □

We point out that we do not need to calculate v_{jk}, only the values (24) are actually needed to compute $\tilde{f}_j$.

*Remark.*1. Note that $\forall k \in \mathbb{K}_j$, $supp(\widehat{v}_{jk}) \subset \Omega_j$ (see (13)). Since, disregarding the overlaps, W_j is nearly a basis of the set of functions with spectrum in Ω_j, the assumption $D^{\alpha}_{*}\psi_{jk} \subset W_j$ is justified. Thus, in this case based on the localization properties of the chosen wavelet basis and on $\widehat{v}_{jk}(\omega) = 2\pi h(\omega)\widehat{\psi}_{jk}(\omega)$, we can assume that $J_{min} \le j \le J_{max}$, $D^{\alpha}_{*}(W_j) \cong W_j$.

Nevertheless, this assumption can be relaxed considering union of wavelet subspaces, i.e., $D^{\alpha}_{*}\psi_{jk} \subset (W_{j-1} \cup W_j \cup W_{j+1})$.

Finally,

$$\tilde{f}(t) = \sum_{J_{min} \leq j \leq J_{max}} \tilde{f}_j(t) = \sum_{J_{min} \leq j \leq J_{max}} \sum_{k \in \mathbb{K}_j} b_{jk} \psi_{jk}(t). \tag{25}$$

Remark.2. Other Fractional Derivatives such as the new Caputo Fabrizio Fractional Derivative can also be approximated using this scheme, [2,29].

5 Numerical Examples

For the integral operator D_*^α as in (8) with kernel $h(\omega) = \dfrac{1}{2\pi}(i\omega)^\alpha$, $\alpha = 0.5$ we illustrate the accuracy of the approximation scheme. In Figures 2 and 3 we show the plot of the functions $v_{3,0}$ and $v_{4,0}$ together with the plot of the wavelet $\psi_{3,0}$ and $\psi_{4,0}$ respectively.

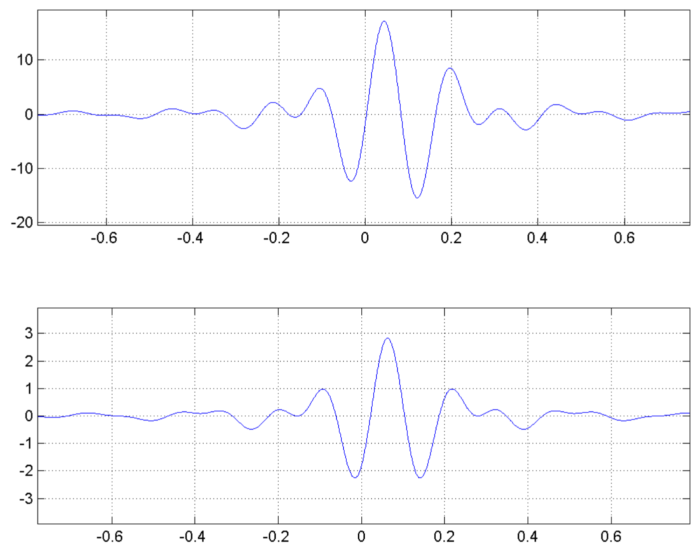

Fig. 2: Functions $v_{3,0}$ (above) and $\psi_{3,0}$ (below)

We generate some v_{jk} for $j = 3$ and $k = 0, 5, 15$ (see Figure 4). The resemblance with the wavelet basis can be observed. In Figures 5 and 6 diagonal dominant matrix are shown.

The whole proposed approximation scheme is implemented in the following example. We choose the data g as the CFD of order $\alpha = 0.5$ of the sampled function

$$f(t) = e^{-t^2/2}(\sin(16\pi t) + \cos(\frac{3}{2}\pi t)).$$

Both plots of f and g are displayed in Figure 7. Wavelet analysis indicates that the energy of the data g is concentrated in the subspaces W_{-1}, W_0, W_3 and W_4, since levels $j = 0, 3, 4$ summarize the 6.9%, 46.1% and 46.3% of it, (see Table 1).

Similarly for f, the energy is concentrated in the subspaces W_{-1}, W_0 and W_4, (see Table 2). In this case, for instance, $A(\cup_{j=3}^4 W_j) \subset \cup_{j=3}^4 W_j$.

Finally, the sum of the reconstruction components $\sum_{j=-1}^0 \tilde{f}_j + \sum_{j=3}^4 \tilde{f}_j$, that is, the approximate solution to the IP, is displayed in Figure 8 along with the projections of the true solution $\sum_{j=-1}^0 f_j + \sum_{j=3}^4 f_j$.

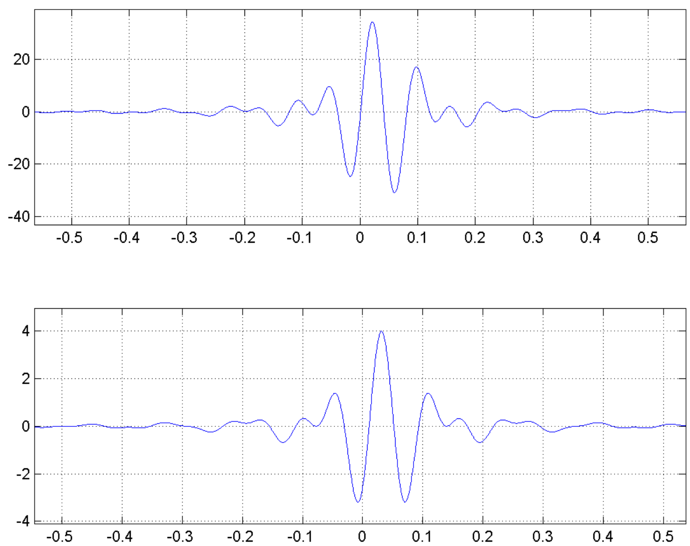

Fig. 3: Functions $v_{4,0}$ (above) and $\psi_{4,0}$ (below)

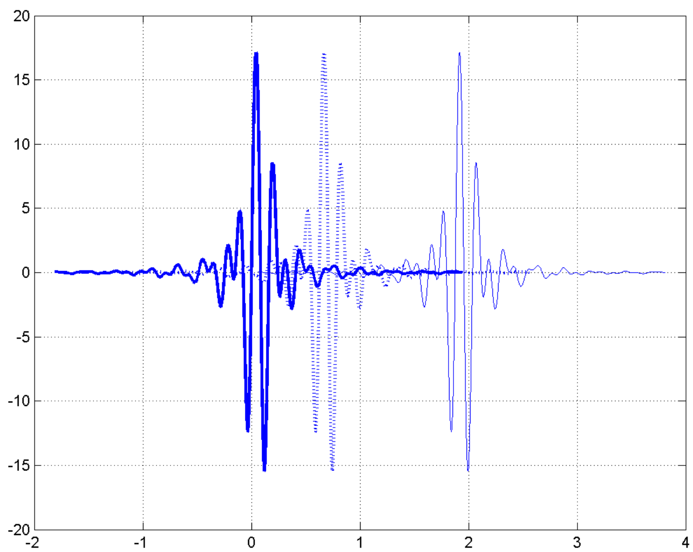

Fig. 4: Functions $v_{3,0}$, $v_{3,5}$ and $v_{3,15}$ from right to left respectively

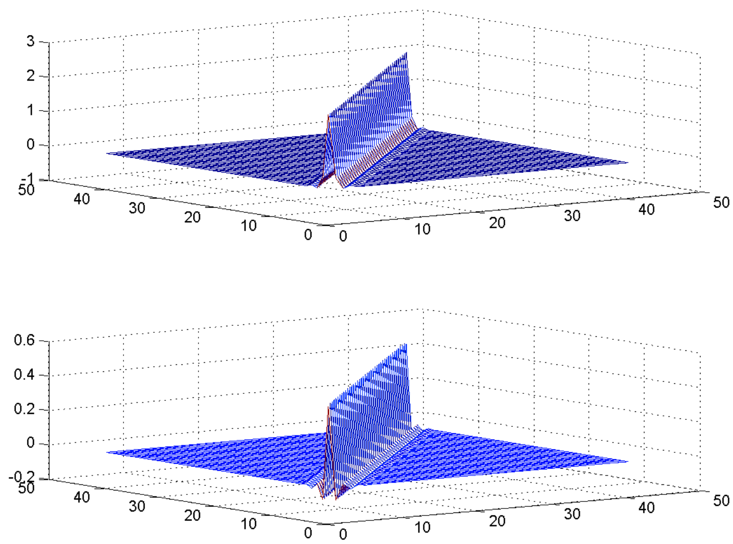

Fig. 5: Matrix M^3 for (above) and its inverse (below)

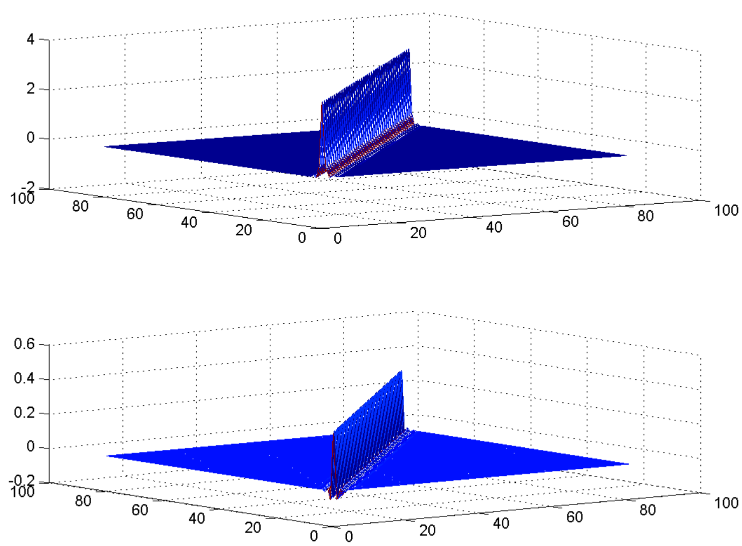

Fig. 6: Matrix M^4 for (above) and its inverse (below)

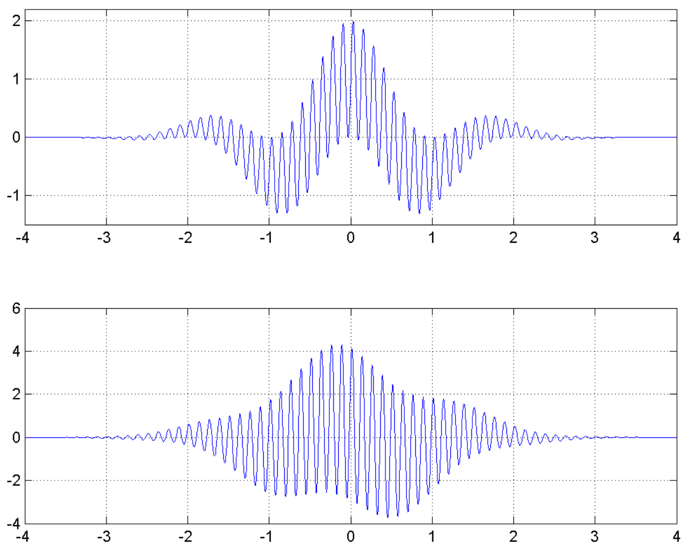

Fig. 7: **Functions** f **(above) and** g **(below)**

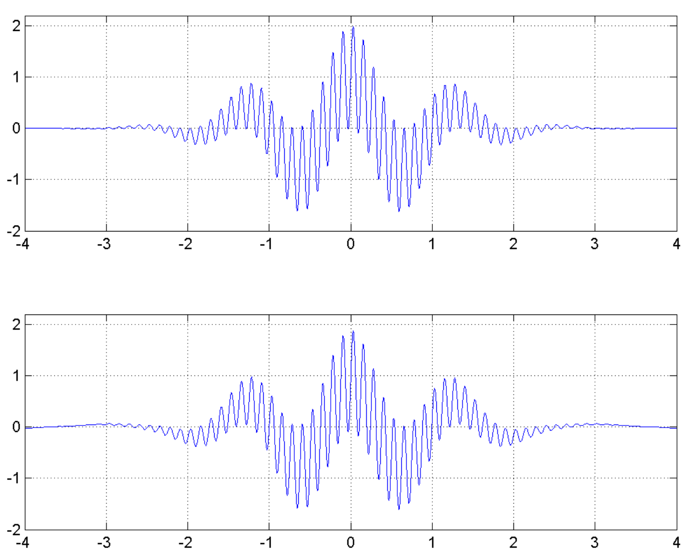

Fig. 8: $\sum_{j=-1}^{0} \tilde{f}_j + \sum_{j=3}^{4} \tilde{f}_j$ **(above) vs** $\sum_{j=-1}^{0} f_j + \sum_{j=3}^{4} f_j$ **(below)**

Table 1: Energy distribution of g

Level j	Energy	Frequencies
5	0.0000	[100.5, 201.0]
4	**0.4636**	[50.2, 100.5]
3	**0.4613**	[25.1, 50.2]
2	0.0000	[12.5, 25.1]
1	0.0002	[6.28, 12.5]
0	**0.0698**	[3.14, 6.28]
-1	0.0052	[1.57 3.14]

Table 2: Energy distribution of f

Level j	Energy	Frequencies
5	0.0000	[100.5, 201.0]
4	**0.4961**	[50.2, 100.5]
3	0.0039	[25.1, 50.2]
2	0.0000	[12.5, 25.1]
1	0.0001	[6.28, 12.5]
0	**0.4257**	[3.14, 6.28]
-1	0.0740	[1.57 3.14]

6 Conclusion

In this article, we construct an approximate solution to an inverse problem that consists in finding a function whose Caputo Fractional Derivative is given. The solution is built by means of a numerical scheme proposed in an previous work. First we rewrite the equation considering the Fourier Transform of the unknown. Afterwards, for a suitable wavelet basis, the data is decomposed and projected in wavelet subspaces and, by a Galerkin scheme, the coefficients of the unknown function in the chosen wavelet basis are calculated. Properties of the basis and the fractional integral operator enables us to warranty that the scheme is efficient and numerically stable. The errors introduced in the approximation can be handled and controlled. We illustrate the results with an example.

We hope that this scheme can be adapted to solve inverse problems associated to other fractional operators and to solve fractional differential equations.

Acknowledgments

This work is dedicated to Dr. E. P. Serrano's memory who proposed the initial idea and guided us in the development. MINCyT- CONICET- CNRS (International Cooperation Programme, CONICET Resol. 147/15) and UBACyT project 20020130100671BA partially supported this work.

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