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Vector Fractional Trigonometric Korovkin Approximation

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Abstract: In this manuscript we study quantitatively with rates the trigonometric fractional convergence of sequences of linear operators applied on Banach space valued functions. We derive pointwise and uniform estimates. To establish our main results we apply an elegant boundedness property of our linear operators by their companion positive linear operators. Our inequalities are trigonometric fractional involving the right and left vector Caputo type fractional derivatives, built in vector moduli of continuity. We consider very general classes of Banach space valued functions. Finally we present applications to vector Bernstein operators.

Keywords: Vector fractional derivative, Bochner integral, vector Fractional Taylor formula, vector modulus of continuity, linear operators, positive linear operators, trigonometric approximation.

1 Motivation

Let $(X, \|\cdot\|)$ be a Banach space, $N \in \mathbb{N}$. Let $g \in C([0,1])$ and the classic Bernstein polynomials

$$\left(\widetilde{B}_N g\right)(t) = \sum_{k=0}^{N} g\left(\frac{k}{N}\right)\binom{N}{k} t^k (1-t)^{N-k}, \quad \forall\, t \in [0,1]. \tag{1}$$

Let also $f \in C([0,1], X)$ and define the vector valued in X Bernstein linear operators

$$(B_N f)(t) = \sum_{k=0}^{N} f\left(\frac{k}{N}\right)\binom{N}{k} t^k (1-t)^{N-k}, \quad \forall\, t \in [0,1]. \tag{2}$$

That is $(B_N f)(t) \in X$.

Here we conclude $\|f\| \in C([0,1])$.

We notice that

$$\|(B_N f)(t)\| \le \sum_{k=0}^{N} \left\| f\left(\frac{k}{N}\right)\right\| \binom{N}{k} t^k (1-t)^{N-k} = \left(\widetilde{B}_N(\|f\|)\right)(t), \tag{3}$$

$\forall\, t \in [0,1]$.

The property

$$\|(B_N f)(t)\| \le \left(\widetilde{B}_N(\|f\|)\right)(t), \quad \forall t \in [0,1], \tag{4}$$

is shared by almost all summation/integration similar operators and motivates our work here.

If $f(x) = c \in X$ the constant function, then

$$(B_N c) = c. \tag{5}$$

If $g \in C([0,1])$ and $c \in X$, then $cg \in C([0,1], X)$ and

$$(B_N(cg)) = c\widetilde{B}_N(g). \tag{6}$$

Again (5), (6) are fulfilled by many summation/integration operators.

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In fact here (6) implies (5), when $g \equiv 1$.

The above can be generalized from $[0,1]$ to any interval $[a,b] \subset \mathbb{R}$. All this discussion motivates us to investigate the following situation.

Let $L_N : C([a,b],X) \hookrightarrow C([a,b],X)$, $(X, \|\cdot\|)$ a Banach space, L_N is a linear operator, $\forall\, N \in \mathbb{N}$, $x_0 \in [a,b]$. Let also $\widetilde{L}_N : C([a,b]) \hookrightarrow C([a,b])$, a sequence of positive linear operators, $\forall\, N \in \mathbb{N}$.

We assume that

$$\|(L_N(f))(x_0)\| \leq \left(\widetilde{L}_N(\|f\|)\right)(x_0), \tag{7}$$

$\forall\, N \in \mathbb{N}$, $\forall\, x_0 \in X$, $\forall\, f \in C([a,b],X)$.

When $g \in C([a,b])$, $c \in X$, we suppose that

$$(L_N(cg)) = c\widetilde{L}_N(g). \tag{8}$$

The special case of

$$\widetilde{L}_N(1) = 1, \tag{9}$$

implies

$$L_N(c) = c, \ \ \forall\, c \in X. \tag{10}$$

We call $\widetilde{L}_N$ the companion operator of L_N.

Based on the above fundamental properties we study over $[-\pi,\pi]$ the trigonometric fractional approximation properties of the sequence of linear operators $\{L_N\}_{N\in\mathbb{N}}$, i.e. their trigonometric fractional convergence to the unit operator. No kind of positivity property of $\{L_N\}_{N\in\mathbb{N}}$ is assumed. Other important motivation comes from [1], [2], [3], [4].

2 Background

All vector integrals here are of Bochner type [5].

We need

Definition 1.([7]) *Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$; $m = \lceil \alpha \rceil \in \mathbb{N}$, ($\lceil \cdot \rceil$ is the ceiling of the number), $f : [a,b] \to X$. We assume that $f^{(m)} \in L_1([a,b],X)$. We call the Caputo-Bochner left fractional derivative of order α:*

$$\left(D_{*a}^{\alpha} f\right)(x) := \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \ \ \forall\, x \in [a,b]. \tag{11}$$

If $\alpha \in \mathbb{N}$, we set $D_{*a}^{\alpha} f := f^{(m)}$ the ordinary X-valued derivative (defined similar to numerical one, see [6], p. 83), and also set $D_{*a}^{0} f := f$.

By [7], $\left(D_{*a}^{\alpha} f\right)(x)$ exists almost everywhere in $x \in [a,b]$ and $D_{*a}^{\alpha} f \in L_1([a,b],X)$.

If $\left\|f^{(m)}\right\|_{L_\infty([a,b],X)} < \infty$, then by [7], $D_{*a}^{\alpha} f \in C([a,b],X)$, hence $\|D_{*a}^{\alpha} f\| \in C([a,b])$.

We mention

Lemma 1.([3]) *Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, $m = \lceil \alpha \rceil$, $f \in C^{m-1}([a,b],X)$ and $f^{(m)} \in L_\infty([a,b],X)$. Then $D_{*a}^{\alpha} f(a) = 0$.*

We mention

Definition 2.([8]) *Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m := \lceil \alpha \rceil$. We assume that $f^{(m)} \in L_1([a,b],X)$, where $f : [a,b] \to X$. We call the Caputo-Bochner right fractional derivative of order α:*

$$\left(D_{b-}^{\alpha} f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\, dz, \ \ \forall\, x \in [a,b]. \tag{12}$$

We observe that $\left(D_{b-}^{m} f\right)(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $\left(D_{b-}^{0} f\right)(x) = f(x)$.

By [8], $\left(D_{b-}^{\alpha} f\right)(x)$ exists almost everywhere on $[a,b]$ and $\left(D_{b-}^{\alpha} f\right) \in L_1([a,b],X)$.

If $\left\|f^{(m)}\right\|_{L_\infty([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, by [8], $D_{b-}^{\alpha} f \in C([a,b],X)$, hence $\left\|D_{b-}^{\alpha} f\right\| \in C([a,b])$.

We need

Lemma 2.([3]) *Let* $f \in C^{m-1}([a,b],X)$, $f^{(m)} \in L_\infty([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$. *Then* $D_{b-}^{\alpha} f(b) = 0$.

We mention the left fractional Taylor formula

Theorem 1.([7]) *Let* $m \in \mathbb{N}$ *and* $f \in C^{m-1}([a,b],X)$, *where* $[a,b] \subset \mathbb{R}$ *and* X *is a Banach space, and let* $\alpha > 0 : m = \lceil \alpha \rceil$. *Set*

$$F_x(t) := \sum_{i=0}^{m-1} \frac{(x-t)^i}{i!} f^{(i)}(t), \quad \forall\, t \in [a,x], \tag{13}$$

where $x \in [a,b]$.

Assume that $f^{(m)}$ *exists outside a* λ*-null Borel set* $B_x \subseteq [a,x]$ (λ *is the Lebesgue measure) such that*

$$h_1(F_x(B_x)) = 0, \ \forall\, x \in [a,b] \tag{14}$$

(h_1 *is the Hausdorff measure of order* 1, *see* [9]). *We also assume that* $f^{(m)} \in L_1([a,b],X)$. *Then*

$$f(x) = \sum_{i=0}^{m-1} \frac{(x-a)^i}{i!} f^{(i)}(a) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-z)^{\alpha-1} (D_{*a}^{\alpha} f)(z)\, dz, \tag{15}$$

$\forall\, x \in [a,b]$.

We also mention the right fractional Taylor formula

Theorem 2.([8]) *Let* $[a,b] \subset \mathbb{R}$, X *be a Banach space,* $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in C^{m-1}([a,b],X)$. *Set*

$$F_x(t) := \sum_{i=0}^{m-1} \frac{(x-t)^i}{i!} f^{(i)}(t), \quad \forall\, t \in [x,b], \tag{16}$$

where $x \in [a,b]$.

Assume that $f^{(m)}$ *exists outside a* λ*-null Borel set* $B_x \subseteq [x,b]$, *such that*

$$h_1(F_x(B_x)) = 0, \ \forall\, x \in [a,b]. \tag{17}$$

We also assume that $f^{(m)} \in L_1([a,b],X)$. *Then*

$$f(x) = \sum_{i=0}^{m-1} \frac{(x-b)^i}{i!} f^{(i)}(b) + \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} \left(D_{b-}^{\alpha} f\right)(z)\, dz, \tag{18}$$

$\forall\, x \in [a,b]$.

We define the following classes of functions:

Definition 3.([3]) *We call* ($x_0 \in [a,b] \subset \mathbb{R}$)$\rightarrow$

$$H_{x_0}^{(1)}([a,b]) := \{ f \in C^{m-1}([a,b],X) : [a,b] \subset \mathbb{R}, (X, \|\cdot\|) \tag{19}$$

is a Banach space, $\alpha > 0 : m = \lceil \alpha \rceil$; $f^{(m)} \in L_\infty([a,b],X)$; $F_x^{(1)}(t) := \sum_{i=0}^{m-1} \frac{(x-t)^i}{i!} f^{(i)}(t)$ *is defined* $\forall\, t \in [x,x_0]$,*with* $x \in [a,x_0]$ *and* $f^{(m)}$ *exists outside a* λ*-null Borel set* $B_x^{(1)} \subseteq [x,x_0]$, *such that* $h_1\left(F_x^{(1)}\left(B_x^{(1)}\right)\right) = 0$, $\forall\, x \in [a,x_0]$; $F_x^2(t) := \sum_{i=0}^{m-1} \frac{(x-t)^i}{i!} f^{(i)}(t)$ *is defined* $\forall\, t \in [x_0,x]$,*with* $x \in [x_0,b]$ *and* $f^{(m)}$ *exists outside a* λ*-null Borel set* $B_x^{(2)} \subseteq [x_0,x]$, *such that* $h_1\left(F_x^{(2)}\left(B_x^{(2)}\right)\right) = 0$, $\forall\, x \in [x_0,b]\}$,

$$H^{(2)}([a,b]) := \{ f \in C^m([a,b],X) : [a,b] \subset \mathbb{R}, \tag{20}$$

X *is a Banach space,* $\alpha > 0 : m = \lceil \alpha \rceil \}$.

Notice that

$$H^{(2)}([a,b]) \subset H_{x_0}^{(1)}([a,b]), \ \forall\, x_0 \in [a,b]. \tag{21}$$

Convention 1 *We assume that*

$$D_{*x_0}^{\alpha} f(x) = 0, \text{ for } x < x_0, \quad \text{and} \quad D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0, \tag{22}$$

for all $x, x_0 \in [a,b]$.

We need

Definition 4. *([3]) Let* $f \in C([a,b], X)$, $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ *a Banach space. We define the first modulus of continuity of* f *as*

$$\omega_1(f, \delta) := \sup_{\substack{x,y \in [a,b]: \\ |x-y| \le \delta}} \|f(x) - f(y)\|, \ \ 0 < \delta \le b - a. \tag{23}$$

If $\delta > b - a$, *then* $\omega_1(f, \delta) = \omega_1(f, b-a)$.
Notice $\omega_1(f, \delta)$ *is increasing in* $\delta > 0$.

Clearly f is uniformly continuous and $\omega_1(f, \delta) < \infty$. For $f \in B([a,b], X)$ (bounded functions) $\omega_1(f, \delta)$ is defined the same way.

Lemma 3. *([3]) We have* $\omega_1(f, \delta) \to 0$ *as* $\delta \downarrow 0$ *iff* $f \in C([a,b], X)$.

We mention

Proposition 2. *([3]) Let* $f \in C^n([a,b], X)$, $n = \lceil \nu \rceil$, $\nu > 0$. *Then* $D_{*a}^{\nu} f(x)$ *is continuous in* $x \in [a,b]$.

Proposition 3. *([3]) Let* $f \in C^m([a,b], X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$. *Then* $D_{b-}^{\nu} f(x)$ *is continuous in* $x \in [a,b]$.

We also mention

Proposition 4. *([3]) Let* $f \in C^{m-1}([a,b], X)$, $f^{(m)} \in L_{\infty}([a,b], X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$ *and*

$$D_{*x_0}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{24}$$

for all $x, x_0 \in [a,b] : x \ge x_0$.
Then $D_{*x_0}^{\alpha} f(x)$ *is continuous in* x_0.

Proposition 5. *([3]) Let* $f \in C^{m-1}([a,b], X)$, $f^{(m)} \in L_{\infty}([a,b], X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$ *and*

$$D_{x_0-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{x}^{x_0} (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta)\, d\zeta, \tag{25}$$

for all $x, x_0 \in [a,b] : x_0 \ge x$.
Then $D_{x_0-}^{\alpha} f(x)$ *is continuous in* x_0.

Corollary 1. *([3]) Let* $f \in C^m([a,b], X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in [a,b]$. *Then* $D_{*x_0}^{a} f(x)$, $D_{x_0-}^{a} f(x)$ *are jointly continuous functions in* (x, x_0) *from* $[a,b]^2$ *into* X, X *is a Banach space.*

We need

Theorem 3. *([3]) Let* $f : [a,b]^2 \to X$ *be jointly continuous,* X *is a Banach space. Consider*

$$G(x) = \omega_1(f(\cdot, x), \delta, [x,b]), \tag{26}$$

$\delta > 0$, $x \in [a,b]$.
Then G *is continuous on* $[a,b]$.

Theorem 4.*([3]) Let $f:[a,b]^2 \to X$ be jointly continuous, X is a Banach space. Then*

$$H(x) = \omega_1 (f(\cdot,x), \delta, [a,x]), \tag{27}$$

$x \in [a,b]$, is continuous in $x \in [a,b]$, $\delta > 0$.

We mention and need

Remark.([3]) Let $f \in C^{n-1}([a,b])$, $f^{(n)} \in L_\infty([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$, $\nu \notin \mathbb{N}$. Then

$$\|D_{*a}^{\nu} f(x)\| \leq \frac{\left\| f^{(n)} \right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)} (x-a)^{n-\nu}, \ \forall x \in [a,b], \tag{28}$$

and it follows that

$$\omega_1 (D_{*a}^{\nu} f, \delta) \leq \frac{2\left\| f^{(n)} \right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}. \tag{29}$$

Similarly, let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, then

$$\omega_1 \left(D_{b-}^{\alpha} f, \delta\right) \leq \frac{2\left\| f^{(m)} \right\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}. \tag{30}$$

So for $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, we find

$$\sup_{x_0 \in [a,b]} \omega_1 \left(D_{*x_0}^{\alpha} f, \delta\right)_{[x_0,b]} \leq \frac{2\left\| f^{(m)} \right\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}, \tag{31}$$

and

$$\sup_{x_0 \in [a,b]} \omega_1 \left(D_{x_0-}^{\alpha} f, \delta\right)_{[a,x_0]} \leq \frac{2\left\| f^{(m)} \right\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}. \tag{32}$$

Lemma 4.*([1], p. 208, Lemma 7.1.1) Let $f \in B([a,b],X)$, $(X, \|\cdot\|)$ is a Banach space. Then*

$$\|f(x) - f(x_0)\| \leq \omega_1(f,h) \left\lceil \frac{|x-x_0|}{h} \right\rceil \leq \omega_1(f,h)\left(1 + \frac{|x-x_0|}{h}\right), \tag{33}$$

$\forall\, x, x_0 \in [a,b]$, $h > 0$.

We make

*Remark.*Let μ be a finite positive measure on Borel σ-algebra of $[-\pi,\pi]$.
Let $\alpha > 0$, then by Hölder's inequality we obtain ($x_0 \in [-\pi,\pi]$),

$$\int_{[-\pi,x_0]} (x_0 - x)^{\alpha} d\mu(x) \leq$$

$$2^{\alpha} \left(\int_{[-\pi,x_0]} \left(\frac{(x_0-x)}{2} \right)^{\alpha+1} d\mu(x) \right)^{\frac{\alpha}{(\alpha+1)}} \mu([-\pi,x_0])^{\frac{1}{(\alpha+1)}} \leq$$

$$(2\pi)^{\alpha} \left(\int_{[-\pi,x_0]} \left(\sin\left(\frac{(x_0-x)}{4} \right) \right)^{\alpha+1} d\mu(x) \right)^{\frac{\alpha}{(\alpha+1)}} \mu([-\pi,x_0])^{\frac{1}{(\alpha+1)}}, \tag{34}$$

by $|t| \leq \pi \sin\left(\frac{|t|}{2}\right)$, $t \in [-\pi,\pi]$.

Similarly we obtain

$$\int_{(x_0,\pi]} (x-x_0)^{\alpha} d\mu(x) \leq$$

$$2^{\alpha}\left(\int_{(x_0,\pi]}\left(\frac{(x-x_0)}{2}\right)^{\alpha+1} d\mu(x)\right)^{\frac{\alpha}{(\alpha+1)}} \mu((x_0,\pi])^{\frac{1}{(\alpha+1)}} \leq$$

$$(2\pi)^{\alpha}\left(\int_{(x_0,\pi]}\left(\sin\left(\frac{(x-x_0)}{4}\right)\right)^{\alpha+1} d\mu(x)\right)^{\frac{\alpha}{(\alpha+1)}} \mu((x_0,\pi])^{\frac{1}{(\alpha+1)}}. \tag{35}$$

Let now $m=\lceil\alpha\rceil$, $\alpha\in\mathbb{N}$, $\alpha>0$, $k=1,...,m-1$. Then again by Hölder's inequality we obtain

$$\int_{[-\pi,\pi]} |x-x_0|^k d\mu(x) \leq$$

$$2^k\left(\int_{[-\pi,\pi]}\left(\frac{|x-x_0|}{2}\right)^{\alpha+1} d\mu(x)\right)^{\frac{k}{(\alpha+1)}} (\mu([-\pi,\pi]))^{\frac{\alpha+1-k}{(\alpha+1)}} \leq$$

$$(2\pi)^k\left(\int_{[-\pi,\pi]}\left(\sin\left(\frac{|x-x_0|}{4}\right)\right)^{\alpha+1} d\mu(x)\right)^{\frac{k}{(\alpha+1)}} \mu([-\pi,\pi])^{\frac{\alpha+1-k}{(\alpha+1)}}. \tag{36}$$

Terminology 1*Let $C([-\pi,\pi])$ denotes all the real valued continuous functions on $[-\pi,\pi]$. Let $\widetilde{L}_N : C([-\pi,\pi]) \to C([-\pi,\pi])$, $N\in\mathbb{N}$, be a sequence of positive linear operators. By Riesz representation theorem (see [10], p. 304) we have*

$$\widetilde{L}_N(f,x_0) = \int_{[-\pi,\pi]} f(t)\, d\mu_{Nx_0}(t), \tag{37}$$

$\forall\, x_0\in[-\pi,\pi]$, *where μ_{Nx_0} is a unique positive finite measure on σ-Borel algebra of $[-\pi,\pi]$. Call*

$$\widetilde{L}_N(1,x_0) = \mu_{Nx_0}([-\pi,\pi]) = M_{Nx_0}. \tag{38}$$

We make

*Remark.*Let $\widetilde{L}_N : C([-\pi,\pi]) \to C([-\pi,\pi])$, $N\in\mathbb{N}$, be a sequence of positive linear operators. Using (37) and (36), we obtain ($x\in[-\pi,\pi]$, $k=1,...,m-1,\ ,m=\lceil\alpha\rceil$, $\alpha\notin\mathbb{N}$, $\alpha>0$) for $k=1,...,m-1$ that

$$\left\|\widetilde{L}_N\left(|\cdot-x|^k, x\right)\right\|_{\infty} \leq$$

$$(2\pi)^k\left(\left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot-x|}{4}\right)\right)^{\alpha+1}, x\right)\right\|_{\infty}^{\frac{k}{(\alpha+1)}}\right)\left\|\widetilde{L}_N 1\right\|_{\infty}^{\left(\frac{\alpha+1-k}{\alpha+1}\right)}. \tag{39}$$

Notice that for any $x\in[-\pi,\pi]$ we have

$$C([-\pi,\pi]) \ni |\cdot-x|\,\chi_{[-\pi,x]}(\cdot) \leq |\cdot-x| \in C([-\pi,\pi]),$$

(χ is the characteristic function)
therefore

$$C([-\pi,\pi]) \ni \left(\sin\left(\frac{|\cdot-x|\,\chi_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1} \leq \left(\sin\left(\frac{|\cdot-x|}{4}\right)\right)^{\alpha+1} \in C([-\pi,\pi]). \tag{40}$$

Consequently, by positivity of $\widetilde{L}_N$ we obtain

$$\left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot-x|\,\chi_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_{\infty} \leq \left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot-x|}{4}\right)\right)^{\alpha+1}, x\right)\right\|_{\infty}. \tag{41}$$

Similarly, for any $x \in [-\pi, \pi]$ we have

$$C([-\pi,\pi]) \ni |\cdot - x| \chi_{[x,\pi]}(\cdot) \leq |\cdot - x| \in C([-\pi,\pi]),$$

thus

$$C([-\pi,\pi]) \ni \left(\sin\left(\frac{|\cdot - x| \chi_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1} \leq \left(\sin\left(\frac{|\cdot - x|}{4}\right)\right)^{\alpha+1} \in C([-\pi,\pi]). \tag{42}$$

Hence

$$\left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot - x| \chi_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty \leq \left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot - x|}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty. \tag{43}$$

So if the right side of each of (41), (43) goes to zero, so do their left hand sides.
In fact we notice that

$$\left(\sin\frac{|\cdot - x|}{4}\right)^{\alpha+1} = \left(\sin\left(\frac{|\cdot - x| \chi_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1} + \left(\sin\left(\frac{|\cdot - x| \chi_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, \tag{44}$$

for every $x \in [-\pi, \pi]$.
Hence it holds

$$\left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot - x|}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty \leq \left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot - x| \chi_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty + \left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot - x| \chi_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty. \tag{45}$$

Consequently, if both

$$\left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot - x| \chi_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty, \quad \left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot - x| \chi_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty \to 0 \tag{46}$$

as $N \to +\infty$, then

$$\left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot - x|}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty \to 0. \tag{47}$$

Here we work on $[a,b] = [-\pi, \pi]$ interval.

3 Main Results

It follows our first main result

Theorem 5. *Let $N \in \mathbb{N}$ and $L_N : C([-\pi,\pi], X) \to C([-\pi,\pi], X)$, where $(X, \|\cdot\|)$ is a Banach space and L_N is a linear operator. Let the positive linear operators $\widetilde{L}_N : C([-\pi,\pi]) \hookrightarrow C([-\pi,\pi])$, such that*

$$\|(L_N(f))(x_0)\| \leq \left(\widetilde{L}_N(\|f\|)\right)(x_0), \tag{48}$$

$\forall\, N \in \mathbb{N}$, where $f \in C([-\pi,\pi], X)$, and $x_0 \in [-\pi, \pi]$.
Furthermore assume that

$$L_N(cg) = c\widetilde{L}_N(g), \quad \forall\, g \in C([-\pi,\pi]), \forall\, c \in X. \tag{49}$$

Here we consider $f \in H_{x_0}^{(1)}([-\pi,\pi])$; $r_1, r_2 > 0$, $0 < \alpha \notin \mathbb{N}$. Furthermore the unique positive finite measure μ_{x_0} is as in (37).

Then

$$\left\| \left(L_N(f)\right)(x_0) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!}\left(\widetilde{L}_N\left((\cdot - x_0)^k\right)\right)(x_0)\right\| \leq$$

$$\frac{(2\pi)^\alpha}{\Gamma(\alpha+1)}\left\{\left[\left(\mu_{Nx_0}([-\pi,x_0])\right)^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1)r_1}\right]\right.$$

$$\left(\int_{[-\pi,x_0]}\left(\sin\left(\frac{x_0-x}{4}\right)\right)^{\alpha+1} d\mu_{Nx_0}(x)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\omega_1\left(D^\alpha_{x_0-}f, r_1\left(\int_{[-\pi,x_0]}\left(\sin\left(\frac{x_0-x}{4}\right)\right)^{\alpha+1} d\mu_{Nx_0}(x)\right)^{\frac{1}{(\alpha+1)}}\right)_{[-\pi,x_0]} +$$

$$\left[\left(\mu_{Nx_0}((x_0,\pi])\right)^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1)r_2}\right] \tag{50}$$

$$\left(\int_{(x_0,\pi]}\left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1} d\mu_{Nx_0}(x)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\left.\omega_1\left(D^\alpha_{*x_0}f, r_2\left(\int_{(x_0,\pi]}\left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1} d\mu_{Nx_0}(x)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,\pi]}\right\}.$$

*Proof.*For a fixed $x_0 \in [-\pi,\pi]$ we have

$$\Delta(x_0) := \left\| \left(L_N(f)\right)(x_0) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!}\left(\widetilde{L}_N\left((\cdot - x_0)^k\right)\right)(x_0)\right\| =$$

$$\left\|\left(L_N\left(f(\cdot) - \sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}(\cdot - x_0)^k\right)\right)(x_0)\right\| \leq \tag{51}$$

$$\left(\widetilde{L}_N\left(\left\|f(\cdot) - \sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}(\cdot - x_0)^k\right\|\right)\right)(x_0) \overset{(37)}{=}$$

$$\int_{[-\pi,\pi]}\left\|f(x) - \sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\right\| d\mu_{Nx_0}(x) =$$

$$\int_{[-\pi,x_0]}\left\|f(x) - \sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\right\| d\mu_{Nx_0}(x) + \tag{52}$$

$$\int_{(x_0,\pi]}\left\|f(x) - \sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\right\| d\mu_{Nx_0}(x) \overset{\text{by } ((18),(15))}{=}$$

$$\frac{1}{\Gamma(\alpha)}\left[\int_{[-\pi,x_0]}\left\|\int_x^{x_0}(z-x)^{\alpha-1}\left(D^\alpha_{x_0-}f\right)(z)\,dz\right\| d\mu_{Nx_0(x)} + \right.$$

$$\left.\int_{(x_0,\pi]}\left\|\int_{x_0}^{x}(x-z)^{\alpha-1}\left(D^\alpha_{*x_0}f\right)(z)\,dz\right\| d\mu_{Nx_0(x)}\right] \leq \tag{53}$$

(above the integrands are continuous functions in x, also $D^\alpha_{x_0-}f, D^\alpha_{*x_0}f \in L_1([-\pi,\pi],X)$)

$$\frac{1}{\Gamma(\alpha)}\left[\int_{[-\pi,x_0]}\left(\int_x^{x_0}(z-x)^{\alpha-1}\left\|\left(D^\alpha_{x_0-}f\right)(z) - \left(D^\alpha_{x_0-}f\right)(x_0)\right\| dz\right) d\mu_{Nx_0(x)} + \right. \tag{54}$$

$$\int_{(x_0,\pi]}\left(\int_{x_0}^{x}(x-z)^{\alpha-1}\left\|\left(D_{*x_0}^{\alpha}f\right)(z)-\left(D_{*x_0}^{\alpha}f\right)(x_0)\right\|dz\right)d\mu_{Nx_0(x)}\Bigg]\leq$$

(let $h_1,h_2>0$, by (33))

$$\frac{1}{\Gamma(\alpha)}\left\{\left[\int_{[-\pi,x_0]}\left(\int_{x}^{x_0}(z-x)^{\alpha-1}\left(1+\frac{x_0-z}{h_1}\right)dz\right)d\mu_{Nx_0}(x)\right]\omega_1\left(D_{x_0-}^{\alpha}f,h_1\right)_{[-\pi,x_0]}\right.$$
$$\left.+\left[\int_{(x_0,\pi]}\left(\int_{x_0}^{x}(x-z)^{\alpha-1}\left(1+\frac{z-x_0}{h_2}\right)dz\right)d\mu_{Nx_0}(x)\right]\omega_1\left(D_{*x_0}^{\alpha}f,h_2\right)_{[x_0,\pi]}\right\}. \tag{55}$$

I.e. it holds

$$\Delta(x_0)\leq\frac{1}{\Gamma(\alpha)}\cdot$$
$$\left\{\left[\int_{[-\pi,x_0]}\left(\int_{x}^{x_0}(z-x)^{\alpha-1}\left(1+\frac{x_0-z}{h_1}\right)dz\right)d\mu_{Nx_0}(x)\right]\omega_1\left(D_{x_0-}^{\alpha}f,h_1\right)_{[-\pi,x_0]}\right.$$
$$\left.+\left[\int_{(x_0,\pi]}\left(\int_{x_0}^{x}(x-z)^{\alpha-1}\left(1+\frac{z-x_0}{h_2}\right)dz\right)d\mu_{Nx_0}(x)\right]\omega_1\left(D_{*x_0}^{\alpha}f,h_2\right)_{[x_0,\pi]}\right\}= \tag{56}$$

$$\frac{1}{\Gamma(\alpha)}\left\{\left[\int_{[-\pi,x_0]}\left(\frac{(x_0-x)^{\alpha}}{\alpha}+\frac{1}{h_1}\left(\int_{x}^{x_0}(x_0-z)^{2-1}(z-x)^{\alpha-1}dz\right)\right)d\mu_{Nx_0}(x)\right]\right.$$
$$\omega_1\left(D_{x_0-}^{\alpha}f,h_1\right)_{[-\pi,x_0]}+$$
$$\left[\int_{(x_0,\pi]}\left(\frac{(x-x_0)^{\alpha}}{\alpha}+\frac{1}{h_2}\left(\int_{x_0}^{x}(x-z)^{\alpha-1}(z-x_0)^{2-1}dz\right)\right)d\mu_{Nx_0}(x)\right]$$
$$\left.\omega_1\left(D_{*x_0}^{\alpha}f,h_2\right)_{[x_0,\pi]}\right\}= \tag{57}$$
$$\frac{1}{\Gamma(\alpha)}\left\{\left[\int_{[-\pi,x_0]}\left(\frac{(x_0-x)^{\alpha}}{\alpha}+\frac{1}{h_1}\frac{(x_0-x)^{\alpha+1}}{\alpha(\alpha+1)}\right)d\mu_{Nx_0}(x)\right]\omega_1\left(D_{x_0-}^{\alpha}f,h_1\right)_{[-\pi,x_0]}+\right.$$
$$\left.\left[\int_{(x_0,\pi]}\left(\frac{(x-x_0)^{\alpha}}{\alpha}+\frac{1}{h_2}\frac{(x-x_0)^{\alpha+1}}{\alpha(\alpha+1)}\right)d\mu_{Nx}(x)\right]\omega_1\left(D_{*x_0}^{\alpha}f,h_2\right)_{[x_0,\pi]}\right\}.$$

Therefore it holds

$$\Delta(x_0)\leq\frac{1}{\Gamma(\alpha)}\cdot$$
$$\left\{\left[\frac{1}{\alpha}\int_{[-\pi,x_0]}(x_0-x)^{\alpha}d\mu_{Nx_0}(x)+\frac{1}{h_1\alpha(\alpha+1)}\int_{[-\pi,x_0]}(x_0-x)^{\alpha+1}d\mu_{Nx_0}(x)\right]\right. \tag{58}$$
$$\omega_1\left(D_{x_0-}^{\alpha}f,h_1\right)_{[-\pi,x_0]}+$$
$$\left[\frac{1}{\alpha}\int_{(x_0,\pi]}(x-x_0)^{\alpha}d\mu_{Nx_0}(x)+\frac{1}{h_2\alpha(\alpha+1)}\int_{(x_0,\pi]}(x-x_0)^{\alpha+1}d\mu_{Nx_0}(x)\right]$$
$$\left.\omega_1\left(D_{*x_0}^{\alpha}f,h_2\right)_{[x_0,\pi]}\right\}.$$

Momentarily we assume positive choices of

$$h_1=r_1\left(\int_{[-\pi,x_0]}\left(\sin\left(\frac{x_0-x}{4}\right)\right)^{\alpha+1}d\mu_{Nx_0}(x)\right)^{\frac{1}{(\alpha+1)}}>0, \tag{59}$$

$$h_2=r_2\left(\int_{(x_0,\pi]}\left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1}d\mu_{Nx_0}(x)\right)^{\frac{1}{(\alpha+1)}}>0. \tag{60}$$

Consequently, by (34), (35) and (58), we obtain

$$\Delta\left(x_0\right) \leq \frac{(2\pi)^{\alpha}}{\Gamma\left(\alpha+1\right)}\cdot$$

$$\left\{\left[\mu_{Nx_0}\left(\left[-\pi,x_0\right]\right)^{\frac{1}{(\alpha+1)}}+\frac{2\pi}{\left(\alpha+1\right)r_1}\right]\left(\frac{h_1}{r_1}\right)^{\alpha}\omega_1\left(D_{x_0-}^{\alpha}f,h_1\right)_{\left[-\pi,x_0\right]}+\right.$$

$$\left.\left[\left(\mu_{Nx_0}\left(\left(x_0,\pi\right]\right)\right)^{\frac{1}{(\alpha+1)}}+\frac{2\pi}{\left(\alpha+1\right)r_2}\right]\left(\frac{h_2}{r_2}\right)^{\alpha}\omega_1\left(D_{*x_0}^{\alpha}f,h_2\right)_{\left[x_0,\pi\right]}\right\}, \tag{61}$$

proving (50).

Next we examine the special cases. If

$$\int_{(x_0,\pi]}\left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1}d\mu_{Nx_0}\left(x\right)=0, \tag{62}$$

then $\sin\left(\frac{x-x_0}{4}\right)=0$, a.e. on $\left(x_0,\pi\right]$, that is $x=x_0$ a.e. on $\left(x_0,\pi\right]$, more precisely $\mu_{Nx_0}\left\{x\in\left(x_0,\pi\right]:x\neq x_0\right\}=0$, hence $\mu_{Nx_0}\left(x_0,\pi\right]=0$. Therefore μ_{Nx_0} concentrates on $\left[-\pi,x_0\right]$. In that case (50) is written and holds as

$$\Delta\left(x_0\right) \leq \frac{(2\pi)^{\alpha}}{\Gamma\left(\alpha+1\right)}\cdot\left\{\left[\mu_{Nx_0}\left(\left[-\pi,x_0\right]\right)^{\frac{1}{(\alpha+1)}}+\frac{2\pi}{\left(\alpha+1\right)r_1}\right]\right.$$

$$\left(\int_{[-\pi,x_0]}\left(\sin\left(\frac{x_0-x}{4}\right)\right)^{\alpha+1}d\mu_{Nx_0}\left(x\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\omega_1\left(D_{x_0-}^{\alpha}f,r_1\left(\int_{[-\pi,x_0]}\left(\sin\left(\frac{x_0-x}{4}\right)\right)^{\alpha+1}d\mu_{Nx_0}\left(x\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{\left[-\pi,x_0\right]} \tag{63}$$

Since $\left(\pi,\pi\right]=\emptyset$ and $\mu\left(\emptyset\right)=0$, in the case of $x_0=\mu$, we get again (63) written for $x_0=\pi$. So inequality (63) is a valid inequality when

$$\int_{[-\pi,x_0]}\left(\sin\left(\frac{x_0-x}{4}\right)\right)^{\alpha+1}d\mu_{Nx_0}\left(x\right)\neq 0, \tag{64}$$

If additionally we assume that

$$\int_{[-\pi,x_0]}\left(\sin\left(\frac{x_0-x}{4}\right)\right)^{\alpha+1}d\mu_{Nx_0}\left(x\right)=0, \tag{65}$$

then $\sin\left(\frac{x_0-x}{4}\right)=0$, a.e. on $\left[-\pi,x_0\right]$, that is $x=x_0$ a.e. on $\left[-\pi,x_0\right]$, which means $\mu_{Nx_0}\left\{x\in\left[-\pi,x_0\right]:x\neq x_0\right\}=0$. Hence $\mu_{Nx_0}=\delta_{x_0}M$, where δ_{x_0} is the unit Dirac measure and $M=\mu_{Nx_0}\left(\left[-\pi,\pi\right]\right)>0$.

In the last case we obtain L.H.S.(63)=R.H.S.(63)$=0$, that is (63) is valid trivially.

At last we go the other way around. Let us assume that

$$\int_{[-\pi,x_0]}\left(\sin\left(\frac{x_0-x}{4}\right)\right)^{\alpha+1}d\mu_{Nx_0}\left(x\right)=0, \tag{66}$$

then reasoning similarly as before, we get that μ_{Nx_0} over $\left[-\pi,x_0\right]$ concentrates at x_0. That is $\mu_{Nx_0}=\delta_{x_0}\mu_{Nx_0}\left(\left[-\pi,x_0\right]\right)$, on $\left[-\pi,x_0\right]$.

In the last case (50) is written and holds as

$$\Delta\left(x_0\right) \leq \frac{(2\pi)^{\alpha}}{\Gamma\left(\alpha+1\right)}\cdot\left\{\left[\mu_{Nx_0}\left(\left(x_0,\pi\right]\right)^{\frac{1}{(\alpha+1)}}+\frac{2\pi}{\left(\alpha+1\right)r_2}\right]\right.$$

$$\left(\int_{(x_0,\pi]}\left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1}d\mu_{Nx_0}\left(x\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\omega_1\left(D^{\alpha}_{*x_0}f, r_2\left(\int_{(x_0,\pi]}\left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1} d\mu_{Nx_0}(x)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,\pi]}\Bigg\}. \tag{67}$$

If $x_0 = -\pi$, then (67) can be redone and rewritten, just replace $(x_0,\pi]$ by $[-\pi,\pi]$ all over. So inequality (67) is valid when

$$\int_{(x_0,\pi]}\left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1} d\mu_{Nx_0}(x) \neq 0. \tag{68}$$

If additionally we assume that

$$\int_{(x_0,\pi]}\left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1} d\mu_{Nx_0}(x) = 0, \tag{69}$$

then as before $\mu_{Nx_0}(x_0,\pi] = 0$. Hence (67) is trivially true, in fact L.H.S.(67)= R.H.S.(67)= 0. The proof of (50) now is completed in all possible cases.

We continue in a special case.
In the Theorem 5, when $r = r_1 = r_2 > 0$, and by calling $M = \mu_{Nx_0}([-\pi,\pi]) \geq \mu_{Nx_0}([-\pi,x_0]), \mu_{x_0}((x_0,\pi])$, we get

Corollary 2. *It holds*

$$\left\|\left(L_N(f)\right)(x_0) - \sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}\left(\widetilde{L}_N\left((\cdot - x_0)^k\right)\right)(x_0)\right\| \leq \tag{70}$$

$$\frac{(2\pi)^{\alpha}}{\Gamma(\alpha+1)}\cdot\left[M^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1)r}\right]$$

$$\Bigg[\left(\int_{[-\pi,x_0]}\left(\sin\left(\frac{x_0-x}{4}\right)\right)^{\alpha+1} d\mu_{Nx_0}(x)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\omega_1\left(D^{\alpha}_{x_0-}f, r\left(\int_{[-\pi,x_0]}\left(\sin\left(\frac{x_0-x}{4}\right)\right)^{\alpha+1} d\mu_{Nx_0}(x)\right)^{\frac{1}{(\alpha+1)}}\right)_{[-\pi,x_0]} +$$

$$\left(\int_{[x_0,\pi]}\left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1} d\mu_{Nx_0}(x)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\omega_1\left(D^{\alpha}_{*x_0}f, r\left(\int_{[x_0,\pi]}\left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1} d\mu_{Nx_0}(x)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,\pi]}\Bigg].$$

Based on Theorem 5, Corollary 2, and (37), we obtain

Theorem 6. *All as in Theorem 5, $r = r_1 = r_2 > 0$. Then*

$$\left\|\left(L_N(f)\right)(x_0) - \sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}\left(\widetilde{L}_N\left((\cdot - x_0)^k\right)\right)(x_0)\right\| \leq$$

$$\frac{(2\pi)^{\alpha}}{\Gamma(\alpha+1)}\cdot\left[\left(\widetilde{L}_N(1,x_0)\right)^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1)r}\right]$$

$$\Bigg[\left(\widetilde{L}_N\left(\left(\sin\left(\frac{|x-x_0|\,\chi_{[-\pi,x_0]}(x)}{4}\right)\right)^{\alpha+1}, x_0\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\omega_1\left(D^{\alpha}_{x_0-}f,r\left(\widetilde{L}_N\left(\left(\sin\left(\frac{|x-x_0|\,\chi_{[-\pi,x_0]}(x)}{4}\right)\right)^{\alpha+1},x_0\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[-\pi,x_0]}+$$

$$\left(\widetilde{L}_N\left(\left(\sin\left(\frac{|x-x_0|\,\chi_{[x_0,\pi]}(x)}{4}\right)\right)^{\alpha+1},x_0\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)} \quad (71)$$

$$\left.\omega_1\left(D^{\alpha}_{*x_0}f,r\left(\widetilde{L}_N\left(\left(\sin\left(\frac{|x-x_0|\,\chi_{[x_0,\pi]}(x)}{4}\right)\right)^{\alpha+1},x_0\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,\pi]}\right].$$

Corollary 3.*(to Theorem 6). It holds*

$$\left\|L_N(f,x_0)-\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}\widetilde{L}_N\left((x-x_0)^k,x_0\right)\right\|\leq$$

$$\frac{(2\pi)^{\alpha}}{\Gamma(\alpha+1)}\left[\left(\widetilde{L}_N(1,x_0)\right)^{\frac{1}{(\alpha+1)}}+\frac{2\pi}{(\alpha+1)r}\right]$$

$$\left[\omega_1\left(D^{\alpha}_{x_0-}f,r\left(\widetilde{L}_N\left(\left(\sin\left(\frac{|x-x_0|}{4}\right)\right)^{\alpha+1},x_0\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[-\pi,x_0]}+\right.$$

$$\left.\omega_1\left(D^{\alpha}_{*x_0}f,r\left(\widetilde{L}_N\left(\left(\sin\left(\frac{|x-x_0|}{4}\right)\right)^{\alpha+1},x_0\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,\pi]}\right] \quad (72)$$

$$\left(\widetilde{L}_N\left(\left(\sin\left(\frac{|x-x_0|}{4}\right)\right)^{\alpha+1},x_0\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}.$$

We need

Definition 5.*([3]) We call ($x_0\in[a,b]\subset\mathbb{R}$)*

$$\widetilde{H}^{(1)}_{x_0}([a,b]):=\{f\in C([a,b],X):[a,b]\subset\mathbb{R},(X,\|\cdot\|) \quad (73)$$

is a Banach space, $0<\alpha<1$; $f'\in L_{\infty}([a,b],X)$; f' exists outside a λ-null Borel set $B^{(1)}_x\subseteq[x,x_0]$, such that $h_1\left(f\left(B^{(1)}_x\right)\right)=0$, $\forall\ x\in[a,x_0]$; f' exists outside a λ-null Borel set $B^{(2)}_x\subseteq[x_0,x]$, such that $h_1\left(f\left(B^{(2)}_x\right)\right)=0$, $\forall\ x\in[x_0,b]\}$.

Notice that $C^1([a,b],X)\subset\widetilde{H}^{(1)}_{x_0}([a,b])$, $\forall\ x_0\in[a,b]$.

The last Definition 5 simplifies a lot Definition 3 when $m=1$.

Because h_1 is an outer measure on the power set $\mathscr{P}(X)$ we can further simplify Definition 5, based on $f(\emptyset)=\emptyset$, $h_1(\emptyset)=0$, and $A\subset B$ implies $h_1(A)\leq h_1(B)$, as follows:

We make

Remark.([3]) Let $x_0\in[a,b]\subset\mathbb{R}$. We have that

$$\widetilde{H}^{(1)}_{x_0}([a,b]):=\{f\in C([a,b],X):(X,\|\cdot\|) \quad (74)$$

is a Banach space, $0<\alpha<1$; $f'\in L_{\infty}([a,b],X)$; f' exists outside a λ-null Borel set $B_a\subseteq[a,x_0]$, such that $h_1(f(B_a))=0$; f' exists outside a λ-null Borel set $B_b\subseteq[x_0,b]$, such that $h_1(f(B_b))=0\}$.

*Remark.*Notice that

$$\|L_N(f,x_0)-f(x_0)\|\leq\left\|L_N(f,x_0)-f(x_0)\widetilde{L}_N(1,x_0)\right\|+$$

$$\|f(x_0)\|\left|\widetilde{L}_N(1,x_0)-1\right|. \quad (75)$$

By Theorem 6 we obtain

Theorem 7. *Let $N \in \mathbb{N}$ and $L_N : C([-\pi,\pi],X) \to C([-\pi,\pi],X)$, where $(X,\|\cdot\|)$ is a Banach space and L_N is a linear operator. Let the positive linear operators $\widetilde{L}_N : C([-\pi,\pi]) \hookrightarrow C([-\pi,\pi])$, such that*

$$\|(L_N(f))(x_0)\| \leq \left(\widetilde{L}_N(\|f\|)\right)(x_0), \tag{76}$$

$\forall\, N \in \mathbb{N}$, $\forall\, f \in C([-\pi,\pi],X)$, and $x_0 \in [-\pi,\pi]$.

Furthermore assume that

$$L_N(cg) = c\widetilde{L}_N(g), \quad \forall\, g \in C([-\pi,\pi]), \forall\, c \in X. \tag{77}$$

Here we consider $f \in \widetilde{H}^{(1)}_{x_0}([-\pi,\pi])$, $0<\alpha<1$, $r>0$. Then

$$\|(L_N(f))(x_0) - f(x_0)\| \leq \|f(x_0)\| \left|\widetilde{L}_N(1,x_0) - 1\right| +$$
$$\frac{(2\pi)^\alpha}{\Gamma(\alpha+1)} \cdot \left[\left(\widetilde{L}_N(1,x_0)\right)^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1)r}\right]$$
$$\left[\left(\widetilde{L}_N\left(\left(\sin\left(\frac{|x-x_0|\chi_{[-\pi,x_0]}(x)}{4}\right)\right)^{\alpha+1}, x_0\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)} \tag{78}$$
$$\omega_1\left(D^\alpha_{x_0-}f, r\left(\widetilde{L}_N\left(\left(\sin\left(\frac{|x-x_0|\chi_{[-\pi,x_0]}(x)}{4}\right)\right)^{\alpha+1}, x_0\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[-\pi,x_0]} +$$
$$\left(\widetilde{L}_N\left(\left(\sin\left(\frac{|x-x_0|\chi_{[x_0,\pi]}(x)}{4}\right)\right)^{\alpha+1}, x_0\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$
$$\left.\omega_1\left(D^\alpha_{*x_0}f, r\left(\widetilde{L}_N\left(\left(\sin\left(\frac{|x-x_0|\chi_{[x_0,\pi]}(x)}{4}\right)\right)^{\alpha+1}, x_0\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,\pi]}\right].$$

We make

Remark. Let $f \in H^{(2)}([-\pi,\pi])$. We observe that

$$\text{R.H.S.(71)} \leq \frac{(2\pi)^\alpha}{\Gamma(\alpha+1)} \cdot \left[\left\|\widetilde{L}_N(1)\right\|_\infty^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1)r}\right]$$
$$\left[\left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot-x|\chi_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\left(\frac{\alpha}{\alpha+1}\right)}$$
$$\sup_{x\in[-\pi,\pi]} \omega_1\left(D^\alpha_{x-}f, r\left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot-x|\chi_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\frac{1}{(\alpha+1)}}\right)_{[-\pi,x]} +$$
$$\left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot-x|\chi_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\left(\frac{\alpha}{\alpha+1}\right)} \tag{79}$$
$$\left.\sup_{x\in[-\pi,\pi]} \omega_1\left(D^\alpha_{*x}f, r\left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot-x|\chi_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\frac{1}{(\alpha+1)}}\right)_{[x,\pi]}\right] =: \theta.$$

So that

$$Z := \left\|\left\|\left\| L_N(f,x_0) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} \widetilde{L}_N\left((\cdot - x)^k, x\right)\right\|\right\|\right\|_{\infty,[-\pi,\pi]} \leq \theta. \tag{80}$$

We further observe that

$$\|L_N(f,x) - f(x)\| \leq Z + \|f(x)\| \left|\widetilde{L}_N(1,x) - 1\right| +$$

$$\sum_{k=1}^{m-1} \frac{\left\|f^{(k)}(x)\right\|}{k!} \left|\widetilde{L}_N\left((\cdot - x)^k, x\right)\right| \leq$$

$$\|f(x)\| \left|\widetilde{L}_N(1,x) - 1\right| + \sum_{k=1}^{m-1} \frac{\left\|f^{(k)}(x)\right\|}{k!} \left|\widetilde{L}_N\left((\cdot - x)^k, x\right)\right| + \theta. \tag{81}$$

We have proved the main result, a Shisha-Mond type ([11]) trigonometric inequality at the fractional level.

Theorem 8. *Let $N \in \mathbb{N}$ and $L_N : C([-\pi,\pi],X) \to C([-\pi,\pi],X)$, where $(X, \|\cdot\|)$ is a Banach space and L_N is a linear operator. Let the positive linear operators $\widetilde{L}_N : C([-\pi,\pi]) \hookrightarrow C([-\pi,\pi])$, such that*

$$\|(L_N(f))(x)\| \leq \left(\widetilde{L}_N(\|f\|)\right)(x), \tag{82}$$

$\forall\, N \in \mathbb{N}$, $\forall\, f \in C([-\pi,\pi],X)$, $\forall\, x \in [-\pi,\pi]$.

Furthermore assume that

$$L_N(cg) = c\widetilde{L}_N(g), \quad \forall\, g \in C([-\pi,\pi]), \forall\, c \in X. \tag{83}$$

Here $f \in H^{(2)}([-\pi,\pi])$, $r > 0$, $0 < \alpha \notin \mathbb{N}$. Then

$$\begin{aligned}
\|\|L_N f - f\|\|_\infty \leq{}& \|\|f\|\|_\infty \left\|\widetilde{L}_N 1 - 1\right\|_\infty + \sum_{k=1}^{m-1} \frac{\left\|\left\|f^{(k)}\right\|\right\|_\infty}{k!} \left\|\widetilde{L}_N\left((\cdot - x)^k, x\right)\right\|_\infty \\
&+ \frac{(2\pi)^\alpha}{\Gamma(\alpha+1)} \cdot \left[\left\|\widetilde{L}_N(1)\right\|_\infty^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1)r}\right] \\
&\left[\left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot - x|\chi_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\left(\frac{\alpha}{\alpha+1}\right)} \right. \\
&\sup_{x\in[-\pi,\pi]} \omega_1\left(D_{x-}^\alpha f, r\left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot - x|\chi_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\frac{1}{(\alpha+1)}}\right)_{[-\pi,x]} + \\
&\left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot - x|\chi_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\left(\frac{\alpha}{\alpha+1}\right)} \\
&\left.\sup_{x\in[-\pi,\pi]} \omega_1\left(D_{*x}^\alpha f, r\left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot - x|\chi_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\frac{1}{(\alpha+1)}}\right)_{[x,\pi]}\right].
\end{aligned} \tag{84}$$

Above it is $\|\cdot\|_\infty = \|\cdot\|_{\infty,[-\pi,\pi]}$.

Next we derive the following trigonometric Korovkin type [12]) convergence result at fractional level.

Theorem 9. *Let all here as in Theorem 8. Assume $\widetilde{L_N} 1 \overset{u}{\to} 1$, uniformly, and $\left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot - x|}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty \to 0$, then $L_N f \overset{u}{\to} f$, uniformly, $\forall\, f \in H^{(2)}([-\pi,\pi])$, $0 < \alpha \notin \mathbb{N}$.*

Proof. Since $\left\|\widetilde{L}_N 1 - 1\right\|_\infty \to 0$ we get $\left\|\widetilde{L}_N 1 - 1\right\|_\infty \le K$, for some $K > 0$. We write $\widetilde{L}_N 1 = \widetilde{L}_N 1 - 1 + 1$, hence

$$\left\|\widetilde{L}_N 1\right\|_\infty \le \left\|\widetilde{L}_N 1 - 1\right\|_\infty + \|1\|_\infty \le K + 1, \ \forall N \in \mathbb{N}.$$

That is $\left\|\widetilde{L}_N 1\right\|_\infty$ is bounded. So we are using inequality (84). By assumption also $\left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot - x|}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty \to 0$, as $N \to \infty$ and (39) we get $\left\|\widetilde{L}_N\left(|\cdot - x|^k, x\right)\right\|_\infty \to 0$ for $k = 1, \ldots, m-1$. Also by (41) and (43) we obtain that

$$\left\|L_N\left(\left(\sin\left(\frac{|\cdot - x|\,\chi_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty,$$

and

$$\left\|L_N\left(\left(\sin\left(\frac{|\cdot - x|\,\chi_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty \to 0, \tag{85}$$

as $N \to \infty$.

Additionally by (31) and (32) we get that

$$\sup_{x\in[-\pi,\pi]} \omega_1\left(D^\alpha_{x-} f, \cdot\right)_{[-\pi,x]}, \sup_{x\in[-\pi,\pi]} \omega_1\left(D^\alpha_{*x} f, \cdot\right)_{[x,\pi]} \le \frac{2\left\|f^{(m)}\right\|_\infty}{\Gamma(m-\alpha+1)} (2\pi)^{m-\alpha}, \tag{86}$$

so they are bounded.

Thus based on the above, from (84), we derive that $\||L_N f - f|\|_\infty \to 0$, proving the claim.

We give

Corollary 4. *(to Theorem 8) Case of* $0 < \alpha < 1$, $r > 0$. *Here* $f \in C^1([-\pi,\pi], X)$. *Then*

$$\begin{aligned}
&\||L_N f - f|\|_\infty \le \||f|\|_\infty \left\|\widetilde{L}_N 1 - 1\right\|_\infty + \\
&\frac{(2\pi)^\alpha}{\Gamma(\alpha+1)}\left[\left\|\widetilde{L}_N(1)\right\|_\infty^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1)r}\right] \\
&\left[\left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot - x|\,\chi_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\left(\frac{\alpha}{\alpha+1}\right)}\right. \\
&\left\{\sup_{x\in[-\pi,\pi]} \omega_1\left(D^\alpha_{x-} f, r\left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot - x|\,\chi_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\frac{1}{(\alpha+1)}}\right)_{[-\pi,x]}\right\} \\
&+\left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot - x|\,\chi_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\left(\frac{\alpha}{\alpha+1}\right)} \\
&\left.\left\{\sup_{x\in[-\pi,\pi]} \omega_1\left(D^\alpha_{*x} f, r\left\|\widetilde{L}_N\left(\left(\sin\left(\frac{|\cdot - x|\,\chi_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\frac{1}{(\alpha+1)}}\right)_{[x,\pi]}\right\}\right].
\end{aligned} \tag{87}$$

Above it is $\|\cdot\|_\infty = \|\cdot\|_{\infty,[-\pi,\pi]}$.

4 Application

Consider the Bernstein operators from $C([-\pi,\pi],X)$ into itself, where $(X,\|\cdot\|)$ is a Banach space:

$$(B_N(f))(x)=\sum_{k=0}^{N}\binom{N}{k}f\left(-\pi+\frac{2\pi k}{N}\right)\left(\frac{x+\pi}{2\pi}\right)^{k}\left(\frac{\pi-x}{2\pi}\right)^{N-k}, \tag{88}$$

$\forall\, N\in\mathbb{N}$, $\forall\, x\in[-\pi,\pi]$, $\forall\, f\in C([-\pi,\pi],X)$.

Consider also the Bernstein polynomials from $C([-\pi,\pi])$ into itself:

$$\left(\widetilde{B}_N(h)\right)(x)=\sum_{k=0}^{N}\binom{N}{k}h\left(-\pi+\frac{2\pi k}{N}\right)\left(\frac{x+\pi}{2\pi}\right)^{k}\left(\frac{\pi-x}{2\pi}\right)^{N-k}, \tag{89}$$

$\forall\, N\in\mathbb{N}$, $\forall\, x\in[-\pi,\pi]$, $\forall\, h\in C([-\pi,\pi])$.

Let here $0<\alpha<1$, $r>0$, and $f\in C^1([-\pi,\pi],X)$.

Setting $g(t)=h(2\pi t-\pi)$, $t\in[0,1]$, we have $g(0)=h(-\pi)$, $g(1)=h(\pi)$, and

$$\left(\widetilde{B}_N g\right)(t)=\sum_{k=0}^{N}\binom{N}{k}g\left(\frac{k}{N}\right)t^k(1-t)^{N-k}=\left(\widetilde{B}_N h\right)(x),\ \ x\in[-\pi,\pi]. \tag{90}$$

Here $x=\varphi(t)=2\pi t-\pi$ is an $1-1$ and onto map from $[0,1]$ onto $[-\pi,\pi]$.

Clearly $g\in C^1([0,1])$ when $h\in C^1([0,1])$.

Notice also that

$$\left(\widetilde{B}_N\left((\cdot-x)^2\right)\right)(x)=\left[\left(\widetilde{B}_N\left((\cdot-t)^2\right)\right)(t)\right](2\pi)^2=\frac{(2\pi)^2}{N}t(1-t)$$

$$=\frac{(2\pi)^2}{N}\left(\frac{x+\pi}{2\pi}\right)\left(\frac{\pi-x}{2\pi}\right)=\frac{1}{N}(x+\pi)(\pi-x)\leq\frac{\pi^2}{N}, \tag{91}$$

$\forall\, x\in[-\pi,\pi]$.

I.e.

$$\left(\widetilde{B}_N\left((\cdot-x)^2\right)\right)(x)\leq\frac{\pi^2}{N},\ \ \forall\, x\in[-\pi,\pi]. \tag{92}$$

In particular $\left(\widetilde{B}_N 1\right)(x)=1$, $\forall\, x\in[-\pi,\pi]$.

Applying Corollary 4 we get

Corollary 5. *It holds*

$$\||\|B_N f-f\||\|_\infty\leq\frac{(2\pi)^\alpha}{\Gamma(\alpha+1)}\left[1+\frac{2\pi}{(\alpha+1)r}\right]$$

$$\left[\left\|\widetilde{B}_N\left(\left(\sin\left(\frac{|\cdot-x|\,\chi_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1},x\right)\right\|_\infty^{\left(\frac{\alpha}{\alpha+1}\right)}\right.$$

$$\left\{\sup_{x\in[-\pi,\pi]}\omega_1\left(D_{x-}^{\alpha}f,r\left\|\widetilde{B}_N\left(\left(\sin\left(\frac{|\cdot-x|\,\chi_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1},x\right)\right\|_\infty^{\frac{1}{(\alpha+1)}}\right)_{[-\pi,x]}\right\}$$

$$+\left\|\widetilde{B}_N\left(\left(\sin\left(\frac{|\cdot-x|\,\chi_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1},x\right)\right\|_\infty^{\left(\frac{\alpha}{\alpha+1}\right)} \tag{93}$$

$$\left.\left\{\sup_{x\in[-\pi,\pi]}\omega_1\left(D_{*x}^{\alpha}f,r\left\|\widetilde{B}_N\left(\left(\sin\left(\frac{|\cdot-x|\,\chi_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1},x\right)\right\|_\infty^{\frac{1}{(\alpha+1)}}\right)_{[x,\pi]}\right\}\right],$$

$\forall\, N\in\mathbb{N}$.

Next let $\alpha = \frac{1}{2}$, and $r = \frac{1}{\alpha+1}$, that is $r = \frac{2}{3}$. Notice $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$.

Corollary 6.*(to Corollary 5) It holds*

$$\left\|\left\|B_N f - f\right\|\right\|_\infty \leq 2\sqrt{2}(2\pi+1)$$

$$\left[\left\|\widetilde{B}_N\left(\left(\sin\left(\frac{|\cdot - x|\chi_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\frac{3}{2}}, x\right)\right\|_\infty^{\frac{1}{3}}\right.$$

$$\left\{\sup_{x\in[-\pi,\pi]} \omega_1\left(D_{x-}^{\frac{1}{2}} f, \frac{2}{3}\left\|\widetilde{B}_N\left(\left(\sin\left(\frac{|\cdot - x|\chi_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\frac{3}{2}}, x\right)\right\|_\infty^{\frac{2}{3}}\right)_{[-\pi,x]}\right\} +$$

$$\left\|\widetilde{B}_N\left(\left(\sin\left(\frac{|\cdot - x|\chi_{[x,\pi]}(\cdot)}{4}\right)\right)^{\frac{3}{2}}, x\right)\right\|_\infty^{\frac{1}{3}} \tag{94}$$

$$\left.\left\{\sup_{x\in[-\pi,\pi]} \omega_1\left(D_{*x}^{\frac{1}{2}} f, \frac{2}{3}\left\|\widetilde{B}_N\left(\left(\sin\left(\frac{|\cdot - x|\chi_{[x,\pi]}(\cdot)}{4}\right)\right)^{\frac{3}{2}}, x\right)\right\|_\infty^{\frac{2}{3}}\right)_{[x,\pi]}\right\}\right],$$

$\forall\, N \in \mathbb{N}$.

By $|\sin x| < |x|$, $\forall\, x \in \mathbb{R} - \{0\}$, in particular $\sin x \leq x$, for $x \geq 0$, we get

$$\left(\sin\left(\frac{|\cdot - x|}{4}\right)\right)^{\frac{3}{2}} \leq \left(\frac{|\cdot - x|}{4}\right)^{\frac{3}{2}} = \frac{1}{8}|\cdot - x|^{\frac{3}{2}}.$$

Hence

$$\left\|\widetilde{B}_N\left(\left(\sin\left(\frac{|\cdot - x|}{4}\right)\right)^{\frac{3}{2}}, x\right)\right\|_\infty \leq \frac{1}{8}\left\|\widetilde{B}_N\left(|\cdot - x|^{\frac{3}{2}}, x\right)\right\|_\infty. \tag{95}$$

We observe that

$$\widetilde{B}_N\left(|\cdot - x|^{\frac{3}{2}}, x\right) = \sum_{k=0}^{N}\left|x + \pi - \frac{2\pi k}{N}\right|^{\frac{3}{2}}\binom{N}{k}\left(\frac{x+\pi}{2\pi}\right)^k\left(\frac{\pi - x}{2\pi}\right)^{N-k}$$

(by discrete Hölder's inequality)

$$\leq \left[\sum_{k=0}^{N}\left(x + \pi - \frac{2\pi k}{N}\right)^2\binom{N}{k}\left(\frac{x+\pi}{2\pi}\right)^k\left(\frac{\pi - x}{2\pi}\right)^{N-k}\right]^{\frac{3}{4}} =$$

$$\left(\widetilde{B}_N\left((\cdot - x)^2, x\right)\right)^{\frac{3}{4}} \leq \frac{\pi^{\frac{3}{2}}}{N^{\frac{3}{4}}}, \ \forall\, x \in [-\pi, \pi]. \tag{96}$$

Consequently it holds

$$\left\|\widetilde{B}_N\left((\cdot - x)^{\frac{3}{2}}, x\right)\right\|_\infty \leq \frac{\pi^{\frac{3}{2}}}{N^{\frac{3}{4}}}, \tag{97}$$

and

$$\left\|\widetilde{B}_N\left(\left(\sin\left(\frac{|\cdot - x|}{4}\right)\right)^{\frac{3}{2}}, x\right)\right\|_\infty \leq \frac{\pi^{\frac{3}{2}}}{8N^{\frac{3}{4}}}, \ \forall\, N \in \mathbb{N}. \tag{98}$$

Therefore we get

$$\left\|\widetilde{B}_N\left(\left(\sin\left(\frac{|\cdot - x|\chi_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\frac{3}{2}}, x\right)\right\|_\infty,$$

$$\left\| \widetilde{B}_N \left(\left(\sin \left(\frac{|\cdot - x| \chi_{[x,\pi]}(\cdot)}{4} \right) \right)^{\frac{3}{2}}, x \right) \right\|_\infty$$

$$\leq \left\| \widetilde{B}_N \left(\left(\sin \left(\frac{|\cdot - x|}{4} \right) \right)^{\frac{3}{2}}, x \right) \right\|_\infty \leq \frac{\pi^{\frac{3}{2}}}{8N^{\frac{3}{4}}}, \ \forall N \in \mathbb{N}. \tag{99}$$

Finally we give

Corollary 7. *It holds*

$$\| \| B_N f - f \| \|_\infty \leq \frac{(2\pi + 1)\sqrt{2\pi}}{\sqrt[4]{N}}$$

$$\left[\sup_{x \in [-\pi,\pi]} \omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{\pi}{6\sqrt{N}} \right)_{[-\pi,x]} + \sup_{x \in [-\pi,\pi]} \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{\pi}{6\sqrt{N}} \right)_{[x,\pi]} \right]. \tag{100}$$

So as $N \to \infty$ we derive that $B_N f \xrightarrow{u} f$, uniformly, with rates.

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