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Some Results on the Composition of Singular Distributions

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Let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. The neutrix composition $F(f(x))$ of F and f is said to exist and equal to the distribution $h(x)$ if the neutrix limit of the sequence $\{F_n(f(x))\}$ is equal to $h(x)$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \ldots$ and $\{\delta_n(x)\}$ is a certain regular sequence converging to the Dirac delta function. In particular, the composition $F(f(x))$ is said to exist and be equal to the distribution h if the sequence $\{F_n(f(x))\}$ converges to h in the normal sense.

In this study it was proved that if $F(x)$ denotes the distribution x^{-1}, then the composition $F(\sinh x)$ exists and given by $F(\sinh x) = \operatorname{coshec} x$. Some further similar results are also deduced.

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1 Introduction

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions φ with compact support and let $\mathcal{D}[a, b]$ be the space of infinitely differentiable functions with support contained in the interval $[a, b]$. A *distribution* is a continuous linear functional defined on $\mathcal{D}$. The set of all distributions defined on $\mathcal{D}$ is denoted by $\mathcal{D}'$ and the set of all distributions defined on $\mathcal{D}[a, b]$ is denoted by $\mathcal{D}'[a, b]$. Two distributions F and G are equal if and only if $\langle F, \varphi\rangle = \langle G, \varphi\rangle$ for all φ in $\mathcal{D}$.

If f is a summable function, it defines a distribution, also denoted by f, by defining $\langle f, \varphi \rangle$, its value at φ as

$$\langle f, \varphi \rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)\, dx.$$

This kind of distributions may be multiplied with real numbers and can be added together, such that they form a real vector space. Thus certain operations on ordinary functions can be extended without difficulty to distributions. In general it is not possible to define other operations such as multiplication, convolution and change of variables for arbitrary distributions except only for particular distributions. For example, in the theory of Schwartz distributions, no meaning were given for $F(f(x))$, where F and f are distributions.

Thus the distributions $\cosh ecx$ and $\coth x$ are defined by $\cosh ecx = [\ln|\tanh(x/2)]'$ and $\coth x = (\ln|\sinh x|)'$, respectively. It follows that if $\varphi(x)$ is an arbitrary function in $\mathcal{D}[-1, 1]$, then

$$\begin{aligned} \langle \operatorname{coshec} x, \varphi(x) \rangle &= \int_{-1}^{1} \operatorname{coshec} x[\varphi(x) - \varphi(0)]\, dx, \\ \langle \coth x, \varphi(x) \rangle &= \int_{-1}^{1} \coth x[\varphi(x) - \varphi(0)]\, dx. \end{aligned}$$

Now let $\rho(x)$ be a function in $\mathcal{D}$ having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that

$$\begin{aligned} \lim_{n\to\infty} \langle \delta_n(x), \varphi(x) \rangle &= \lim_{n\to\infty} \int_{-1/n}^{1/n} \delta_n(x)\varphi(x)\, dx \\ &= \lim_{n\to\infty} \int_{-1}^{1} \rho(t)\varphi(t/n)\, dt = \langle \delta, \varphi \rangle = \varphi(0), \end{aligned}$$

for arbitrary φ in $\mathcal{D}$ and so we see that $\{\delta_n(x)\}$ is a sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. More generally, $\{\delta_n^{(r)}(x)\}$ is a sequence of infinitely differentiable functions converging to $\delta^{(r)}(x)$.

If F is a distribution, it is the r-th derivative, for some r, of a summable function f on a bounded interval (a, b). We can therefore define the convolution $(F * \delta_n)(x) = F_n(x)$ by

$$(F * \delta_n)(x) = \langle F(x - t), \delta_n(t) \rangle = \langle f(x - t), \delta_n^{(r)}(t) \rangle = \int_{-1/n}^{1/n} f(x - t)\delta_n^{(r)}(t)\, dt$$

on the interval (a, b). It follows that $\{F_n(x)\}$ is a sequence of infinitely differentiable functions converging to $F(x)$ on the interval (a, b).

Now let $f(x)$ be an infinitely differentiable function having a single simple root at the point $x = x_0$. Gel'fand and Shilov defined the distribution $\delta(f(x))$ by the equation

$$\delta(f(x)) = \frac{1}{|f'(x_0)|}\delta(x - x_0)$$

and more generally,

$$\delta^{(s)}(f(x)) = \frac{1}{|f'(x_0)|}\left[\frac{1}{f'(x)}\frac{d}{dx}\right]^s \delta(x - x_0)$$

for $s = 1, 2, \ldots$, see [9].

Note that some certain divergent integrals can be interpreted as distributions, see [2] and [13]. Then it is a diffucult task to give a meaning to the expression $F(f(x))$, where F and f are singular distributions.

In order to give a more general definition for the composition of distributions, the following definition was given in [3] and was originally called the composition of distributions. Note that taking the neutrix limit of a function $f(n)$, is equivalent to taking the usual limit of Hadamard's finite part of $f(n)$, see [14].

Definition 1. *Let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. We say that the neutrix composition $F(f(x))$ exists and is equal to h on the open interval (a, b) if*

$$\mathrm{N}-\lim_{n\to\infty}\int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ in $\mathcal{D}[a, b]$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \ldots$ and N is the neutrix, see [1], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n,\ \ln^r n : \qquad \lambda > 0,\ r = 1, 2, \ldots$$

and all functions which converge to zero in the usual sense as n tends to infinity. In particular, we say that the composition $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ in $\mathcal{D}[a, b]$.

The following theorems were proved in [3] and [4] respectively.

Theorem 1. *The neutrix composition* $\delta^{(s)}(\operatorname{sgn} x|x|^\lambda)$ *exists and*

$$\delta^{(s)}(\operatorname{sgn} x|x|^\lambda) = 0$$

for $s = 0, 1, 2, \ldots$ *and* $(s+1)\lambda = 1, 3, \ldots$ *and*

$$\delta^{(s)}(\operatorname{sgn} x|x|^\lambda) = \frac{(-1)^{(s+1)(\lambda+1)} s!}{\lambda[(s+1)\lambda - 1]!}\delta^{((s+1)\lambda-1)}(x)$$

for $s = 0, 1, 2, \ldots$ *and* $(s+1)\lambda = 2, 4, \ldots.$

Theorem 2. *The neutrix composition* $(x_+^\mu)_-^\lambda$ *exists and*

$$(x_+^\mu)_-^\lambda = \frac{(-1)^{\lambda\mu}\pi \cos ec(\pi\lambda)}{2\mu(-\lambda\mu-1)!}\delta^{(-\lambda\mu-1)}(x)$$

for $\mu > 0,\ \lambda \neq -1, -2, \ldots$ *and* $\lambda\mu = -1, -2, \ldots.$

Main Results

We now prove the following theorem.

Theorem 3. *If* $F(x)$ *denotes the distribution* x^{-1}, *then the composition* $F(\sinh x)$ *exists and*

$$F(\sinh x) = \operatorname{coshec} x. \tag{1.1}$$

Proof: Putting

$$F_n(x) = x^{-1} * \delta_n(x) = \int_{-1/n}^{1/n} \ln|x-t|\delta_n'(t)\,dt,$$

we have

$$F_n(\sinh x) = \int_{-1/n}^{1/n} \ln|\sinh x - t|\delta_n'(t)\,dt.$$

We note that

$$\int_{-1}^{1} F_n(\sinh x)\,dx = 0, \tag{1.2}$$

since $F_n(\sinh x)$ is an odd function and

$$\int_{-1}^{1} x\int_{-1/n}^{1/n} \ln|\sinh x - t|\delta_n'(t)\,dt\,dx = 2\int_{0}^{1} x\int_{-1/n}^{1/n} \ln|\sinh x - t|\delta_n'(t)\,dt\,dx, \tag{1.3}$$

since $F_n(\sinh x)$ is odd. We have

$$\begin{aligned}\int_0^1 x\int_{-1/n}^{1/n}\ln|\sinh x-t|\delta_n'(t)\,dt\,dx &= \int_{-1}^{1}\delta_n'(t)\int_0^{\sinh^{-1}n^{-1}} x\ln|\sinh x-t|\,dx\,dt\\ &\quad+\int_{-1}^{1}\delta_n'(t)\int_{\sinh^{-1}n^{-1}}^{1} x\ln|\sinh x-t|\,dx\,dt\\ &=\int_{-1}^{1}\rho'(v)\int_0^1(1+u^2/n^2)^{-1/2}\sinh^{-1}(u/n)\ln|(u-v)/n|\,du\,dv\\ &\quad+\int_{-1}^{1}\rho'(v)\int_1^n(1+u^2/n^2)^{-1/2}\sinh^{-1}(u/n)\ln|(u-v)/n|\,du\,dv\\ &= J_1+J_2,\end{aligned}$$

where we have made the substitutions $u = n\sinh x$ and $v = nt$. Now

$$\sinh^{-1}(u/n) = O(n^{-1})$$

and it follows that for any continuous function $\psi(x)$,

$$\lim_{n\to\infty}\int_{-1}^{1}\delta_n'(t)\int_0^{\sinh^{-1}n^{-1}} x\psi(x)\ln|\sinh x-t|\,dx\,dt = 0. \tag{1.4}$$

Similarly

$$\lim_{n\to\infty}\int_{-1}^{1}\delta_n'(t)\int_{-\sinh^{-1}n^{-1}}^{0} x\psi(x)\ln|\sinh x-t|\,dx\,dt = 0. \tag{1.5}$$

Now if $\sinh^{-1}n^{-1} < \eta < 1$, then

$$\begin{aligned}&\int_{\sinh^{-1}n^{-1}}^{\eta} x\left|\int_{-1/n}^{1/n}\ln(\sinh x-t)\delta_n'(t)\,dt\right|dx\\ =\ &\int_{\sinh^{-1}n^{-1/2}}^{\eta} x^2\left|\int_{-1}^{1} n\ln\Big(1-\frac{v}{n\sinh x}\Big)\rho'(v)\,dv\right|\\ &= O(\eta)+O(n^{-1}).\end{aligned}$$

It follows that

$$\lim_{n\to\infty}\int_{\sinh^{-1}n^{-1}}^{\eta} x\psi(x)\left|\int_{-1/n}^{1/n}\ln(\sinh x-t)\delta_n'(t)\right|dx = O(\eta). \tag{1.6}$$

Similarly

$$\lim_{n\to\infty}\int_{-\eta}^{-\sinh^{-1}n^{-1}} x\psi(x)\left|\int_{-1/n}^{1/n}\ln(\sinh x-t)\delta_n'(t)\right|dx = O(\eta). \tag{1.7}$$

Now let $\varphi(x)$ be an arbitrary function in $\mathcal{D}[-1,1]$. Then by using Taylor's Theorem we have

$$\varphi(x) = \varphi(0) + x\varphi'(\xi x),$$

where $0 < \xi < 1$. Then

$$\begin{aligned}
\langle F_n(\sinh x), \varphi(x)\rangle &= \varphi(0)\int_{-1}^{1} F_n(\sinh x)\,dx + \int_{-1/n}^{1/n} xF_n(\sinh x)\varphi'(\xi x)\,dx \\
&\quad + \int_{1/n}^{\eta} xF_n(\sinh x)\varphi'(\xi x)\,dx + \int_{-\eta}^{-1/n} xF_n(\sinh x)\varphi'(\xi x)\,dx \\
&\quad + \int_{\eta}^{1} xF_n(\sinh x)\varphi'(\xi x)\,dx + \int_{-1}^{-\eta} xF_n(\sinh x)\varphi'(\xi x)\,dx.
\end{aligned}$$

Using equations (1.2) to (1.7) it follows that

$$\begin{aligned}
\lim_{n\to\infty}\langle F_n(\sinh x), \varphi(x)\rangle &= \int_{\eta}^{1} xF(\sinh x)\varphi'(\xi x)\,dx + \int_{-1}^{-\eta} xF(\sinh x)\varphi'(\xi x)\,dx \\
&\qquad + O(\eta) \\
&= \int_{\eta}^{1} x\operatorname{cosech} x\varphi'(\xi x)\,dx + \int_{-1}^{-\eta} x\operatorname{cosech} x\varphi'(\xi x)\,dx \\
&\qquad + O(\eta),
\end{aligned}$$

since $\sinh x$ is an infinitely differentiable function on the intervals $[\eta, 1]$ and $[-1, \eta]$ and so $F(\sinh x) = \operatorname{cosech} x$ on these intervals. Since η can be made arbitrarily small, it follows that

$$\begin{aligned}
\lim_{n\to\infty}\langle F_n(\sinh x), \varphi(x)\rangle &= \int_{-1}^{1} x\operatorname{cosech} x\varphi'(\xi x)\,dx \\
&= \int_{-1}^{1} \operatorname{cosech} x[\varphi(x) - \varphi(0)]\,dx = \langle \operatorname{cosech} x, \varphi(x)\rangle
\end{aligned}$$

and equation (1.1) follows on the interval $[-1, 1]$. Since $F(\sinh x) = \operatorname{cosech}^{-1} x$ on any closed intervals not containing the origin, it follows that equation (1.1) holds on the real line, completing the proof of the theorem.

Theorem 4. *If $F(x)$ denotes the distribution x^{-1}, then the composition $F(\tanh x)$ exists and*

$$F(\tanh x) = \coth x.$$

The proof of this theorem is similar to the proof of Theorem 3 and is left as an exercise for the reader.

Theorem 5. *The neutrix composition* $\delta^{(s)}(\sinh^{-1} x_+)$ *exists and*

$$\delta^{(s)}(\sinh^{-1} x_+) = \sum_{k=0}^{s}\sum_{i=0}^{k}(-1)^{s+i+k}\binom{k}{i}\frac{(k-2i+1)^s+(k-2i-1)^s}{2^k k!}\delta^{(k)}(x), \quad (1.8)$$

for $s = 0, 1, 2, \ldots$.

Proof: To prove equation (1.8), we first of all evaluate

$$\int_{-1}^{1} x^k \delta_n^{(s)}(\sinh^{-1} x_+)\, dx.$$

We have

$$\begin{aligned}\int_{-1}^{1} x^k \delta_n^{(s)}(\sinh^{-1} x_+)\, dx &= n^{s+1}\int_{-1}^{1} x^k \rho^{(s)}(\sinh^{-1} x_+)\, dx \\ &= n^{s+1}\int_{0}^{1} x^k \rho^{(s)}(\sinh^{-1} x_+)\, dx \\ +n^{s+1}\int_{-1}^{0} x^k \rho^{(s)}(\sinh^{-1} x_+)\, dx &= I_1 + I_2. \end{aligned} \quad (1.9)$$

It is obvious that

$$\mathrm{N}-\lim_{n\to\infty} I_2 = \mathrm{N}-\lim_{n\to\infty}\int_{-1}^{0} x^k \delta_n^{(s)}(\sinh^{-1} x_+)\, dx = 0. \quad (1.10)$$

Making the substitution $t = n\sinh^{-1} x$, we have for large enough n

$$\begin{aligned} I_1 &= n^s \int_0^1 \sinh^k(t/n)\cosh(t/n)\rho^{(s)}(t)\, dt \\ &= \frac{n^s}{2^{k+1}}\int_0^1 \exp[(k+1)t/n][1-\exp(-2t/n)]^k[1+\exp(-2t/n)]\rho^{(s)}\, dt \\ &= \frac{n^s}{2^{k+1}}\sum_{i=0}^{k}\binom{k}{i}(-1)^i\int_0^1\{\exp[(k-2i+1)t/n)+\exp[(k-2i-1)t/n)\}\rho^{(s)}(t)\, dt, \end{aligned}$$

where

$$\begin{aligned} n^s\int_0^1\{\exp[(k-2i+1)t/n)+\exp[(k-2i-1)t/n)\}\rho^{(s)}(t)\, dt &= \\ = \sum_{p=0}^{\infty}\int_0^1 \frac{(k-2i+1)^p+(k-2i-1)^p}{p!n^{p-s}}t^p\, \rho^{(s)}(t)\, dt. \end{aligned}$$

It follows that

$$\begin{aligned}\mathrm{N}-\lim_{n\to\infty} n^s\int_0^1\{\exp[(k-2i+1)t/n)+\exp[(k-2i-1)t/n)\}\rho^{(s)}(t)\,dt &=\\ =\int_0^1\frac{(k-2i+1)^s+(k-2i-1)^s}{s!}t^p\,\rho^{(s)}(t)\,dt\\ =\frac{(-1)^s[(k-2i+1)^s+(k-2i-1)^s]}{2}\end{aligned}$$

and so

$$\begin{aligned}\mathrm{N}-\lim_{n\to\infty} n^{s+1}\int_0^1 x^k\rho^{(s)}(n\sinh^{-1}x_+)\,dx &=\\ =\frac{1}{2^{k+2}}\sum_{i=0}^{k}\binom{k}{i}(-1)^{s+i}[(k-2i+1)^s+(k-2i-1)^s],\end{aligned}\tag{1.11}$$

for $k=0,1,2,\ldots$.

When $k=s+1$, we have

$$\begin{aligned}|I_1| &= \int_0^1\left|x^{s+1}\delta_n^{(s)}(\sinh^{-1}x_+)\right|\,dx=n^{s+1}\int_0^1\left|x^{s+1}\rho^{(s)}(n\sinh^{-1}x_+)\right|\,dx\\ &\le \frac{n^s\exp(s+2)}{2^{s+2}}\int_0^1\left|[1-\exp(-2t/n)]^{s+1}\rho^{(s)}(t)\right|\,dt\\ &= \frac{n^s\exp(s+2)}{2^{s+2}}\int_0^1[2t/n+O(n^{-2})]^{s+1}\left|\rho^{(s)}(t)\right|\,dt\\ &\le n^{-1}\exp(s+2)\int_0^1[1+O(n^{-1})]\left|\rho^{(s)}(t)\right|\,dt\\ &= O(n^{-1}).\end{aligned}\tag{1.12}$$

Thus if $\psi(x)$ is an arbitrary continuous function, then

$$\lim_{n\to\infty}\int_0^1 x^{s+1}\delta_n^{(s)}(\sinh^{-1}x_+)\psi(x)\,dx=0.\tag{1.13}$$

We also have

$$\int_{-1}^0\delta_n^{(s)}(\sinh^{-1}x_+)\psi(x)\,dx=n^{s+1}\int_{-1}^0\rho^{(s)}(0)\psi(x)\,dx$$

and it follows that

$$\int_{-1}^0\delta_n^{(s)}(\sinh^{-1}x_+)\psi(x)\,dx=0.\tag{1.14}$$

Now similarly we let $\varphi(x)$ be an arbitrary function in $\mathcal{D}[-1,1]$. Then by Taylor's Theorem we have

$$\varphi(x)=\sum_{k=0}^{s}\frac{\varphi^{(k)}(0)}{k!}x^k+\frac{x^{s+1}}{(s+1)!}\varphi^{(s+1)}(\xi x),$$

where $0 < \xi < 1$. On using equations (1.9) to (1.14), it follows that

$$\begin{aligned}
\mathrm{N} \quad - \quad & \lim_{n\to\infty} \langle \delta_n^{(s)}(\sinh^{-1} x_+), \varphi(x) \rangle = \mathrm{N} - \lim_{n\to\infty} \sum_{k=0}^{s} \frac{\varphi^{(k)}(0)}{k!} \int_0^1 x^k \delta_n^{(s)}(\sinh^{-1} x_+)\, dx \\
+ \quad & \mathrm{N} - \lim_{n\to\infty} \sum_{k=0}^{s} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^0 x^k \delta_n^{(s)}(\sinh^{-1} x)\, dx \\
+ \quad & \lim_{n\to\infty} \frac{1}{(s+1)!} \int_0^1 x^{s+1} \delta_n^{(s)}(\sinh^{-1} x_+) \varphi^{(s+1)}(\xi z)\, dx \\
+ \quad & \lim_{n\to\infty} \frac{1}{(s+1)!} \int_{-1}^0 x^{s+1} \delta_n^{(s)}(\sinh^{-1} x_+) \varphi^{(s+1)}(\xi z)\, dx \\
& = \sum_{k=0}^{s} \sum_{i=0}^{k} \binom{k}{i} (-1)^{s+i+k} \frac{(k-2i+1)^s + (k-2i-1)^s}{2^{k+2} k!} \langle \delta^{(k)}(x), \varphi(x) \rangle.
\end{aligned}$$

This proves equation (1.8) on the interval $(-1, 1)$. Now it is clear that $\delta^{(s)}(\sinh^{-1} x_+) = 0$ for $x > 0$ and so equation (1.8) holds for $x > -1$. Further now suppose that $\varphi(x)$ is an arbitrary function in $\mathcal{D}(a, b)$, where $a < b < 0$. Then

$$\int_a^b \delta_n^{(s)}(\sinh^{-1} x_+) \varphi(x)\, dx = n^{s+1} \int_a^b \rho^{(s)})(0) \varphi(x)\, dx$$

and so

$$\mathrm{N} - \lim_{n\to\infty} \int_a^b \delta_n^{(s)}(\sinh^{-1} x_+) \varphi(x)\, dx = 0.$$

It follows that $\delta^{(s)}(\sinh^{-1} x_+) = 0$ on the interval (a, b). Since a and b are arbitrary, we see that equation (1.8) holds on the real line. This completes the proof of the theorem.

Corollary 5.1 *The neutrix composition $\delta^{(s)}(\sinh^{-1}|x|)$ exists and*

$$\begin{aligned}
\delta^{(s)}(\sinh^{-1}|x|) = \sum_{k=0}^{s} \sum_{i=0}^{k} & (-1)^{s+i+k} [(-1)^{s+i+k} + (-1)^{s+i}] \binom{k}{i} \\
& \times \frac{(k-2i+1)^s + (k-2i-1)^s}{2^{k+2} k!} \delta^{(k)}(x), \qquad (1.15)
\end{aligned}$$

for $s = 0, 1, 2, \ldots$.

Proof: To prove equation (1.15), we note that

$$\begin{aligned}
\int_{-1}^1 x^k \delta_n^{(s)}(\sinh^{-1}|x|\, dx &= n^{s+1} \int_{-1}^1 x^k \rho_n^{(s)}(\sinh^{-1}|x|)\, dx \\
&= n^{s+1}[1 + (-1)^k] \int_0^1 x^k \rho_n^{(s)}(\sinh^{-1}|x|)\, dx
\end{aligned}$$

and equation (21) now follows. For further results on the neutrix products and convolutions of distributions, see [12], [13], [14], [16], and [17].

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