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Numerical Solutions for Solving Special Tenth Order Linear Boundary Value Problems using Legendre Galerkin method

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Abstract: Galerkin method is used for the numerical solutions of linear tenth-order boundary value problems, with two point boundary conditions, using Legendre polynomials as basis functions over the interval $[-1,\ 1]$. To examine the accuracy of the method some examples have been considered, which shows that the method provides much better accuracy as compared with [Viswanadham and Ballem, Int. J. Appl. Sci. and Eng., **13(3)**, 247-260 (2015)] and [Viswanadham and Ballem, Int. J. Appl. Math. Stat. Sci., **3(3)**, 17-30 (2014)].

Keywords: Boundary value problems, Galerkin method, Legendre polynomials

1 Introduction

The finite element method (FEM) has become a very powerful tool for solving the boundary value problems arise in the complex geometry. In FEM, the approximate solution can be written as a linear combination of basis functions that form a basis for the approximation space under consideration. FEM comprises variational methods like Galerkin, Petrov-Galerkin, Collocation and Least Squares *etc*.
In Galerkin method, the residual of approximation is made orthogonal to the basis functions. A weak form of approximate solution for a given differential equation exists and is sole under apt conditions [1,2] regardless of properties of a given differential operator on using Galerkin method. Further, a weak solution also tends to a classical solution of given differential equation, provided sufficient consideration is given to the boundary conditions [3]. That means the basis functions should vanish on the boundary where the Dirichlet type of boundary conditions are prescribed. Generally, Petrov-Galerkin methods [4] have come to be known as stablized formulations, because they prevent the spatial oscillations and sometimes yield nodally exact solutions where the classical method would fail. The difference between Galerkin and Petrov-Galerkin methods is that the test and trial functions in Galerkin method are the same, while in Petrov Galerkin method, they are not. Collocation methods [5,6] have become progressively popular for solving differential equations, since they are beneficial in providing highly accurate results to nonlinear differential equations. Petrov-Galerkin method is also broadly used for solving ordinary and partial differential equations, as in [7,8,9,10].
In applied science almost all researchers meet some special classical orthogonal functions such as Legendre, Hermite and Laguerre [11,12] polynomials. Legendre polynomials have largely been used in physics and engineering, particularly. As, Legendre and associate Legendre polynomials are broadly used in determining of wave functions of electrons in the orbits of an atom and in determining of potential functions in the spherically symmetric geometry *etc*.
Overview the literature of the boundary value problems (BVPs), distinguished that fourth, sixth and eighth order boundary value problems are frequently encountered in plate deflection theory [13], astrophysics [14] and torsional vibration of uniform beams [16] respectively. Moreover, tenth order boundary value problems arise in the mathematical modeling of viscoelastic flows,

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hydrodynamics and many other fields of mathematical, physical and engineering sciences. Sinclair *et al.* [15] observed tenth and twelfth order equations arise in coupled flight-dynamic and low-order aero elastic model for a slender launch vehicle.
Chandrasekhar [17] pointed out that when an infinite horizontal layer of fluid is heated from below with the assumption that a uniform magnetic field is also used across in the same direction as gravity and the fluid is under the action of rotation, instability sets in. When instability sets in as ordinary convection, the ordinary differential equation is tenth order. A class of characteristic value problems of high order (as high as twenty four) are known to arise in hydrodynamic and hydro magnetic stability analysis, also. Agarwal [18] presented the existence and uniqueness theorem of solutions of such boundary value problems, in detail. In general, most of the ordinary differential equations have no exact solution. In this case, finding analytical solutions have become the goal of many researchers/ mathematicians. Some of these methods including finite difference methods [19,20,21], spline solutions [22], non-ploynomial spline techniques [23,24], eleventh degree spline [25], Galerkin method with sextic B-splines [26], Galerkin method with septic B-splines [27], differential transform method [28], modified Adomain decomposition method [29] and variational iteration method [30]. Elahi *et al.* [31] constructed numerical solution for solving special eighth order linear boundary value problems using Legendre Galerkin method, recently.
Aim of the paper is, to give an algorithm for handling the linear tenth order boundary value problems based on Legendre Galerkin method.
Consider the following general linear tenth-order boundary value problem

$$Lu(x) = u^{(10)}(x) + \sum_{i=0}^{9} a_i(x)u^{(i)}(x) = f(x),\ x \in [-1,\ 1], \quad (1)$$

subject to the boundary conditions

$$u^{(j)}(-1) = u^{(j)}(1) = 0, \qquad j = 0,1,2,3,4, \quad (2)$$

where $u(x)$ and $f(x)$ are continuous functions in the space $\ell^2]-1,\ 1[$ and $a_i(x) = x^i$.

Preliminaries of Legendre polynomials are discussed in Section 2. In Section 3, Legendre Galerkin method is elaborated to obtain the discrete system. Convergence and error analysis of the method is presented in Section 4. In Section 5, handling of nonhomogeneous boundary conditions and change of interval are discussed. In Section 6, some numerical examples are considered that prove the accuracy of the method.

2 Preliminaries

Legendre polynomial is an important orthogonal polynomial among the class of classical polynomials, and is also considered as the eigenfunctions of singular Sturm-Liouville [32] problems. Legendre polynomials are solutions to the following Legendre differential equation

$$D\left[(x^2-1)D[u(x)]\right] - \lambda u(x) = 0, \quad -1 \le x \le 1,$$

where $D = \frac{d}{dx}$ and the eigenvalue λ equals $n(n+1)$.
Legendre polynomials may also satisfy the following recurrence relations

$$(2n+1)L_n(x) = L'_{n+1}(x) - L'_{n-1}(x), \quad (3)$$
$$n\,L_n(x) = x\,L'_n(x) - L'_{n-1}(x). \quad (4)$$

Few of the Legendre polynomials are listed below

(i) $L_0(x) = 1,$

(ii) $L_1(x) = x,$

(iii) $L_2(x) = \frac{1}{2}(3x^2-1),$

(iv) $L_3(x) = \frac{1}{2}(5x^3-3x),$

(v) $L_4(x) = \frac{1}{8}(35x^4-30x^2+3),$

The graph of first five Legendre polynomials is shown in Figure 1.

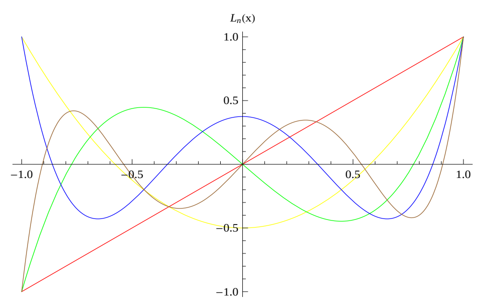

Fig. 1: Legendre polynomials over the interval $[-1,\ 1]$

2.1 Properties of Legendre polynomials

For convenience, some important properties of Legendre polynomials are presented which are useful in Lemmas and Theorems.

(i) Legendre polynomials are orthogonal over the interval $[-1,\ 1]$. *i.e.*,

$$\int_{-1}^{1} L_m(x)L_n(x)dx = \begin{cases} \frac{2}{2n+1}, & \text{if } m = n \\ 0, & \text{if } m \neq n, \end{cases} \quad (5)$$

and

$$\int_{-1}^{1} L_n(x)dx = \begin{cases} 2, \text{ if } n=0 \\ 0, \text{ if } n>0. \end{cases} \tag{6}$$

(ii) Legendre polynomials are even or odd accordingly n is even or odd.

(iii) Legendre polynomials are bounded by 1. *i.e.*, $|L_n(x)| \leq 1$.

(iv) $L_n(\pm 1) = (\pm 1)^n$ for all n.

Lemma 2.1.1 Let n and m be any two integers such that $n-m<N$ and $m>0$, then

$$\int_{-1}^{1} L_n(x)L''_{n-m}(x)dx = 0.$$

Proof Integrating left hand side by parts and using Eq.(6), yields the result.

Lemma 2.1.2 Let n and m be any two integers such that $n \geq m$, then

$$\int_{-1}^{1} L_n(x)L'_m(x)dx = 0.$$

Proof The proof is divided into two parts.

Case I: For $n=m$, we have

$$\int_{-1}^{1} L_n(x)L'_n(x)dx = \left[\frac{1}{2}\{L_n(x)\}^2\right]_{-1}^{1} = 0.$$

Case II: For $n>m$. From Eq.(3) and Eq.(6), it can be written as

$$\begin{aligned} \int_{-1}^{1} L_n(x)L'_m(x)dx &= \int_{-1}^{1} \left[(2m-1)L_{m-1}(x) + L'_{m-2}(x)\right] L_n(x)dx \\ &= \int_{-1}^{1} L_n(x)L'_{m-2}(x)dx \\ &= \int_{-1}^{1} L_n(x)L'_{m-4}(x)dx \\ &\;\;\vdots \\ &= \begin{cases} \int_{-1}^{1} L_n(x)L'_0(x)dx, \; if \; m = \text{even} \\ \int_{-1}^{1} L_n(x)L'_1(x)dx, \; if \; m = \text{odd} \end{cases} \\ &= 0. \end{aligned}$$

Theorem 2.1.1 Let n and m be any two integers such that $n, \; m \leq N$, then

$$(i) \; \int_{-1}^{1} L'_n(x)L_m(x)dx = \begin{cases} 2, \; if \; n=m+i \\ 0, \; if \; n \neq m+i \; or \; n \leq m, \end{cases}$$

$$(ii) \; \int_{-1}^{1} L''_n(x)L_m(x)dx = \begin{cases} n(n+1)-m(m+1), & if \; n \neq m+i \\ 0, & if \; n = m+i \; or \; n \leq m, \end{cases}$$

where $i = 1,3,5,\ldots,2k+1 \leq N-m$.

Proof (i) Integrating left hand side by parts, we obtain

$$\begin{aligned} \int_{-1}^{1} L'_n(x)L_m(x)dx &= [L_n(x)L_m(x)]_{-1}^{1} - \int_{-1}^{1} L_n(x)L'_m(x)dx \\ &= \left[1+(-1)^{n+m+1}\right] - \int_{-1}^{1} L_n(x)L'_m(x)dx. \end{aligned} \tag{7}$$

For $n=m+i, \;\; i=1,3,5,\ldots,2k+1 \leq N-m$, and using Lemma 2.1.2, yields

$$\int_{-1}^{1} L'_n(x)L_m(x)dx = 2.$$

For $n=m+i, \;\; i=0,2,4,\ldots,2k \leq N-m$, Eq.$(7)$ yields

$$\int_{-1}^{1} L'_n(x)L_m(x)dx = 0.$$

For $n \leq m$, and considering the previous cases with Lemma 2.1.2, yield $\int_{-1}^{1} L'_n(x)L_m(x)dx = 0$.

(ii) The proof is divided into four parts.

(a) For $n=m+i, \;\; i=2,4,6,\ldots,2k \leq N-m$, then

$$\begin{aligned} \int_{-1}^{1} L''_n(x)L_m(x)dx &= \left[L'_n(x)L_{n-i}(x)\right]_{-1}^{1} - \int_{-1}^{1} L'_n(x)L'_{n-i}(x)dx \\ &= n(n+1) - \left[L_n(x)L'_{n-i}(x)\right]_{-1}^{1} + \int_{-1}^{1} L_n(x)L''_{n-i}(x)dx \\ &= n(n+1) - \left[L_n(x)L'_{n-i}(x)\right]_{-1}^{1}, \\ &\quad [\text{using Lemma 2.1.1}] \\ &= n(n+1)-m(m+1). \end{aligned}$$

(b) For $n=m+i, \;\; i=1,3,5,\ldots,2k+1 \leq N-m$, then $\int_{-1}^{1} L''_n(x)L_m(x)dx = 0$.

(c) For $n=m$, then

$$\begin{aligned} \int_{-1}^{1} L''_n(x)L_m(x)dx &= \left[L'_n(x)L_{n-i}(x)\right]_{-1}^{1} - \int_{-1}^{1} L'_n(x)L'_{n-i}(x)dx \\ &= n(n+1)-m(m+1) \\ &= 0. \end{aligned}$$

(d) For $n<m$, then integrating $\int_{-1}^{1} L''_n(x)L_m(x) \; dx$ by parts and using Eq.(6) leads to $\int_{-1}^{1} L''_n(x)L_m(x) \; dx = 0$.

3 Description of the Method

In order to solve the linear tenth order boundary value problem (1) by the Galerkin method along with Legendre basis, u(x) is approximated as

$$u(x) = \sum_{j=0}^{n} \alpha_j \, L_j(x). \tag{8}$$

Where the Legendre coefficients $\alpha_j, \ \ j = 0,1,2,...,n$ in Eq.(8) will be determined by orthogonalizing the residual with respect to the basis functions. *i.e.,*

$$\langle u^{(10)}(x), L_r(x)\rangle + \sum_{i=0}^{9} \langle a_i(x)u^{(i)}(x), L_r(x)\rangle - \langle f(x), L_r(x)\rangle = 0, \tag{9}$$

where

$$\langle \phi, \ \psi \rangle = \int_{-1}^{1} \phi(x) \ \psi(x) \ dx. \tag{10}$$

Approximating the integrals in Eq.(9) by integrating by parts such that all derivatives transfer from u to L_r. For convenience, few of the inner products can be calculated, as

$$\langle a_4(x)u^{(4)}(x), L_r(x)\rangle = \int_{-1}^{1} u(x)\,[a_4(x)L_r(x)]^{(4)}\,dx, \tag{11}$$

$$\langle a_3(x)u^{(3)}(x), L_r(x)\rangle = -\int_{-1}^{1} u(x)\,[a_3(x)L_r(x)]^{(3)}\,dx, \tag{12}$$

$$\langle a_2(x)u^{(2)}(x), L_r(x)\rangle = \int_{-1}^{1} u(x)\,[a_2(x)L_r(x)]^{(2)}\,dx, \tag{13}$$

$$\langle a_1(x)u'(x), L_r(x)\rangle = -\int_{-1}^{1} u(x)\,[a_1(x)L_r(x)]'\,dx, \tag{14}$$

$$\langle a_0(x)u(x), L_r(x)\rangle = \int_{-1}^{1} a_0(x)u(x)L_r(x)dx, \tag{15}$$

$$\text{and } \langle \mathrm{f(x)}, \mathrm{L_r(x)}\rangle \simeq \sum_{k=0}^{m} \frac{2\,f(x_k)\ L_r(x_k)}{[(1-x_k^2)\ (L'_m(x_k))^2]}. \tag{16}$$

Lemma 3.1 The following relations hold:

1. $$\langle u^{(10)}(x), L_r(x)\rangle = \int_{-1}^{1} u(x)L_r^{(10)}(x)dx + \sum_{k=5}^{9}(-1)^{k+1}\left[u^{(k)}(x)L_r^{(9-k)}(x)\right]_{-1}^{1}, \tag{17}$$

2. $$\langle a_9(x)u^{(9)}(x), L_r(x)\rangle = -\int_{-1}^{1} u(x)\{a_9(x)L_r(x)\}^{(9)}dx + \sum_{k=5}^{8}(-1)^{k}\left[u^{(k)}(x)\{a_9(x)L_r(x)\}^{(8-k)}\right]_{-1}^{1}, \tag{18}$$

3. $$\langle a_8(x)u^{(8)}(x), L_r(x)\rangle = \int_{-1}^{1} u(x)\{a_8(x)L_r(x)\}^{(8)}dx + \sum_{k=5}^{7}(-1)^{k+1}\left[u^{(k)}(x)\{a_8(x)L_r(x)\}^{(7-k)}\right]_{-1}^{1}, \tag{19}$$

4. $$\langle a_7(x)u^{(7)}(x), L_r(x)\rangle = -\int_{-1}^{1} u(x)\{a_7(x)L_r(x)\}^{(7)}dx + \sum_{k=5}^{6}(-1)^{k}\left[u^{(k)}(x)\{a_7(x)L_r(x)\}^{(6-k)}\right]_{-1}^{1}, \tag{20}$$

5. $$\langle a_6(x)u^{(6)}(x), L_r(x)\rangle = \int_{-1}^{1} u(x)\{a_6(x)L_r(x)\}^{(6)}dx + \left[u^{(5)}(x)a_6(x)L_r(x)\right]_{-1}^{1}, \tag{21}$$

6. $$\langle a_5(x)u^{(5)}(x), L_r(x)\rangle = -\int_{-1}^{1} u(x)\{a_5(x)L_r(x)\}^{(5)}dx. \tag{22}$$

Proof 1. As

$$\langle u^{(10)}(x), L_r(x)\rangle = \int_{-1}^{1} u^{(10)}(x)L_r(x)dx.$$

Integrating right hand side by parts leads to the equality

$$\langle u^{(10)}(x), L_r(x)\rangle = B_{T,10} + \sum_{k=5}^{9}(-1)^{k+1}\left[u^{(k)}(x)L_r^{(9-k)}(x)\right]_{-1}^{1} + \int_{-1}^{1} u(x)L_r^{(10)}(x)dx,$$

where the boundary term

$$B_{T,10} = \sum_{k=0}^{4}(-1)^{k+1}\left[u^{(k)}(x)L_r^{(9-k)}(x)\right]_{-1}^{1}$$

is zero using the boundary conditions defined in Eq.(2), yield the relation.

2. Integration of $\{a_9(x)u^{(9)}(x)\}$ with $L_r(x)$ may be handled in a similar manner leads to the equality

$$\langle a_9(x)u^{(9)}(x), L_r(x)\rangle = B_{T,9} + \sum_{k=5}^{8}(-1)^{k}\left[u^{(k)}(x)\{a_9(x)L_r(x)\}^{(8-k)}\right]_{-1}^{1} - \int_{-1}^{1} u(x)\{a_9(x)L_r(x)\}^{(9)}dx,$$

where the boundary term

$$B_{T,9} = \sum_{k=0}^{4}(-1)^{k}\left[u^{(k)}(x)L_r^{(8-k)}(x)\right]_{-1}^{1} = 0,$$

yield the relation. The other relations can be obtained similarly.

The following theorem is obtained by replacing each term of Eq.(9) with the approximation defined in Eq.(11)–Eq.(22).

Theorem 3.1 If Eq.(8) is the assumed approximate solution of the boundary value problem $(1)-(2)$, then the discrete system for determining the coefficients

$\{\alpha_j\}_{j=0}^{n}$ is given

$$\sum_{j=0}^{n}\left[\sum_{q=0}^{10}(-1)^q\int_{-1}^{1}L_j(x)\{a_q(x)L_r(x)\}^{(q)}dx\right.$$
$$+\sum_{k=5}^{6}(-1)^k\left[L_j^{(k)}(x)\{a_7(x)L_r(x)\}^{(6-k)}\right]_{-1}^{1}$$
$$+\sum_{k=5}^{7}(-1)^{k+1}\left[L_j^{(k)}(x)\{a_8(x)L_r(x)\}^{(7-k)}\right]_{-1}^{1}$$
$$+\sum_{k=5}^{8}(-1)^k\left[L_j^{(k)}(x)\{a_9(x)L_r(x)\}^{(8-k)}\right]_{-1}^{1}.$$
$$+\sum_{k=5}^{9}(-1)^{k+1}\left[L_j^{(k)}(x)L_r^{(9-k)}(x)\right]_{-1}^{1}$$
$$\left.+\left[L_j^{(5)}(x)a_6(x)L_r(x)\right]_{-1}^{1}\right]\alpha_j$$
$$=\sum_{k=0}^{m}\frac{2f(x_k)\,L_r(x_k)}{[(1-x_k^2)\,(L_m'(x_k))^2]},\ 0\le r\le n.$$

It can be written, in matrix form, as

$$AX=B, \tag{23}$$

where

$$A=\begin{pmatrix}\mu_{0,0}+\nu_{0,0} & \mu_{1,0}+\nu_{1,0} & \mu_{2,0}+\nu_{2,0} & \cdots & \mu_{n,0}+\nu_{n,0}\\ \mu_{0,1}+\nu_{0,1} & \mu_{1,1}+\nu_{1,1} & \mu_{2,1}+\nu_{2,1} & \cdots & \mu_{n,1}+\nu_{n,1}\\ \mu_{0,2}+\nu_{0,2} & \mu_{1,2}+\nu_{1,2} & \mu_{2,2}+\nu_{2,2} & \cdots & \mu_{n,2}+\nu_{n,2}\\ \vdots & \vdots & \vdots & & \vdots\\ \mu_{0,n}+\nu_{0,n} & \mu_{1,n}+\nu_{1,n} & \mu_{2,n}+\nu_{2,n} & \cdots & \mu_{n,n}+\nu_{n,n}\end{pmatrix}$$

and

$$\mu_{j,r}=\sum_{q=0}^{10}(-1)^q\int_{-1}^{1}\{a_q(x)L_r(x)\}^{(q)}\,L_j(x)\,dx,\ \ a_{10}(x)=1,$$

$$\nu_{j,r}=[\sum_{k=5}^{6}(-1)^kL_j^{(k)}(x)\{a_7(x)L_r(x)\}^{(6-k)}+\sum_{k=5}^{7}(-1)^{k+1}L_j^{(k)}(x)\{a_8(x)L_r(x)\}^{(7-k)}$$
$$+L_j^{(5)}(x)a_6(x)L_r(x)$$
$$+\sum_{k=5}^{8}(-1)^kL_j^{(k)}(x)\{a_9(x)L_r(x)\}^{(8-k)}$$
$$+\sum_{k=5}^{9}(-1)^{k+1}L_j^{(k)}(x)L_r^{(9-k)}(x)$$
$$+L_j^{(5)}(x)a_6(x)L_r(x)\Big]_{-1}^{1}.]$$

The term $\mu_{j,r}$ can be calculated using the results given in Sect. 2, while the boundary term $\nu_{j,r}$ can be calculated as

$$\left[L_j^{(5)}(x)a_6(x)L_r(x)\right]_{-1}^{1}=\frac{1}{3840}\left\{1+(-1)^{n+r+j}\right\}\prod_{i=0}^{9}(j-i+5),$$

$$\sum_{k=5}^{9}(-1)^{k+1}\left[L_j^{(k)}(x)L_r(x)^{(9-k)}\right]_{-1}^{1}$$
$$=\frac{1}{185794560}\left\{1+(-1)^{r+j}\right\}\prod_{i=0}^{17}(j-i+9)$$
$$-\frac{1}{20643840}\left\{1-(-1)^{r+j-1}\right\}\{r(r+1)\}$$
$$\times\prod_{i=0}^{15}(j-i+8)+\frac{1}{5160960}\left\{1-(-1)^{r+j-1}\right\}$$
$$\times\prod_{i=0}^{3}(r-i+2)\prod_{i=0}^{13}(j-i+7)-\frac{1}{2211840}$$
$$\times\left\{1-(-1)^{r+j-1}\right\}\prod_{i=0}^{5}(r-i+3)\prod_{i=0}^{11}(j-i+6)$$
$$+\frac{1}{1474560}\left\{1-(-1)^{r+j-1}\right\}\prod_{i=0}^{7}(r-i+4)$$
$$\times\prod_{i=0}^{9}(j-i+5),$$

$$\sum_{k=5}^{8}(-1)^k\left[L_j^{(k)}(x)\{a_9(x)L_r(x)\}^{(8-k)}\right]_{-1}^{1}$$
$$=\frac{1}{10321920}\left\{1-(-1)^{n+r+j}\right\}\prod_{i=0}^{15}(j-i+8)$$
$$-\frac{1}{1290240}\{2n+r(r+1)\}\left\{1+(-1)^{n+r+j-1}\right\}$$
$$\times\prod_{i=0}^{13}(j-i+7)+\frac{1}{368640}\{8n(n-1)+8nr(r+1)$$
$$+(r+2)(r+1)(r)(r-1)\}\left\{1-(-1)^{n+r+j}\right\}$$
$$\times\prod_{i=0}^{11}(j-i+6)-\frac{1}{184320}\left\{1-(-1)^{n+r+j-1}\right\}$$
$$\times\{48n(n-1)(n-2)+72nr(n-1)(r+1) \tag{24}$$
$$+18nr(r+2)(r+1)(r-1)+(r+3)(r+2)$$
$$\times(r+1)(r)(r-1)(r-2)\}\prod_{i=0}^{9}(j-i+5),$$

$$\sum_{k=5}^{7}(-1)^{k+1}\left[L_j^{(k)}(x)\{a_8(x)L_r(x)\}^{(7-k)}\right]_{-1}^{1}$$
$$=\frac{1}{645120}\left\{1+(-1)^{n+r+j}\right\}\prod_{i=0}^{13}(j-i+7)$$
$$-\frac{1}{92160}(2n+r(r+1))\left\{1+(-1)^{n+r+j}\right\}$$
$$\times\prod_{i=0}^{11}(j-i+6)+\frac{1}{30720}\left\{1-(-1)^{n+r+j}\right\}$$
$$\times\{8n(n-1)+8nr(r+1)+(r+2)(r+1)(r)$$
$$\times(r-1)\}\prod_{i=0}^{9}(j-i+5),$$

$$\sum_{k=5}^{6}(-1)^k\left[L_j^{(k)}(x)\{a_7(x)L_r(x)\}^{(6-k)}\right]_{-1}^{1}$$
$$=\frac{1}{46080}\left\{1-(-1)^{n+r+j}\right\}\prod_{i=0}^{11}(j-i+6)$$
$$-\frac{1}{7680}(2n+r(r+1))\left\{1+(-1)^{n+r+j-1}\right\}$$
$$\times\prod_{i=0}^{9}(j-i+5).$$

After solving the linear system (24) having $(n+1)$ equations with $(n+1)$ unknowns, yield the column vector $X=(\alpha_0,\ \alpha_1,\ \alpha_2,\dots,\ \alpha_n)^T$. Thus, $u(x)$ can now be approximated by Eq.(8).

4 Convergence and Error Analysis

In this section, convergence and error analysis of the Legendre Galerkin method have been discussed.

4.1 Convergence of the method

Lemma 4.1.1 Let $x(t)\in H^k]-1,\ 1[$ (a Sobolev space) and let $x_n(t)=\sum_{i=0}^{n}c_iL_i(t)$ be the best approximation polynomial of $x(t)$ in the ℓ^2-norm, then

$$||x(t)-x_n(t)||_{\ell^2[-1,\ 1]}\leq\ c_0\ n^{-k}\ ||x(t)||_{H^k]-1,\ 1[}\ ,$$

and c_0 is a non-negative constant which depends on the selected norm and is free from $x(t)$ and n.

Proof [33,34,35].

Theorem 4.1.1 Assume $\kappa: X\to$ X is bounded, with X a Banach space, and $\lambda-\kappa: X\to X$ is bijective. Further assume

$$||\kappa-\kappa L_n||\to 0\ \text{ as }\ \mathrm{n}\to\infty,$$

then for all sufficiently large n, say $n\geq N$, the operator $(\lambda-\kappa L_n)^{-1}$ exists as a bounded operator from X to X. Moreover, it is uniformly bounded such that

$$\sup_{n\geq N}||(\lambda-\kappa L_n)^{-1}||<\infty.$$

For the solution of

$$(\lambda-\kappa L_n)x_m=L_ny,\ x_m\in X \text{ and } (\lambda-\kappa)\mathrm{x}=\mathrm{y};$$

$$x-x_m=\lambda(\lambda-L_n\kappa)^{-1}(x-L_n(x)),$$

$$\frac{|\lambda|}{||\lambda-\kappa L_n||}||x-L_n(x)||\leq||x-x_n||\leq|\lambda|\ ||(\lambda-\kappa L_n)^{-1}||$$
$$||x-L_n(x)||.$$

Proof [36].

Consequently, the approximation rate of Legendre polynomials is n^{-k} with respect to Lemma 4.1.1 also from Theorem 4.1.1, $||x-x_n||$ converge to zero as soon as $||x-L_n||$.

4.2 Error analysis of the method

In this subsection, an error estimator for tenth order boundary value problems using Legendre Galerkin approximation has been discussed.

Consider $e_n(x)=u(x)-u_n(x)$ as the error function of Legnedre approximation $u_n(x)$ to $u(x)$, where $u(x)$ is the exact solution of Eq.(1) with boundary conditions defined in Eq.(2). So, $u_n(x)$ satisfies the following problem

$$u_n^{(10)}(x)+\sum_{i=0}^{9}a_i(x)u_n^{(i)}(x)=f(x)+P_n(x),\ x\in[-1,\ 1],\quad(25)$$

with boundary conditions

$$u_n^{(i)}(-1)=u_n^{(i)}(1)=0,\ \ i=0,1,2,3,4,\qquad(26)$$

where $P_n(x)$ is a perturbation term linked with $u_n(x)$ obtained as follows

$$P_n(x)=u_n^{(10)}(x)+\sum_{i=0}^{9}a_i(x)u_n^{(i)}(x)-f(x),\ \ i=0,1,2,3,4.\qquad(27)$$

To determine an approximation $e_{n,N}(x)$ to $e_n(x)$, in the same way as in Sect.3 for the solution of Eq.(1) with Eq.(2). Subtracting Eq.(22) and Eq.(23) from Eq.(1) and Eq.(2) respectively, yields the error function of the form

$$P_n(x)=-e_n^{(10)}(x)-\sum_{i=0}^{9}a_i(x)e_n^{(i)}(x),\qquad(28)$$

and

$$e_n^{(i)}(-1)=e_n^{(i)}(1)=0,\quad i=0,1,2,3,4.\qquad(29)$$

Solving this problem using the Legendre Galerkin method to get the approximation $e_{n,N}(x)$.

5 Handling of Boundary Conditions and Solution Domain

If the boundary conditions are nonhomogeneous or the solution domain is [a, b], then these conditions are converted to homogeneous conditions through an interpolation with a known function and the domain of solution must be converted to $[-1,\ 1]$. Consider

$$Lu(t)=u^{(10)}(t)+\sum_{i=0}^{9}a_i(t)u^{(i)}(t)=f(t),\ \ t\in[a,\ b],\quad(30)$$

subject to the following boundary conditions

$$u^{(j)}(a)=\theta_j,\ \ u^{(j)}(b)=\phi_j,\ \ j=0,1,2,3,4.\qquad(31)$$

Using the linear transformation $t=\frac{b-a}{2}x+\frac{b+a}{2}$, then Eq.(30) takes the form

$$Lu(x)=\left(\frac{2}{b-a}\right)^{10}u^{(10)}(x)+\sum_{i=0}^{9}a_i(\chi)\left(\frac{2}{b-a}\right)^{i}u^{(i)}(x)$$
$$=f(\chi),\quad x\in[-1,\ 1],\quad(32)$$

where

$$\chi = \frac{b-a}{2}x + \frac{b+a}{2}$$

subject to the following boundary conditions

$$u^{(j)}(-1) = \left(\frac{2}{b-a}\right)^j \theta_j = \Theta_j,\ \ u^{(j)}(1) = \left(\frac{2}{b-a}\right)^j \phi_j = \Phi_j,\ \ j = 0,1,2,3,4. \quad (33)$$

To transform the nonhomogeneous boundary conditions (33) to homogeneous boundary conditions, replacing

$$u(x) = \Psi(x) + \Omega(x) \quad (34)$$

where $\Psi(x)$ is the interpolating polynomial such that $\Psi^{(j)}(-1) = \Theta_j$ and $\Psi^{(j)}(1) = \Phi_j$, $j = 0,1,2,3,4$. Also

$$\Omega(x) = \sum_{j=0}^{9} \eta_j\, x^j.$$

$$\eta_0 = \frac{1}{768}(384\Theta_0 + 279\Theta_1 + 87\Theta_2 + 14\Theta_3 + \Theta_4 + 384\Phi_0 - 279\Phi_1 + 87\Phi_2 - 14\Phi_3 + \Phi_4),$$

$$\eta_1 = \frac{1}{768}(-945\Theta_0 - 561\Theta_1 - 141\Theta_2 - 18\Theta_3 - \Theta_4 + 945\Phi_0 - 561\Phi_1 + 141\Phi_2 - 18\Phi_3 + \Phi_4),$$

$$\eta_2 = \frac{1}{192}(-105\Theta_1 - 57\Theta_2 - 12\Theta_3 - \Theta_4 + 105\Phi_1 - 57\Phi_2 + 12\Phi_3 - \Phi_4),$$

$$\eta_3 = \frac{1}{192}(315\Theta_0 + 315\Theta_1 + 105\Theta_2 + 16\Theta_3 + \Theta_4 - 315\Phi_0 + 315\Phi_1 - 105\Phi_2 + 16\Phi_3 - \Phi_4),$$

$$\eta_4 = \frac{1}{128}(35\Theta_1 + 35\Theta_2 + 10\Theta_3 + \Theta_4 - 35\Phi_1 + 35\Phi_2 - 10\Phi_3 + \Phi_4),$$

$$\eta_5 = \frac{1}{128}(-189\Theta_0 - 189\Theta_1 - 77\Theta_2 - 14\Theta_3 - \Theta_4 + 189\Phi_0 - 189\Phi_1 + 77\Phi_2 - 14\Phi_3 + \Phi_4),$$

$$\eta_6 = \frac{1}{192}(-21\Theta_1 - 21\Theta_2 - 8\Theta_3 - \Theta_4 + 21\Phi_1 - 21\Phi_2 + 8\Phi_3 - \Phi_4),$$

$$\eta_7 = \frac{1}{192}(135\Theta_0 + 135\Theta_1 + 57\Theta_2 + 12\Theta_3 + \Theta_4 - 135\Phi_0 + 135\Phi_1 - 57\Phi_2 + 12\Phi_3 - \Phi_4),$$

$$\eta_8 = \frac{1}{768}(15\Theta_1 + 15\Theta_2 + 6\Theta_3 + \Theta_4 - 15\Phi_1 + 15\Phi_2 - 6\Phi_3 + \Phi_4),$$

$$\eta_9 = \frac{1}{768}(-105\Theta_0 - 105\Theta_1 - 45\Theta_2 - 10\Theta_3 - \Theta_4 + 105\Phi_0 - 105\Phi_1 + 45\Phi_2 - 10\Phi_3 + \Phi_4).$$

The problem takes the form

$$L\Psi(x) = \Psi^{(10)}(x) + \sum_{i=0}^{9} a_i(x)\Psi^{(i)}(x) = f^*(x),\ x \in [-1,1], \quad (35)$$

subject to the following boundary conditions

$$\Psi^{(j)}(-1) = 0,\ \ \Psi^{(j)}(1) = 0,\ \ j = 0,1,2,3,4 \quad (36)$$

where

$$f^*(x) = f(x) - L\Omega(x) = f(x) - \sum_{i=0}^{9} a_i(x)\, \Omega^{(i)}(x).$$

Consider

$$\Psi(x) = \sum_{j=0}^{n} \alpha_j\, L_j(x), \quad (37)$$

be an approximate solution of Eq.(35). Then,

$$u(x) = \sum_{j=0}^{n} \alpha_j\, L_j(x) + \Omega(x), \quad (38)$$

be the approximate solution of (34). Using the inverse linear transformation $x = \frac{2}{b-a}t - \frac{b+a}{b-a}$ in (38), yields the approximate solution $u(t)$ of Eq.(30).

6 Numerical Examples

The reliability and effectiveness of the present method is examined through some examples that appreciate the results.

Example 1 Consider the following differential equation

$$u^{(10)}(x) - u^{(2)}(x) + xu(x) = (-8 + x - x^2)e^x,\ x \in [0,\ 1], \quad (39)$$

subject to the boundary conditions

$$u(0) = 1,\ u(1) = 0,\ u'(0) = 0,\ u'(1) = -e,\ u''(0) = -1,\ u''(1) = -2e,\ u'''(0) = -2,\ u'''(1) = -3e,\ u^{(4)}(1) = -3,\ u^{(4)}(1) = -4e. \quad (40)$$

The exact solution of the problem is $u(x) = (1-x)e^x$.

It is evident from the Table 1 that the errors (in absolute value) obtained by the proposed method (for $n = 10$) are less than those developed by Viswanadham and Ballem (Viswanadham and Ballem, [26, 27]). Whereas, the absolute errors for Example 1 are shown in Figure 2 .

Table 1: Numerical results for Example 1

x	LG-Errors	[26]	[27]
0.1	2.47834×10^{-11}	2.324581×10^{-6}	1.537800×10^{-5}
0.2	4.44401×10^{-10}	1.090765×10^{-5}	4.452467×10^{-5}
0.3	1.74799×10^{-9}	3.635883×10^{-5}	3.331900×10^{-5}
0.4	3.44192×10^{-9}	4.988909×10^{-5}	3.552437×10^{-5}
0.5	4.26365×10^{-9}	4.547834×10^{-5}	9.477139×10^{-6}
0.6	3.51165×10^{-9}	3.725290×10^{-5}	2.586842×10^{-5}
0.7	1.81954×10^{-9}	1.931190×10^{-5}	3.975630×10^{-5}
0.8	4.71966×10^{-10}	8.672476×10^{-6}	3.531575×10^{-5}
0.9	2.6854×10^{-11}	7.867813×10^{-6}	2.214313×10^{-5}

Table 2: Numerical results for Example 2

x	LG-Errors	[26]
0.1	1.47672×10^{-11}	1.788139×10^{-7}
0.2	2.6114×10^{-10}	1.192093×10^{-7}
0.3	1.01268×10^{-9}	2.875924×10^{-6}
0.4	1.96537×10^{-9}	3.606081×10^{-6}
0.5	2.39889×10^{-9}	2.44370×10^{-6}
0.6	1.94626×10^{-9}	3.516674×10^{-6}
0.7	9.93086×10^{-10}	2.250075×10^{-6}
0.8	2.53598×10^{-10}	1.639128×10^{-6}
0.9	1.42011×10^{-11}	2.145767×10^{-6}

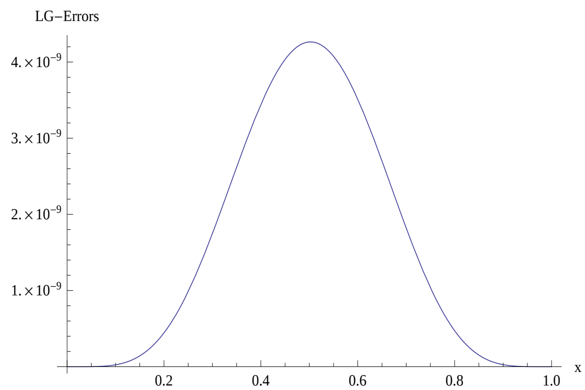

Fig. 2: Absolute Errors for Example 1

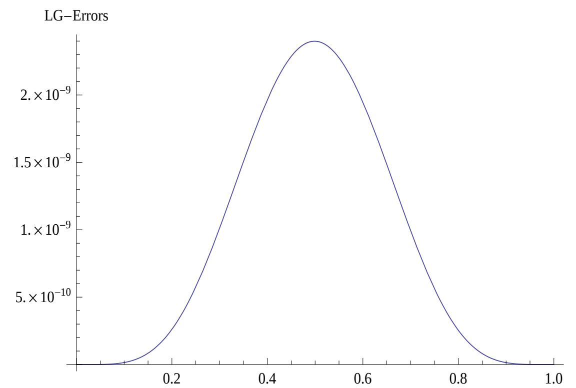

Fig. 3: Absolute Errors for Example 2

Example 2 Consider the following differential equation

$$u^{(10)}(x) + 5u(x) = 10\ cosx + 4(x-1)\ sinx,\ x \in [0,\ 1], \quad (41)$$

subject to the boundary conditions

$$\begin{aligned} &u(0) = 0,\ u(1) = 0,\ u'(0) = -1,\ u'(1) = sin1, \\ &u''(0) = 2,\ u''(1) = 2cos1,\ u'''(0) = 1,\ u'''(1) = -3sin1, \\ &u^{(4)}(0) = -4,\ u^{(4)}(1) = -4cos1. \end{aligned} \quad (42)$$

The exact solution of the problem is

$$u(x) = (x-1)\ sinx.$$

It is evident from the Table 2 that the errors (in absolute value) obtained by the proposed method (for $n = 10$) are less than those developed by Viswanadham and Ballem (Viswanadham and Ballem, [26]). Whereas, the absolute errors for Example 2 are shown in Figure 3.

7 Conclusion

In this paper, Legendre polynomials, as basis functions, have been introduced to manipulate the Galerkin method for approximating the linear tenth order boundary value problems with two point boundary conditions. The nonhomogeneous boundary conditions are converted to the homogeneous boundary conditions while the solution domain is transformed to $[-1,\ 1]$. The main advantage of the presented algorithm is to achieve higher accuracy in the solutions, using few number of terms in the suggested approximation. The obtained results are comparable with the solutions available in the literature. Consequently, the proposed method is suitable choice for getting encouraging results of higher order boundary value problems.

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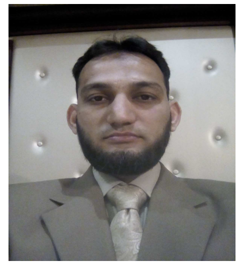

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