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Generalized Differential Transform Method for Solving Liquid-Film Mass Transfer Equation of Fractional Order

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Abstract: Several methods solve linear fractional partial differential equations. In this paper, it is presented a fractional model of the liquid-film mass transfer equation to compare numerical solution between fractional and simple model of the equation. Then the approximate of the generalized differential transform method is compared with the exact solution of the equation in integer orders. Furthermore, the approximates will be obtained in the fractional orders by limiting the intervals of the coefficient and variables. The results show that we can achieve the same result with the proposed fractional model and the method has a high accuracy.

Keywords: Generalized differential transform, liquid-film mass transfer equation, fractional order.

1 Introduction

In recent years, the study of stability and numerical methods of fractional differential equations by using various methods is helped to improve engineering and physics [1,2,3,4,5,6,7,8,9]. There are definitions of the fractional derivative and integral, such as Grunwald- Letnikov, Riemann-Liouville and Caputo. The Caputo fractional derivative of order α is defined as follow

$$D^{\alpha} f(t) = \frac{1}{\Gamma(-\alpha+m)} \int_a^t (t-\tau)^{-\alpha+m-1} f^{(m)}(\tau) d\tau, \tag{1}$$

where $m-1 < \alpha \leq m, m \in Z^+$. For more study see [10,11,12]. One of these methods is the differential transform method [13,14]. The differential transform of function f is

$$F(x) = \frac{1}{k!} \frac{d^k f(x)}{dx^k} |_{x=x_0}, \tag{2}$$

where f is the original function and $F(k)$ is the transformed function. The differential inverse transform of $F(k)$ is

$$f(x) = \sum_{k=0}^{\infty} x^k F(k). \tag{3}$$

Substituting (2) into (3), we get

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{d^k f(x)}{dx^k} |_{x=x_0} . \tag{4}$$

But, this method has developed in [15]. They have presented the generalized differential transform of function f as follow

$$F_{\alpha}(k) = \frac{1}{\Gamma(\alpha k+1)} [(D^{\alpha})^k f(x)]_{x=x_0}, \tag{5}$$

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where $0 < \alpha \leq 1$ and $(D^{\alpha})^k = D^{\alpha}.D^{\alpha}.\cdots.D^{\alpha}(k - times)$. The differential inverse transform of $F_{\alpha}(k)$ is defined by

$$f(x) = \sum_{k=0}^{\infty} F_{\alpha}(k)(x - x_0)^{\alpha k}. \tag{6}$$

Substituting (5) into (6), we get

$$f(x) = \sum_{k=0}^{\infty} \frac{(x - x_0)^{\alpha k}}{\Gamma(\alpha k + 1)}((D^{\alpha})^k f)(x_0). \tag{7}$$

Using theorem (4) in [16], we will obtain an approximate function $f(x)$ from the finite series as

$$f(x) \cong \sum_{k=0}^{n} F_{\alpha}(k)(x - x_0)^{\alpha k}. \tag{8}$$

In case $\alpha = 1$, the generalized differential transform (5) changes to the differential transform. Also, GDTM is presented for solving the linear fractional partial differential equations [17]. Many equations have been studied by using fractional methods. The liquid-film mass transfer equation with the boundary conditions (10) is as

$$(1 - y^2)\frac{\partial \omega}{\partial x} = a\frac{\partial^2 \omega}{\partial y^2}, \tag{9}$$

$$\omega = 0, x = 0, (0 < y \leqslant 1); \omega = 1, y = 0, (x > 0); \frac{\partial \omega}{\partial y} = 0, y = 1, (x > 0). \tag{10}$$

where ω, is a dimensionless temperature; x and y are dimensionless measured coordinates, respectively. $a = \frac{1}{Pe}$ where Pe is the Peclet number [18,19,20,21,22,23,24]. Mixed boundary conditions are commonly encountered in practical purposes. The solution of (9) is given by

$$\omega(x,y) = 1 - \sum_{m=1}^{\infty} A_m \, exp(-a\lambda_m^2 x) \, F_m(y), \tag{11}$$

$$F_m(y) = y \, exp(-\frac{1}{2}\lambda_m y^2) \, \Phi(\frac{3}{4} - \frac{1}{4}\lambda_m, \frac{3}{2}; \lambda_m y^2), \tag{12}$$

where the function F_m and the coefficients A_m and λ_m are independent of the parameter a. The eigenvalues of λ_m for $m = 1, 2, \ldots$ are solutions of the transcendental equation:

$$\lambda_m \Phi(\frac{3}{4} - \frac{1}{4}\lambda_m, \frac{3}{2}; \lambda_m) - \Phi(\frac{3}{4} - \frac{1}{4}\lambda_m, \frac{1}{2}; \lambda_m) = 0, \tag{13}$$

where $\phi(M,N;Z) = 1 + \sum_{m=1}^{\infty} \frac{M(M+1)\cdots(M+m-1)}{N(N+1)\cdots(N+m-1)} \frac{Z^m}{m!}$. The series coefficients A_m are calculated as follows

$$A_m = \frac{\int_0^1 (1 - y^2)F_m(y)dy}{\int_0^1 (1 - y^2)(F_m(y))^2 dy}, \qquad m = 1, 2, \cdots. \tag{14}$$

Table (1) shows the first ten eigenvalues λ_m and coefficients A_m[18].

The organization of this paper is followed by: In Section 2, we described GDTM for linear PDEs. In Section 3, the approximate of the fractional model (9) is compared with (11). In Section 4 we conclude our work.

2 Generalized Two-Dimensional Differential Transform Method

In this section, we provide some important definitions and theorems of GDTM for linear PDEs. First, we consider the function of two variables $u(x,y)$. Also, suppose that $u(x,y) = f(x)g(y)$. According to [25,26], the function $u(x,y)$ can be represented as

$$u(x,y) = \sum_{k=0}^{\infty} F_{\alpha}(k)(x - x_0)^{k\alpha} \sum_{h=0}^{\infty} G_{\beta}(h)(y - y_0)^{h\beta} = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U_{\alpha,\beta}(k,h)(x - x_0)^{k\alpha}(y - y_0)^{h\beta}, \tag{15}$$

Table 1: Eigenvalue λ_m and coefficients A_m in Equation (11).

m	λ_m	A_m
1	2.2631	1.3382
2	6.2977	-0.5455
3	10.3077	0.3589
4	14.3128	-0.2721
5	18.3159	0.2211
6	22.3181	-0.1873
7	26.3197	0.1631
8	30.3209	-0.1449
9	34.3219	0.1306
10	38.3227	-0.1191

where $\alpha > 0$ and $\beta \leqslant 1$. $U_{\alpha,\beta}(k,h) = F_\alpha(k)G_\beta(h)$ is the spectrum of $u(x,y)$. The generalized two-dimensional differential transform of $u(x,y)$ is as follows

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}[(D^\alpha_{x_0})^k(D^\beta_{y_0})^h u(x,y)]_{x_0,y_0}. \tag{16}$$

According to (15) and (16), the following results obtain.
Theorem 2.1. Suppose that $U_{\alpha,\beta}(k,h), V_{\alpha,\beta}(k,h)$, and $W_{\alpha,\beta}(k,h)$ are the differential transformations of $u(x,y)$, $v(x,y)$, and $w(x,y)$, respectively:

(a) if $u(x,y) = v(x,y) \pm w(x,y)$, then $U_{\alpha,\beta}(k,h) = V_{\alpha,\beta}(k,h) \pm W_{\alpha,\beta}(k,h)$,
(b) if $u(x,y) = cv(x,y), c \in \mathbb{R}$, then $U_{\alpha,\beta}(k,h) = cV_{\alpha,\beta}(k,h)$,
(c) if $u(x,y) = v(x,y)w(x,y)$, then $U_{\alpha,\beta}(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h} V_{\alpha,\beta}(k,h)W_{\alpha,\beta}(k,h)$,
(d) if $u(x,y) = (x-x_0)^{n\alpha}(y-y_0)^{m\beta}$, then $U_{\alpha,\beta}(k,h) = \delta(k-n)\delta(h-m)$,
(e) if $u(x,y) = D^\alpha_{x_0} v(x,y)$, $0 < \alpha \leqslant 1$, then $U_{\alpha,\beta}(k,h) = \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} U_{\alpha,\beta}(k+1,h)$.

Theorem 2.2. If $u(x,y) = f(x)g(y)$ and the function $f(x) = x^\lambda h(x)$, where $\lambda > -1$, $h(x)$ has the generalized's series expansion $h(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^{\alpha k}$ [16], and
(I) $\beta < \lambda + 1$ and α is arbitrary, or
(II) $\beta \geqslant \lambda+1, \alpha$ is arbitrary and a_n for $n = 0,1,2,\cdots,m-1$, where $m-1 < \beta \leqslant m$. Then the generalized differential transform (16) becomes

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}[(D^{\alpha k}_{x_0})(D^\beta_{y_0})^h u(x,y)]_{x_0,y_0}. \tag{17}$$

Theorem 2.3. If $v(x,y) = f(x)g(y)$, the function $f(x)$ satisfies the conditions of theorem 2.2 and $u(x,y) = D^\alpha_{x_0} v(x,y)$, then

$$U_{\alpha,\beta}(k,h) = \frac{\Gamma(\alpha(k+1)+\gamma)}{\Gamma(\alpha k+1)} V_{\alpha,\beta}(k+\frac{\gamma}{\alpha},h). \tag{18}$$

The proofs and the convergence of GDTM maybe found in [15,27].

3 Numerical Results

In this section, first we introduce the fractional liquid-film mass transfer equation. Then by using the mentioned definitions and theorems, we compare an approximate of GDTM with the exact solution in the integer and fractional orders cases. We consider the fractional liquid-film mass transfer equation with following boundary conditions

$$(1-y^2)D^\alpha_x \omega = aD^\gamma_y \omega, \tag{19}$$

$$\omega(0,y) = 0, \omega(x,0) = 1, \ \frac{\partial\omega}{\partial y}(x,1) = -437.11 + 2.9130\,10^5 ax - 1.4271\,10^8 a^2 x^2, \tag{20}$$

where $0<\alpha\leqslant 1$, $1<\gamma\leqslant 2$, and $a=1$. It can be obtained an ordinary system by sitting $\alpha=1$, $\beta=1$, and $\gamma=2$. The generalized differential transform of (19) is as follows

$$\Omega_{1,1}(k,h+2)=\frac{1}{a(h+1)(h+2)}[(k+1)\Omega_{1,1}(k+1,h)-\sum_{r=0}^{k}\sum_{s=0}^{h}(\delta(k-r)\delta(s-2)(k-r+1)\Omega_{1,1}(k-r+1,s))]. \quad (21)$$

Also, we can write the generalized differential transform of the boundary conditions (20) as

$$\Omega(k,0)=1,\ \Omega(k,1)=-437.11+2.913010^{5}\delta(k-1)\delta(h)-1.427110^{8}\delta(k-2)\delta(h). \quad (22)$$

Therefore, we have the solution $\omega(x,y)$ up to $O(x^{18})$ as follows:

$$\omega(x,y)=1-437.11x+\frac{x^2}{2a}-72.85166667\frac{x^3}{a}+\left(1-437.11x+\frac{x^2}{a}-145.7033333\frac{x^3}{a}+\frac{x^4}{6a^2}\right)y^2 \quad (23)$$
$$+\left(1-437.11x+\frac{3x^2}{2a}-218.5550000\frac{x^3}{a}+\frac{x^4}{2a^2}\right)y^4+\left(1-437.11x+\frac{2x^2}{a}-291.4066667\frac{x^3}{a}+\frac{3x^4}{4a^2}\right)y^6$$
$$+\left(1-437.11x+\frac{5x^2}{2a}-364.2583333\frac{x^3}{a}+\frac{7x^4}{6a^2}\right)y^8+\left(1-437.11x+\frac{3x^2}{a}-437.11\frac{x^3}{a}+\frac{7x^4}{4a^2}\right)y^{10}$$
$$+\left(1-437.11x+\frac{7x^2}{2a}-509.9616667\frac{x^3}{a}+\frac{9x^4}{4a^2}\right)y^{12}+\left(1-437.11x+\frac{4x^2}{a}-582.8133333\frac{x^3}{a}+\frac{35x^4}{12a^2}\right)y^{14}$$
$$+\left(1-437.11x+\frac{9x^2}{2a}-655.6650000\frac{x^3}{a}+\frac{11x^4}{3a^2}\right)y^{16}+\left(1-437.11x+\frac{5x^2}{a}-728.5166667\frac{x^3}{a}+\frac{9x^4}{2a^2}\right)y^{18}$$

Figures (1a) and (1b) show the exact solution and approximate of (19), respectively. Also, the interval a in figure (1a) and (1b) is $[0,1]$ and $[0.6,0.8]$, respectively. As we see, the approximate is compared with the exact solution by modifying in the interval a.

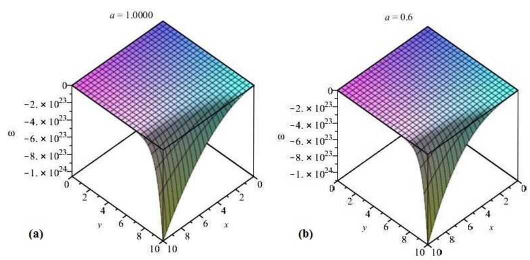

Fig. 1: The solution $\omega(x,y)$ of (19) and (20). (a) exact solution and $a\in[0,1]$. (b) approximate solution when $\alpha=\beta=1$, $\gamma=2$, and $a\in[0.6,0.8]$.

Now, we suppose $\alpha=0.5$, $\beta=1$, and $\gamma=2$. The generalized differential transform of (19) is

$$\Omega_{0.5,1}(k,h+4)=\frac{\Gamma(0.5h+1)}{a\Gamma(0.5k+2.5)}[\frac{\Gamma(0.5k+1.5)}{\Gamma(0.5k+1)}\Omega_{0.5,1}(k+1,h)$$
$$-\sum_{r=0}^{k}\sum_{s=0}^{h}\delta(k-r)\delta(s-2)\frac{\Gamma(0.5(k-r)+1.5)}{\Gamma(0.5(k-r)+1)}\Omega_{0.5,1}(k-r+1,s)]. \quad (24)$$

Also, the transformed boundary conditions (20) is

$$\Omega(k,0)=\delta(k),\Omega(k,1)=-437.11\delta(k)+2.913010^{5}\delta(k-2)-1.427110^{8}\delta(k-4),$$
$$\Omega(k,2)=0,\Omega(k,3)=10.138\delta(k)-337.58\,a\delta(k-2)+8688\,a^2\delta(k-4). \quad (25)$$

Therefore, we have the solution $\omega(x,y)$ up to $O(x^8)$.

$$\omega(x,y)=1-437.11x^{0.5}+10.138x^{1.5}+(145650.0\frac{x^{2.5}}{a}-84.39500000x^{3.5})y^2$$
$$+(291300.0x^{0.5}-337.58\,ax^{1.5})y^4+(-95140000.02\frac{x^{2.5}}{a}+2896\,ax^{3.5})y^6$$
$$+(-142710000.0x^{0.5}+8688\,a^2x^{1.5})y^8. \quad (26)$$

It should be noted that increasing more components of the series solution results in increasing error and changes the solution. Although, α is decreased but the approximate will be obtained by limiting in the intervals x,y, and a. Figure (2) shows the comparison of the exact solution and GDTM approximate.

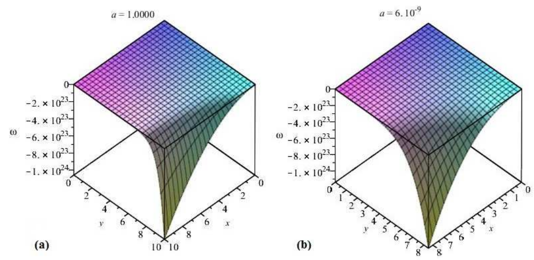

Fig. 2: The solution $\omega(x,y)$ of (19) and (20). (a) exact solution and $a \in [0,1]$. (b) approximate solution when $\alpha = 0.5$, $\beta = 1$, $\gamma = 2$, $a \in [6\cdot 10^{-9}, 8\cdot 10^{-9}]$, and $x, y \in [0, 8.4]$.

Finally, in the last case, we suppose that $\alpha = 0.5$, $\beta = 1$, and $\gamma = 1.5$. The transform of (19) is

$$\Omega_{0.5,1}(k,h+4) = \frac{\Gamma(0.5h+1)}{a\Gamma(0.5k+2)}\Big[\frac{\Gamma(0.5k+1.5)}{\Gamma(0.5k+1)}\Omega_{0.5,1}(k+1,h) - \sum_{r=0}^{k}\sum_{s=0}^{h}\delta(k-r)\delta(s-2) \frac{\Gamma(0.5(k-r)+1.5)}{\Gamma(0.5(k-r)+1)}\Omega_{0.5,1}(k-r+1,s)\Big]. \quad (27)$$

Since $\alpha = 0.5$, therefore, we obtain the solution $\omega(x,y)$ up to $O(x^6)$ with the transformed boundary conditions (25).

$$\omega(x,y) = 1 - 437.11x^{0.5} + 10.138x^{1.5} + (109565.6171x^{2.5} - 152.3672957x^{3.5})y^{1.5} + (291300.0x^{0.5} - 337.58ax^{1.5})y^3 + (-143138658.7\frac{x^{2.5}}{a} + 5228.457710ax^{3.5})y^{4.5} + (-142710000.0x^{0.5} + 8688a^2x^{1.5})y^6. \quad (28)$$

The last case is shown in figure (3b). As we see, both orders are decreased and GDTM approximate obtains with smaller intervals than the previous cases.

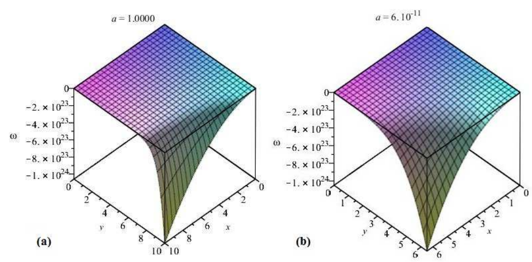

Fig. 3: The solution $\omega(x,y)$ of (19) and (20). (a) exact solution, $a \in [0,1]$ and $x, y \in [0,10]$. (b) approximate solution when $\alpha = 0.5$, $\beta = 1$, $\gamma = 1.5$, $a \in [6\cdot 10^{-11}, 8\cdot 10^{-11}]$, and $x, y \in [0, 6.4]$.

4 Conclusion

In this work, we introduced the fractional model of the liquid-film mass transfer equation. The generalized differential transform method is powerful tool in order to solve linear PDEs. It was used for solving the mentioned equation. In the first case, the interval a has been limited, x and y are fixed. This action makes approximates even if the orders be fractional. As we have seen, when both of orders were decreased, the intervals became limited.

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