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Nonclassical properties of a model for modulated damping under the action of an external force

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We introduce a modified model for the problem of a damped harmonic oscillator in presence of a driving force. The model is treated from the point of view of quantum mechanics. The wave function in the Schrödinger picture is obtained. The connection between the quasicoherent state and pseudostationary state is discussed. The constants of motion are also introduced and the eigenfunctions and the corresponding eigenvalues for pair of quadratic invariants are obtained. The phenomena of squeezing and the Poissonian distribution are considered. It is shown that the system is sensitive to the variation of the damping factor, γ, as well as of the modulation factor, μ.

Keywords: Damping harmonic oscillator, Poissonian distribution.

1 Introduction

The problem of time-dependent harmonic oscillator has been widely studied since the middle of the last century, see for example [1–4]. Even so the problem still attracts much attention nowadays. This is due to the fact that, the existence of the time factor in this particular problem leads to the appearance of the second harmonic generation and consequently the system is converted to a model for the degenerate parametric amplifier [5, 6]. Also, the realization of the symmetry group for this system gives us the opportunity to consider the Lie algebraic approach of such a problem [7, 8]. However, the key to deal with any dynamical system is to find either the closed-form solution to the wave function in the Schrödinger picture or to obtain the solution for the equations of motion in the Heisenberg picture. Therefore most of the attempts during this period were too limited due to the absence of enough methods in addition to the limitation of number of functions from which the solution can be obtained in a compact form. In this context there is question which may arise. This is why does one come back to study such a problem and try to resurrect it. The answer is not just a matter to reconsider a certain problem and try to recover some of the gaps in it. In fact this particular problem has opened the door to consider different aspects in the classical as well as in the quantum mechanics. For instance the observation of nonclassical phenomena in the laboratory, particularly the squeezing phenomenon and its connection to the second harmonic generation, encouraged us to return to the problem. Doubtless the appearance of the second harmonic generation would lead us to think of the nonclassical properties for such system. Therefore, as apart of our duty in this context is to discuss the squeezing phenomenon, however, from the point view of the SU$(1,1)$ Lie Algebra [9,10], we extend our discussion to include the correlation function from which we are able to discuss the Poissonian distribution of the system. Here we may refer to previous works on the problem of quantizing the damped motion of a particle in a quadratic field. The problem is usually trated as an oscillator with constant mass and stiffness placed in the presence of a dissipative force $F = -\gamma\dot{X}$, where γ being constant.

In addition there exists a substantial body of work concerning the study of a classical, undamped harmonic oscillator with arbitrary dependence in its parameters, see for example [1, 10–13]. In the present work we aim at unifying these aspects in order to provide a treatment of a quantal oscillator in the presence of a dissipation mechanism in the most general situation in which the mass is time-dependent. Our consideration is extended to include a time-dependent driving-force which acts on the Hamiltonian model, see for example [4, 10, 14]. In what follows we concentrate on a particular time-dependent mass law which is given by

$$M(t) = \frac{m\exp(2\gamma t)}{(1-\mu\exp(2\gamma t))^2} \tag{1.1}$$

where γ is the damping or growth factor depending upon whether γ has positive or negative value, respectively, m is a constant mass and μ is an arbitrary parameter. In fact the above model can be regarded as a modified damping or growth described by a damping constant multiplied by a certain time-dependent factor. As is well known, the Hamiltonian which usually describes the time-dependent harmonic oscillator in presence of an external driving force is given by

$$\hat{H}(t) = \frac{\hat{P}^2}{2M(t)} + \frac{1}{2}\omega_0^2 M(t)\hat{Q}^2 + G(t)\hat{Q} \tag{1.2}$$

where $M(t)$ is a time-dependent mass and ω_0 is the oscillator frequency, while $G(t)$ is a time-dependent external force. $\hat{P}$ and $\hat{Q}$ are the momentum and position coordinates which satisfy the commutation relation $\left[\hat{Q},\hat{P}\right] = i\hbar$. Before we go further, we introduce canonical coordinates, position $\hat{q}$ and momentum $\hat{p}$ such that

$$\hat{q} = \sqrt{\frac{M(t)}{m}}\hat{Q}, \qquad \hat{p} = \sqrt{\frac{m}{M(t)}}\hat{P}, \tag{1.3}$$

and obey the condition $[\hat{q},\hat{p}] = \left[\hat{Q},\hat{P}\right] = i\hbar$. In this case the Hamiltonian (1.2) takes the form

$$\hat{K}(\hat{q},\hat{p},t) = \frac{\hat{p}^2}{2m} + \frac{m\omega_0^2}{2}\hat{q}^2 + \frac{\varepsilon(t)}{2}(\hat{q}\hat{p}+\hat{p}\hat{q}) + \tilde{G}(t)\hat{q}. \tag{1.4}$$

where we have used the explicity time-dependent generating function and defined

$$\varepsilon(t) = \gamma f(t), \qquad f(t) = \left(\frac{1+\mu\exp(2\gamma t)}{1-\mu\exp(2\gamma t)}\right). \tag{1.5}$$

We introduce the time-dependent function $\tilde{G}(t)$ through

$$\tilde{G}(t) = (\exp(-2\gamma t) - \mu\exp(2\gamma t))\, G(t) \tag{1.6}$$

It is worthwhile to remark that, the present model is especially interesting in the strong damping regime $0 \ll \mu < 1$. For small values of γt the modulation factor becomes arbitrary large when μ increases from $zero$ to $1-\epsilon$ ($\epsilon > 0$). This can describe an arbitrarily heavy initial damping (or growth) without γ exceeding its critical value of ω_0. The case of a heavy growth could have application in quantum optics for a superradiant cavity. Since one of our main task is to introduce the solution of the wave function in the Schrödinger picture, we devote the next section to derive its expression of this function in addition to the corresponding wave function in the quasicoherent state. In Section **3** we introduce two classes of the quadratic invariants and present the eigenfunction and the corresponding eigenvalue for each constant. Section **4** is devoted to the nonclassical properties where the phenomena of the squeezing as well as the Poissonian distribution are given. Finally we give our conclusion in Section **5**.

2 The wave function

In this section we give the expression for the wave function in the Schrödinger picture corresponding to the pseudostationary. Also we introduce the accurate definition of the Dirac operator from which the wave function in the quasicoherent states can be obtained. In this context we give the connection between pseudostationary and quasicoherent stats.

2.1 Schrödinger picture

The time-dependent wave function in the Schrödinger picture is given by the solution of

$$\hat{K}\psi(q,t) = i\hbar\frac{\partial}{\partial t}\psi(q,t), \tag{2.1}$$

where $\hat{K}$ is the time-dependent Hamiltonian given by equation (1.4). From equations (1.4) and (2.1) the wave function satisfies

$$\frac{\partial^2\psi}{\partial q^2} - \frac{m^2\omega_0^2}{\hbar^2}q^2\psi + \frac{im\varepsilon(t)}{\hbar}\left[2q\frac{\partial}{\partial q}+1\right]\psi - \frac{2m\tilde{G}(t)}{\hbar^2}q\psi = -\frac{2im}{\hbar}\frac{\partial\psi}{\partial t}. \tag{2.2}$$

In order to solve the above equation we introduce the substitution

$$q=\left(\frac{x}{\Omega(t)}+\zeta(t)\right), \tag{2.3}$$

where $\zeta(t)$ is an arbitrary time-dependent function to be determined below while $\Omega(t)$ is defined by

$$\Omega(t)=\omega_0\left(\omega^2+\varepsilon^2(t)\right)^{-\frac{1}{2}}, \qquad \omega=\sqrt{\omega_0^2-\gamma^2}. \tag{2.4}$$

Then $\psi(q,t)\rightarrow\phi\left(x,t\right)$ and consequently equation (2.2) becomes

$$\begin{aligned}
&\frac{\partial^2\phi}{\partial x^2}-\frac{m^2\omega_0^2}{\hbar^2\Omega^4(t)}\left(x+\Omega(t)\zeta(t)\right)^2\phi \\
&+\frac{2im}{\hbar}\left(\varepsilon(t)x+v\right)\frac{\partial\phi}{\partial x}-\frac{2m\tilde{G}(t)}{\hbar\Omega^3(t)}\left(x+\Omega(t)\zeta(t)\right)\phi \\
= \quad & -\frac{2im}{\hbar\Omega^2(t)}\left(\frac{\partial}{\partial t}+\frac{\varepsilon(t)}{2}\right)\phi
\end{aligned} \tag{2.5}$$

where $v(t)$ is the time-dependent function given by

$$v(t)=\frac{\varepsilon(t)\zeta(t)-\dot{\zeta}(t)}{\Omega(t)}. \tag{2.6}$$

We seek a separation of the form

$$\phi(x,t)=X(x)\,T(t)\,\exp\left[-\frac{im}{2\hbar}\left[\varepsilon(t)x^2+2v(t)\,x\right]\right], \tag{2.7}$$

then from equations (2.5) and (2.7) we have

$$\begin{aligned}
\frac{1}{X}\frac{d^2X}{dx^2}-\frac{m^2\omega^2}{\hbar^2}x^2 \quad = \quad & -\frac{2im}{\hbar\Omega^2(t)}\frac{1}{T}\frac{dT}{dt}+\frac{im\dot{\Omega}(t)}{\hbar\Omega^3(t)}+\frac{2m\tilde{G}(t)}{\hbar^2\Omega^2(t)}\zeta(t) \\
& -\frac{m^2}{\hbar^2}v^2+\frac{m^2\omega_0^2}{\hbar^2\Omega^2(t)}\zeta^2(t),
\end{aligned}$$

where we choose the function $\zeta(t)$ to satisfy the equation

$$\ddot{\zeta}\left(t\right)+\left[\omega_0^2-\varepsilon^2\left(t\right)-\dot{\varepsilon}\left(t\right)\right]\zeta\left(t\right)=-\frac{\tilde{G}\left(t\right)}{m}, \tag{2.8}$$

After a straightforward calculation we find, with λ a constant of separation, that

$$\frac{d^2X}{dx^2}+\left(\lambda-\frac{m^2\omega^2}{\hbar^2}x^2\right)X=0, \qquad \lambda>0, \tag{2.9}$$

with solution

$$X_n(x)=H_n\left[\sqrt{\frac{m\omega}{\hbar}}x\right]\exp\left[-\frac{m\omega}{2\hbar}\,x^2\right], \tag{2.10}$$

in which we have taken

$$\lambda = \frac{m\,\omega}{\hbar}\left(2n+1\right), \qquad\qquad n = 0, 1, 2, ... \tag{2.11}$$

Also we find that

$$T_n(t) = N_n\sqrt{\Omega\left(t\right)}\exp\left\{-i\omega\left(n+\frac{1}{2}\right)\left[t-\frac{1}{\omega}\tan^{-1}\left(\frac{\varepsilon\left(t\right)}{\omega}\right)\right]-i\frac{m}{2\hbar}I\left(t\right)\right\}\,, \tag{2.12}$$

where N_n is the normalization constant and

$$I\left(t\right) = \int_0^t\left[\frac{2\tilde{G}\left(t\right)}{m}\zeta\left(t\right)+\omega_0^2\zeta^2\left(t\right)-\left(\varepsilon(t)\zeta\left(t\right)-\dot{\zeta}\left(t\right)\right)^2\right]dt. \tag{2.13}$$

After simple algebra we can write the general solution for the wave function in its final form as

$$\begin{aligned}\psi_n(q,t) \quad = \quad & \left[\frac{m\omega}{\pi\hbar}\Omega^2\left(t\right)\right]^{\frac{1}{4}}\frac{2^{-\frac{n}{2}}}{\sqrt{n!}}H_n\left[\sqrt{\frac{m\omega}{\hbar}}\Omega(t)\;\left(q-\zeta\left(t\right)\right)\right]\\ & \times\;\exp\left[-\frac{m\omega_0}{2\hbar}\Omega\left(t\right)\left(q-\zeta\left(t\right)\right)^2\exp\left(i\tan^{-1}\left(\frac{\varepsilon(t)}{\omega}\right)\right)\right]\\ & \times\exp\left[-\frac{im}{\hbar}v(t)\Omega(t)\left(q-\zeta\left(t\right)\right)\right]\\ & \times\exp\left\{-i\omega\left(n+\frac{1}{2}\right)\left[t-\frac{1}{\omega}\tan^{-1}\left(\frac{\varepsilon\left(t\right)}{\omega}\right)\right]-i\frac{m}{2\hbar}I\left(t\right)\right\}\end{aligned} \tag{2.14}$$

Having obtained the wave function in the number state we are in a position to find the wave function in the quasicoherent state. This is seen in the next subsection

2.2 Quasi-Coherent State

Here we employ the result obtained in the previous subsection to derive the wave function in the quasicoherent $|\alpha\rangle$ state. To reach our goal we use the time-dependent coherent sate, $|\alpha, t\rangle$ given by

$$|\alpha,t\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n,t\rangle, \tag{2.15}$$

which leads to

$$\psi_\alpha(q,t) = \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\psi_n(q,t). \tag{2.16}$$

If we now insert the wave function in the number state given by equation (2.14) into the above equation and run the sum over n, we obtain

$$\begin{aligned}\psi_\alpha(q,t) &= \left[\frac{m\omega}{\pi\hbar}\Omega^2(t)\right]^{\frac{1}{4}}\exp\left(\sqrt{\frac{2m\omega}{\hbar}}\Omega(t)\left(q-\zeta(t)\right)\alpha(t)\right)\\ &\times\exp\left(-\frac{1}{2}\left(|\alpha|^2+\alpha^2(t)\right)\right)\exp\left(-\frac{i}{2}\left(f(t)+\frac{m}{\hbar}I(t)\right)\right)\\ &\times\ \exp\left[-\frac{m\omega_0}{2\hbar}\Omega(t)\left(q-\zeta(t)\right)^2\exp\left(i\tan^{-1}\left(\frac{\varepsilon(t)}{\omega}\right)\right)\right]\\ &\times\exp\left[-\frac{im}{\hbar}v(t)\Omega(t)\left(q-\zeta(t)\right)\right], \end{aligned}\tag{2.17}$$

where

$$\alpha(t)=\alpha(0)\exp\left(-if(t)\right),\qquad f(t)=\omega\left[t-\frac{1}{\omega}\tan^{-1}\left(\frac{\varepsilon(t)}{\omega}\right)\right].\tag{2.18}$$

On the other hand we can reach the same target using a different method. This can be achieved if we introduce the Dirac operators from which we are able to diagonalize the Hamiltonian (1.4). To this end we introduce the operator

$$\hat{\mathcal{A}}=\left(2m\omega_0^2\hbar\omega\right)^{-\frac{1}{2}}\left[m\omega_0^2\hat{q}(t)+i\left(\omega-i\varepsilon(t)\right)\hat{p}(t)-\mathcal{L}(t)\right],\tag{2.19}$$

which satisfies the commutation relation $\left[\hat{\mathcal{A}},\hat{\mathcal{A}}^\dagger\right]=1$. The function $\mathcal{L}(t)$ in the above equation is given by

$$\mathcal{L}(t)=\left(\dot{k}(t)+\tilde{G}(t)-i\omega k(t)\right)-\left(\dot{k}(0)+\tilde{G}(0)-i\omega k(0)\right)e^{-i\omega t}\tag{2.20}$$

where

$$k(t)=\frac{1}{\omega}\int_0^t\left(\varepsilon(\tau)\tilde{G}(\tau)-\frac{d}{d\tau}\tilde{G}(\tau)\right)\sin\omega\left(t-\tau\right)d\tau.\tag{2.21}$$

It is interesting to point out that the operator $\hat{\mathcal{A}}$ is constructed by using the solution for the equations of motion in the Heisenberg picture. The effect of the operator $\hat{\mathcal{A}}$ on the coherent state (2.15) gives us $\hat{\mathcal{A}}|\alpha,t\rangle=\alpha|\alpha,t\rangle$ from which we have

$$\alpha\phi_\alpha\left(q,t\right)=\left(2m\omega_0^2\hbar\omega\right)^{-\frac{1}{2}}\left[m\omega_0^2\hat{q}(t)+i\left(\omega-i\varepsilon(t)\right)\hat{p}(t)-\mathcal{L}(t)\right]\phi_\alpha\left(q,t\right).\tag{2.22}$$

Using the fact that $\hat{p}=-i\hbar\partial/\partial q$ we obtain

$$\phi_\alpha\left(q,t\right)=\mathcal{N}_\alpha\exp\left(-\frac{m\omega_0}{2\hbar}q^2+\left(\sqrt{\frac{2m\omega}{\hbar}}\alpha(t)+\frac{\mathcal{L}_1(t)}{\hbar\omega_0}\right)q\right)\Omega(t),\tag{2.23}$$

where $\mathcal{N}_\alpha$ is the normalization constant given by

$$\begin{aligned}\mathcal{N}_\alpha &= \sqrt[4]{\frac{m\omega\Omega^2(t)}{\pi\hbar}}\exp\left(-\frac{1}{2}\left(|\alpha|^2+\alpha^2(t)\right)\right)\\ &\times\exp\left(-\left(\mathcal{L}_1+\mathcal{L}_1^*\right)\alpha(t)-\frac{1}{2}\left(\mathcal{L}_1^2+|\mathcal{L}_1|^2\right)\right)\end{aligned}\tag{2.24}$$

in which $\alpha(t)$ is given by equation (2.18) and

$$\mathcal{L}_1(t) = \mathcal{L}(t)\exp\left(i\tan^{-1}\left(\frac{\varepsilon(t)}{\omega}\right)\right). \tag{2.25}$$

It should be noted that without loss of generality we have dropped the term $\sqrt{2m\omega_0^2\hbar\omega}$ as a multiplier of $\mathcal{L}(t)$. If we use the generating function of the Hermite polynomial

$$\exp\left(2xg - g^2\right) = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}g^n, \tag{2.26}$$

together with the coherent state given by equation (2.15), we can write the wave function in the number state as

$$\begin{aligned}\phi_n(q,t) &= \sqrt[4]{\frac{m\omega\Omega^2(t)}{\pi\hbar}}2^{-n/2}(n!)^{-\frac{1}{2}}H_n\left(\sqrt{\frac{m\omega}{\hbar}}\Omega(t)q - \sqrt{2}Re\mathcal{L}_1(t)\right)\\ &\times\exp\left[-\frac{m}{2\hbar}(\omega + i\varepsilon(t))\Omega(t)q^2 + \sqrt{\frac{2m\omega}{\hbar}}\mathcal{L}_1(t)\Omega(t)q\right]\\ &\times\exp\left(-\frac{1}{2}\left(\mathcal{L}_1^2(t) + |\mathcal{L}_1(t)|^2\right)\right)\\ &\times\exp\left(-i\omega n\left[t - \frac{1}{\omega}\tan^{-1}\left(\frac{\varepsilon(t)}{\omega}\right)\right]\right).\end{aligned} \tag{2.27}$$

In absence of the external driving force we can easily show that the wave functions given by equations (2.14) and (2.27) are identical. However, in the presence of the external force we have to use the identity

$$\mathcal{L}(t) = m\left[\omega_0^2\zeta(t) - i(\omega - i\varepsilon(t))\left(\varepsilon(t)\zeta(t) - \dot{\zeta}(t)\right)\right], \tag{2.28}$$

from which we obtain the agreement between the two equations, (2.14) and (2.27).

3 The constants of motion

The use of explicitly time-dependent invariants in applications of quantum theory has received little attention. Presumably the reason for this lack of attention has been the dearth of examples in which the use of such quantities was both possible and fruitful. However, a class of exact invariants for time-dependent harmonic oscillators, both classical and quantum, was reported [15–19]. The simplicity of the rules for constructing these invariants and the instructive relation of the invariant theory have stimulated an interest in using the invariants [20, 21]. In this section we focus upon finding the quadratic invariant from which we are able to introduce classes of the wave functions constructed on a new definition of the Dirac operator. In this context we calculate the eigenvalue and corresponding eigenfunction of these invariants. However, we restrict our treatment to the case in which the driving force is absent.

3.1 Quadratic invariants

We begin by seeking an invariant of the second degree,

$$\hat{I}^{(2)} = \lambda(t)\hat{p}^2(t) + \nu(t)\hat{q}^2(t) + \delta\,(t)\,\hat{p}(t)\hat{q}(t), \tag{3.1}$$

we require that

$$\frac{d\hat{I}^{(2)}}{dt} = \frac{\partial \hat{I}^{(2)}}{\partial t} + \frac{1}{i\hbar}\left[\hat{I}^{(2)}, \hat{K}\right] = 0, \tag{3.2}$$

From equations (3.1) and (3.2) together with equation (1.4) we find that

$$\begin{aligned} \dot{\lambda}(t) - 2\varepsilon\,(t)\,\lambda\,(t) &= -\delta\,(t)\,/m, \qquad \dot{\nu}(t) + 2\varepsilon\,(t)\,\nu(t) = m\omega_0^2\delta\,(t)\,, \\ \dot{\delta}\,(t) + 2\nu(t)/m &= 2m\omega_0^2\lambda\,(t)\,. \end{aligned} \tag{3.3}$$

After we perform some minor algebra, equations (3.3) give us

$$\lambda(t)\nu(t) = \frac{\delta^2(t)}{4} + c \tag{3.4}$$

where c is an arbitrary constant. If we now put $\nu(t) = \sigma^2(t)$ in equations (3.3), then after some calculations we have

$$\frac{d^2}{dt^2}\sigma\,(t) + \left[\omega_0^2 + \dot{\varepsilon}\,(t) - \varepsilon^2\,(t)\right]\sigma\,(t) = \frac{m^2\omega_0^4 c}{\sigma^3\,(t)}, \tag{3.5}$$

The nonlinear differential equation above is the Pinney equation and can be solved to take the form

$$\sigma\,(t) = \sqrt{A + B\sin\left(2\omega t + \varphi\right)}, \qquad B = \sqrt{A^2 - c\left(\frac{m\omega_0}{\omega}\right)^2}, \tag{3.6}$$

where A and φ are arbitrary constant and phase, respectively. The first class of quadratic invariants may therefore be expressed in the form

$$\hat{I}_p^{(2)} = \left\{\sigma\,(t)\,\hat{q} + \frac{1}{m\omega_0^2}\left[\dot{\sigma}\,(t) + \sigma\,(t)\,\varepsilon\,(t)\right]\hat{p}\right\}^2 + \frac{c}{\sigma^2\,(t)}\hat{p}^2 \tag{3.7}$$

A similar procedure leads to the second family of quadratic invariants. In this case, if we write $\lambda(t) = \rho^2$ and eliminate the function $\nu(t)$ from equations (3.3), we obtain the equation

$$\frac{d^2}{dt^2}\rho\,(t) + \left[\omega_0^2 - \dot{\varepsilon}\,(t) - \varepsilon^2\,(t)\right]\rho\,(t) = \frac{c}{\rho^3\,(t)}, \tag{3.8}$$

which has the solution

$$\rho\,(t) = \frac{1}{m\omega_0}\left[A + \frac{\varepsilon^2(t)}{\omega_0^2}\left\{A + B\left(\sin\left(2\omega t + \varphi\right) + \frac{2\omega}{\varepsilon(t)}\cos\left(2\omega t + \varphi\right)\right)\right\}\right]^{\frac{1}{2}}. \tag{3.9}$$

Consequently the second class of the quadratic invariants can be constructed in the form

$$\hat{I}_q^{(2)} = \left[\rho\,(t)\,\hat{p} + m\left(\rho\,(t)\,\varepsilon\,(t) - \dot{\rho}\,(t)\right)\hat{q}\right]^2 + \frac{c}{\rho^2\,(t)}\hat{q}^2. \tag{3.10}$$

Having obtained the constants of motion we are in a position to consider the eigenfunctions and the corresponding eigenvalues of these invariants.

3.2 The eigenfunctions of the invariants

We devote this subsection to find the eigenfunctions and the corresponding eigenvalues of the operator $\hat{I}(t)$. We start with its eigenstates [22, 23]. The eigenstates of the invariant operator $\hat{I}(t)$ may be found by an operator technique that is completely analogous to the method introduced by Dirac for diagonalizing the Hamiltonian. In this case we define the time-dependent canonical lowering operator

$$\hat{\mathcal{B}}(t) = \left(2\hbar\sqrt{c}\right)^{-\frac{1}{2}} \left\{ \left[\frac{\sqrt{c}}{\sigma(t)} - \frac{i}{m\omega_0^2} \left[\sigma(t)\varepsilon(t) + \dot{\sigma}(t)\right] \right] \hat{p} - i\sigma(t)\hat{q} \right\}, \tag{3.11}$$

which satisfies with its complex conjugate the canonical relation $\left[\hat{\mathcal{B}}(t), \hat{\mathcal{B}}^{\dagger}(t)\right] = 1$. This means that the operator $\hat{\mathcal{B}}^{\dagger}(t)\hat{\mathcal{B}}(t)$ is a number operator with nonnegative integer eigenvalues. Therefore the invariant operator given by (3.11) can be written in term of the operators $\hat{\mathcal{B}}(t)$ and $\hat{\mathcal{B}}^{\dagger}(t)$ thus

$$\hat{I}_p^{(2)} = 2\hbar\sqrt{c}\left[\hat{\mathcal{B}}^{\dagger}(t)\hat{\mathcal{B}}(t) + \frac{1}{2}\right]. \tag{3.12}$$

The eigenstate of the above invariant can be obtained from the coherent state which has the property $\hat{\mathcal{B}}(t)|\beta, t\rangle = \beta|\beta, t\rangle$, where β is a complex parameter. Similarly we can define another operator $\hat{\mathcal{C}}$ corresponding to the second class of the constant of motion $\hat{I}_q^{(2)}$ such that

$$\hat{\mathcal{C}}(t) = \left(2\hbar\sqrt{c}\right)^{-\frac{1}{2}} \left\{ \left[\frac{\sqrt{c}}{\rho(t)} + im\left[\rho(t)\varepsilon(t) - \dot{\rho}(t)\right] \right] \hat{q} + i\rho(t)\hat{p} \right\}, \tag{3.13}$$

which also satisfies with its complex conjugate the commutation relation $\left[\hat{\mathcal{C}}(t), \hat{\mathcal{C}}^{\dagger}(t)\right] = 1$. In this case the invariant operator takes the form

$$\hat{I}_q^{(2)} = 2\hbar\sqrt{c}\left[\hat{\mathcal{C}}^{\dagger}(t)\hat{\mathcal{C}}(t) + \frac{1}{2}\right]. \tag{3.14}$$

The eigenstates can also be obtained from the coherent state $|\gamma\rangle$ which has definition similar to that in equation (2.15) and satisfies the condition $\hat{\mathcal{C}}(t)|\gamma, t\rangle = \gamma|\gamma, t\rangle$. To find the eigenfunction and the corresponding eigenvalue for the constants of motion $\hat{I}_p^{(2)}$ and $\hat{I}_q^{(2)}$ we have to use the operators $\hat{\mathcal{B}}(t)$ and $\hat{\mathcal{C}}(t)$ given by equations (3.11) and (3.13), respectively. In terms of the momentum the wave function corresponding to the invariant $\hat{I}_p^{(2)}$ is given by

$$\begin{aligned} \chi_{\beta}(p,t) \quad = \quad & \left(\frac{\sqrt{c}}{\hbar\pi\sigma^2(t)}\right)^{\frac{1}{4}} \exp\left[-\frac{1}{2}\left(|\beta|^2 + \beta^2\right)\right] \exp\left[\beta\sqrt{\frac{2\sqrt{c}}{\hbar\delta^2(t)}}p\right] \\ & \times \exp\left\{-\frac{1}{2\hbar}\left[\frac{\sqrt{c}}{\sigma^2(t)} - \frac{i}{m\omega_0^2}\left(\varepsilon(t) + \frac{\dot{\sigma}(t)}{\sigma(t)}\right)\right]p^2\right\}. \end{aligned} \tag{3.15}$$

When one uses the definition of the coherent state, the wave function in the number state takes the form

$$\chi_r\left(p,t\right) = \left(\frac{\sqrt{c}}{\hbar\pi\sigma^2\left(t\right)}\right)^{\frac{1}{4}} 2^{-\frac{r}{2}}\frac{1}{\sqrt{r!}}H_r\left[\sqrt{\frac{\sqrt{c}}{2\hbar\tilde{\sigma}^2\left(t\right)}}p\right] \times \exp\left\{-\frac{1}{2\hbar}\left[\frac{\sqrt{c}}{\sigma^2\left(t\right)}-\frac{i}{m\omega_0^2}\left(\varepsilon\left(t\right)+\frac{\dot{\sigma}(t)}{\sigma\left(t\right)}\right)\right]p^2\right\}. \tag{3.16}$$

The wave function for the second constant of motion corresponding to the coherent state $|\gamma\rangle$ can be written in terms of the coordinate as

$$\Phi_\gamma(q,t) = \left(\frac{\sqrt{c}}{\pi\hbar\rho^2\left(t\right)}\right)^{\frac{1}{4}}\exp\left[-\frac{1}{2}\left(|\gamma|^2+\gamma^2\right)\right]\exp\left[\gamma\sqrt{\frac{2\sqrt{c}}{\hbar\rho^2\left(t\right)}}q\right] \times \exp\left[-\frac{1}{2\hbar}\left\{\frac{\sqrt{c}}{\rho^2\left(t\right)}+im\left(\varepsilon\left(t\right)-\frac{\dot{\rho}}{\rho(t)}\right)\right\}q^2\right] \tag{3.17}$$

while the wave function in the number state is given by

$$\Phi_s(q,t) = \left(\frac{\sqrt{c}}{\pi\hbar\rho^2\left(t\right)}\right)^{\frac{1}{4}}\frac{2^{-\frac{s}{2}}}{\sqrt{s!}}H_s\left[\sqrt{\frac{\sqrt{c}}{2\hbar\rho^2\left(t\right)}}q\right] \times \exp\left[-\frac{1}{2\hbar}\left\{\frac{\sqrt{c}}{\rho^2\left(t\right)}+im\left(\varepsilon\left(t\right)-\frac{\dot{\rho}}{\rho(t)}\right)\right\}q^2\right]. \tag{3.18}$$

Here we may remark that the eigenfunction for the quadratic invariant gives us a flexibility to have classes of wave functions. This in fact is due to the existences of the time-dependent function which is a solution of a nonlinear differential equation.

4 Nonclassical properties

We devote this section to discuss some statistical properties for the present system, more precisely the phenomenon of squeezing, however, from the Lie algebra point of view of the $\mathrm{SU}(1,1)$ [24–26]. This would give us an advantage to use the Perelomove coherent state when the system acquires fluctuations as result of the existence of the time-dependent function. For this reason we employ the Casimir operator to describe the Hamiltonian model given by equation (1.4). On the other hand we extend our discussion to include the Poissonian distribution and this depends upon the examination of the second-order correlation function and the use of the number state as a basis. Since the external driving force in not random, its effect is negligible and consequently we concentrate on the case in which the driving force is absent. To do so we define the operators $\hat{K}_\pm$ and $\hat{K}_z$ such that

$$\left(\hat{K}_+-\hat{K}_-\right)=\frac{1}{2i}(\hat{q}\hat{p}+\hat{p}\hat{q}), \qquad K_z=\frac{1}{2\omega_0}\left(\frac{\hat{p}^2}{2m}+\frac{m\omega_0^2}{2}\hat{q}^2\right), \tag{4.1}$$

where

$$\left[\hat{K}_+, \hat{K}_-\right] = -2\hat{K}_z, \qquad \left[\hat{K}_z, \hat{K}_\pm\right] = \pm\hat{K}_\pm. \tag{4.2}$$

In this case we can write the Hamiltonian (1.4) in the form

$$\frac{\hat{H}}{\hbar} = 2\omega_0 \hat{K}_z - i\varepsilon\left(t\right)(\hat{K}_- - \hat{K}_+). \tag{4.3}$$

Therefore the equations of motion in the Heisenberg picture can be written as

$$\frac{d\hat{K}_z}{dt} = 2\varepsilon\left(t\right)\hat{K}_x, \qquad \frac{d\hat{K}_x}{dt} = -2\omega_0\hat{K}_y\left(t\right) + 2\varepsilon\left(t\right)\hat{K}_z\left(t\right), \qquad \frac{d\hat{K}_y}{dt} = 2\omega_0\hat{K}_x\left(t\right), \tag{4.4}$$

where we have defined $\hat{K}_\pm = \left(\hat{K}_x \pm i\hat{K}_y\right)$. It should be noted that the set of the generators $\left\{\hat{K}_x, \hat{K}_y, \hat{K}_z\right\}$ satisfies the commutation relations

$$\left[\hat{K}_x, \hat{K}_y\right] = -i\hat{K}_z, \qquad \left[\hat{K}_y, \hat{K}_z\right] = i\hat{K}_x, \qquad \left[\hat{K}_z, \hat{K}_x\right] = i\hat{K}_y, \tag{4.5}$$

and the associated Heisenberg uncertainty relation regarding to the first commutator takes the form

$$\langle\left(\Delta\hat{K}_x\right)^2\rangle\langle\left(\Delta\hat{K}_y\right)^2\rangle \geqslant \frac{1}{4}\left|\langle\hat{K}_z\rangle\right|^2 \tag{4.6}$$

where

$$\langle\left(\Delta\hat{K}_j\right)^2\rangle = \langle\hat{K}_j^2\rangle - \langle\hat{K}_j\rangle^2, \qquad j = x, y \tag{4.7}$$

After some manipulations the general solution for the equations of motion (4.4) is given by

$$\begin{pmatrix} \hat{K}_x(t) \\ \hat{K}_y(t) \\ \hat{K}_z(t) \end{pmatrix} = \begin{pmatrix} \mathfrak{f}_-(t) & \mathfrak{f}_0(t) & \mathfrak{f}_+(t) \\ \mathfrak{g}_-(t) & \mathfrak{g}_0(t) & \mathfrak{g}_+(t) \\ \mathfrak{h}_-(t) & \mathfrak{h}_0(t) & \mathfrak{h}_+(t) \end{pmatrix} \begin{pmatrix} \hat{K}_x(0) \\ \hat{K}_y(0) \\ \hat{K}_z(0) \end{pmatrix} \tag{4.8}$$

where we have used the abbreviations

$$\begin{aligned} \mathfrak{f}_\pm(t) &= \frac{1}{2}\left[\left(e_3^2 - e_1^2\right) \pm \left(e_4^2 - e_2^2\right)\right], \qquad \mathfrak{f}_0(t) = \left(e_3e_4 - e_1e_2\right), \\ \mathfrak{g}_\pm(t) &= \left(e_1e_3 \pm e_2e_4\right), \qquad \mathfrak{g}_0(t) = \left(e_2e_3 + e_1e_4\right), \\ \mathfrak{h}_\pm(t) &= \frac{1}{2}\left[\left(e_3^2 + e_1^2\right) \pm \left(e_2^2 + e_4^2\right)\right], \qquad \mathfrak{h}_0(t) = \left(e_1e_2 + e_3e_4\right). \end{aligned} \tag{4.9}$$

and defined the time-dependent functions $e_i(t), i = 1, 2, 3, 4$ such that

$$\begin{aligned} e_1\left(t\right) &= \left(\frac{\omega_0}{\omega}\right)\sin\left(\omega t\right), \\ e_2\left(t\right) &= \left[\cos\left(\omega t\right) - \frac{\varepsilon\left(0\right)}{\omega}\sin\left(\omega t\right)\right], \qquad e_3\left(t\right) = \left[\cos\left(\omega t\right) + \frac{\varepsilon\left(t\right)}{\omega}\sin\left(\omega t\right)\right], \\ e_4\left(t\right) &= \left[\left(\frac{\varepsilon\left(t\right) - \varepsilon\left(0\right)}{\omega_0}\right)\cos\left(\omega t\right) - \left(\frac{\omega}{\omega_0}\right)\left[1 + \frac{\varepsilon\left(t\right)\varepsilon\left(0\right)}{\omega^2}\right]\sin\left(\omega t\right)\right]. \end{aligned} \tag{4.10}$$

Since we have obtained the explicit time-dependent dynamical operators, we are able to discuss some statistical properties related to the system.

4.1 The squeezing phenomenon

As we have previously stated, the present system is explicitly time-dependent and therefore it is natural for the appearance of the second harmonic generation to be seen. Consequently this encourages us to consider the phenomenon of squeezing as an example of the nonclassical effect. To measure the phenomenon of squeezing we have to calculate the quadrature variances $\left\langle \left(\Delta\, \hat{K}_j\right)^2 \right\rangle, j = x, y$ with respect to a certain suitable state. The choice of such a state should be consistent with the system under consideration. For the present case the best state we have to use is the state $|\bar{m};\tilde{k}\rangle$, where $\bar{m}$ is any nonnegative integer and $\tilde{k}$ is the Bargmann index. However, we may also calculate the quadrature variances with respect to the Perelomov $\mathrm{SU}(1,1)$ coherent state $|\xi_1;\tilde{k}\rangle$ which can regarded as a generalization of the coherent state [27,28]. This state is defined by

$$|\xi_1;\tilde{k}\rangle = (1-|\xi_1|^2)^{\tilde{k}} \sum_{\bar{m}=0}^{\infty} \sqrt{\frac{\Gamma(\bar{m}+2\tilde{k})}{\bar{m}!\Gamma(2\tilde{k})}} \xi_1^{\bar{m}} |\bar{m};\tilde{k}\rangle, \tag{4.11}$$

where Γ stands for the gamma function and the state $|\bar{m};\tilde{k}\rangle$ has the properties

$$\begin{aligned} \hat{K}^2|\bar{m};\tilde{k}\rangle &= \tilde{k}\left(\tilde{k}-1\right)|\bar{m};\tilde{k}\rangle, \qquad \hat{K}_z|\bar{m};\tilde{k}\rangle = \left(\bar{m}+\tilde{k}\right)|\bar{m};\tilde{k}\rangle, \\ \hat{K}_+|\bar{m};\tilde{k}\rangle &= \sqrt{(\bar{m}+1)\left(\bar{m}+2\tilde{k}\right)}|\bar{m}+1;\tilde{k}\rangle, \\ \hat{K}_-|\bar{m};\tilde{k}\rangle &= \sqrt{\bar{m}\left(\bar{m}+2\tilde{k}-1\right)}|\bar{m}-1;\tilde{k}\rangle. \end{aligned} \tag{4.12}$$

The operator $\hat{K}$ is known as a Casimir operator and is given by

$$\hat{K}^2 = \hat{K}_z^2 - \left(\hat{K}_+\hat{K}_- + \hat{K}_-\hat{K}_+\right), \tag{4.13}$$

with the commutation relation

$$\left[\hat{K}^2, \hat{K}_\pm\right] = \left[\hat{K}^2, \hat{K}_z\right] = 0. \tag{4.14}$$

The Bargmann index $\tilde{k}$ is either $\frac{1}{4}$ (even parity) or $\frac{3}{4}$ (odd parity). This means that for $\tilde{k} = \frac{1}{4}$ the basis for the irreducible unitary representation space is a set of states with an even boson number and for $\tilde{k} = \frac{3}{4}$ the basis is a set of states with an odd boson number. To measure the squeezing we define the functions

$$F_j = \langle\left(\Delta\, \hat{K}_j\right)^2\rangle - \frac{1}{2}\left|\langle\hat{K}_z\rangle\right| \qquad j = x, y. \tag{4.15}$$

Since squeezing is the phenomenon in which the fluctuation of the $\hat{K}_x$ or $\hat{K}_y$ components occurs if $F_x < 0$ or $F_y < 0$, respectively, one can find the maximum squeezing is reached when $\left\langle \left(\Delta\, \hat{K}_x(t)\right)^2 \right\rangle = 0$, or $\left\langle \left(\Delta\, \hat{K}_y(t)\right)^2 \right\rangle = 0$. We now employ the Perelomov state $|\xi_1;\tilde{k}\rangle$, to calculate the quadrature variances $\left\langle\left(\Delta\,\hat{K}_j(t)\right)^2\right\rangle, j = x, y$ as well as $\left\langle\hat{K}_z(t)\right\rangle$. In this case we have

$$\begin{aligned}
\left\langle\left(\Delta\hat{K}_x\,(t)\right)^2\right\rangle &= \frac{\tilde{k}}{2}\left[\mathfrak{f}_+^2\,(t)+\frac{1}{2}\left(\mathfrak{f}_-^2\,(t)-\mathfrak{f}_0^2\,(t)\right)\cos 2\theta+\mathfrak{f}_-\,(t)\,\mathfrak{f}_0\,(t)\sin 2\theta\right]\ \sinh^2 r \\
&+\frac{\tilde{k}}{4}\left(\cosh^2 r+1\right)\left(\mathfrak{f}_-^2\,(t)+\mathfrak{f}_0^2\,(t)\right) \\
&-\frac{\tilde{k}}{2}\left(\mathfrak{f}_-\,(t)\cos\theta+\mathfrak{f}_0\,(t)\sin\theta\right)\mathfrak{f}_+\,(t)\sinh 2r \\
\left\langle\left(\Delta\hat{K}_y\,(t)\right)^2\right\rangle &= \frac{\tilde{k}}{2}\left[\mathfrak{g}_+^2\,(t)+\frac{1}{2}\left(\mathfrak{g}_-^2\,(t)-\mathfrak{g}_0^2\,(t)\right)\cos 2\theta-\mathfrak{g}_-\,(t)\,\mathfrak{g}_0\,(t)\sin 2\theta\right]\sinh^2 r \\
&+\frac{\tilde{k}}{4}\left(\cosh^2 r+1\right)\left(\mathfrak{g}_-^2\,(t)+\mathfrak{g}_0^2\,(t)\right) \\
&+\frac{\tilde{k}}{4}\left(\mathfrak{g}_-\,(t)\cos\theta-\mathfrak{g}_0\,(t)\sin\theta\right)\mathfrak{g}_+\,(t)\sinh 2r \\
\left\langle\hat{K}_z(t)\right\rangle &= \tilde{k}\left[\mathfrak{h}_-(t)\cos\theta-\mathfrak{h}_0(t)\sin\theta\right]\sinh r+\tilde{k}\mathfrak{h}_+(t)\cosh r
\end{aligned}$$

where we have considered

$$\xi_1 = \exp(i\theta)\tanh\left(\frac{r}{2}\right), \qquad |\xi_1| \in (0,1), \qquad r \in (-\infty,\infty), \qquad \theta \in (0,2\pi). \tag{4.16}$$

It should be noted that at $t = 0$, the functions $F_j, j = x, y$, take the forms

$$\begin{aligned}
F_x(r,\theta,t &= 0) = 2\tilde{k}\sinh^2(\frac{r}{2})\left[\cos^2\theta\cosh^2(\frac{r}{2})-\frac{1}{2}\right], \\
F_y(r,\theta,t &= 0) = 2\tilde{k}\sinh^2(\frac{r}{2})\left[\sin^2\theta\cosh^2(\frac{r}{2})-\frac{1}{2}\right].
\end{aligned} \tag{4.17}$$

Consequently the condition for squeezing at $t = 0$ is

$$\cos\theta\cosh(\frac{r}{2}) < \pm\frac{\sqrt{2}}{2}, \qquad \text{or} \qquad \sin\theta\cosh(\frac{r}{2}) < \pm\frac{\sqrt{2}}{2}. \tag{4.18}$$

However, to discuss the squeezing for $t > 0$ we plot some figures to display the behaviour of the quadrature variances $F_x(t)$ and $F_y(t)$. For this reason we have plotted figures (1) for fixed values of $r = \omega_0 = 1$ and $\theta = \pi/4$ but for different values of the parameters γ and μ. For example, when we consider $\gamma = -0.25$ and $\mu = 0.25$, the phenomenon of squeezing is firstly observed in the second quadrature $F_y(t)$ (solid line), but it is pronounced in the first quadrature $F_x(t)$ (dashed line). Also there are regular fluctuations in both quadratures with a symmetry around $zero$, see Fig.(1a). If we decrease the value of the damping factor and consider $\gamma = -0.5$, we can see reduction in the number of the fluctuations in both quadratures. However, the function $F_x(t)$ decreases its value and this refers to an increase

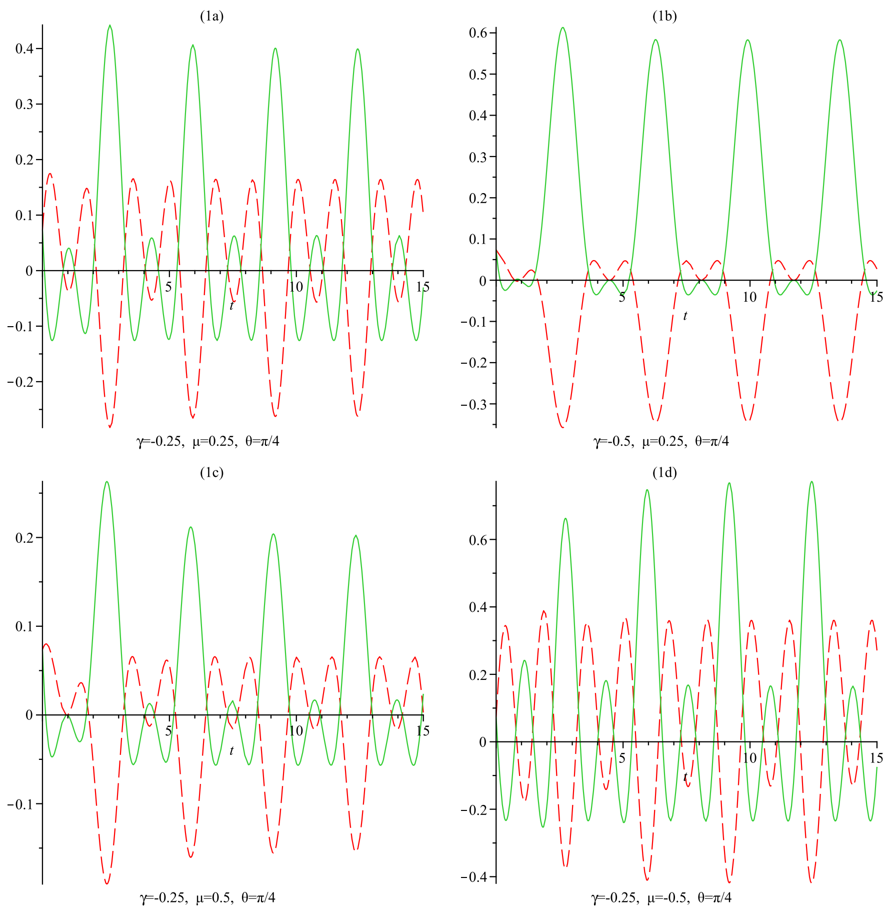

Figure 4.1: F_x (dashed line) and F_y (solid line) against the time t for $r = \omega_0 = 1$ and $\theta = \pi/4$ (a) For $\gamma = -0.25$ and $\mu = 0.25$. (b) As (a) but for $\gamma = -0.5$. (c) As (a) but for $\mu = 0.5$. (d) As (a) but for $\mu = -0.5$.

in the amount of squeezing. In the meantime the function $F_y(t)$ increases its value which refers to reduction in the amount of squeezing in this quadrature, see Fig.(1b). On the other hand, when we consider the case in which $\mu = 0.5$, it is easily to realize that the amount of squeezing in both quadratures is decreased compared with the case displayed in Fig.(1a). However, there is no change in the behaviour of both quadratures, see Fig.(1c). Finally the amount of squeezing is increased in both quadratures when we consider μ with a negative value. In this case and for $\mu = -0.5$ the maximum of the squeezing in the first quadrature approaches the value ~ -0.42 after a considerable period of the time, see Fig.(1d). Here we may point out that more squeezing can be seen as long as we increase the negative value of the parameter μ. However, this increment shows stability when the squeezing reaches its maximum for the case in which $\mu = -2$ (not displayed here).

4.2 The Correlation function

In this Subsection we consider another kind of nonclassical effect, that is the Poissonian distribution. To discuss such a kind of distribution we have to examine the second-order correlation function which leads to better understanding for the nonclassical behaviour of the system. In fact the correlation function is usually used to discuss the sub-Poissonian and super-Poissonian behaviour of the photon distribution from which we can distinguish between the classical and nonclassical behavior. Therefore to discuss the behaviour of the system under consideration we use the Glauber second-order correlation function defined by

$$g^{(2)}(t) = 1 + \frac{\langle (\Delta \hat{n}\ (t)^2 \rangle - \langle \hat{n}(t) \rangle}{\langle \hat{n}(t) \rangle^2}, \tag{4.19}$$

where $\left\langle (\Delta \hat{n}\ (t)^2 \right\rangle$ and $\langle \hat{n}(t) \rangle$ are the variance and the mean photon number at $t > 0$, respectively. To calculate these quantities we have to find the explicit expression of the operator $\hat{a}(t)$ which can be obtained from the solution of the equations of motion in the Heisenberg picture for the Hamiltonian (1.4). After straightforward calculations we can write the operator $\hat{a}(t)$ in the form

$$\hat{a}(t) = \frac{1}{2}\left[(e_3 + e_2) - i\,(e_1 + e_4)\right]\hat{a}(0) + \frac{1}{2}\left[(e_3 - e_2) - i\,(e_1 - e_4)\right]\hat{a}^{\dagger}(0) \tag{4.20}$$

where $e_i, i = 1, 2, 3, 4$ are time-dependent functions given by (4.10). Using this equation together with the Fock state $|n\rangle$ we can obtain the required quantities to discuss the correlation function. Since this is a simple task, we turn our attention to plot the correlation function $g^{(2)}(t)$ against the time t. In this context we have plotted Figures (2) for a fixed value of the frequency $\omega_0 = 1$ and of the squeeze parameter $r = 1$, however, for different values of the other parameters. For instance in Fig.(2a) we plot the correlation function for $\gamma = -0.25, n = 1$ and $\mu = -1$ (solid line) and $\mu = -2$ (dashed line). In both cases the function shows sub-Poissoian behaviour at all values of t. It increases its value as the time increases. However, it is faster for the case in which $\mu = -2$ compared to the case of $\mu = -1$. To examine the effect of the damping factor γ, we consider two cases $\gamma = -0.5$ (solid line) and $\gamma = -0.75$ (dashed line) but for a fixed value of $\mu = -2$. In these two cases the function starts from zero and increases its maximum to display sub-Poissonian, Poissonian and super-Poissonian behaviours for the case in which $\gamma = -0.75$. However, for $\gamma = -0.5$ the function only exhibits sub-Poissonian behaviour, see Fig.(2b). This means that under the negative value of the μ-parameter the correlation function is sensitive to the variation in the damping factor. When we consider the positive value of the μ-parameter, $\mu = 2$ (solid line) and 4 (dashed line) and take $\gamma = -0.75$ and $n = 1$, the function reaches its maximum for both cases after the onset of the interaction showing super-Poissonian

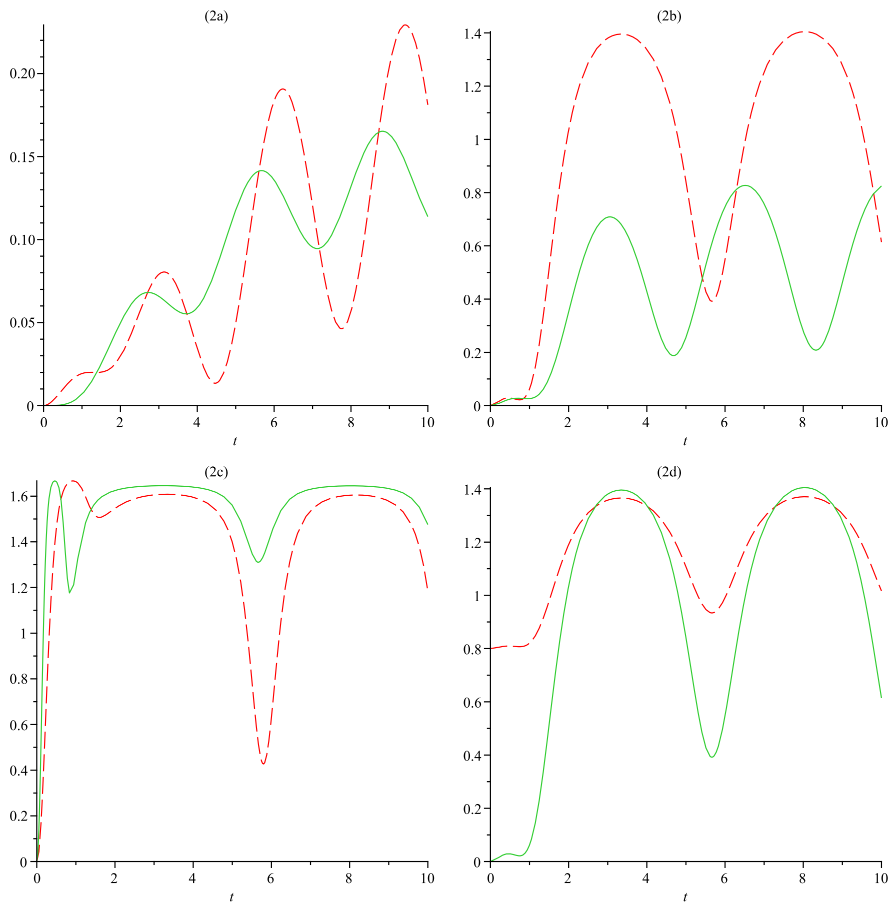

Figure 4.2: The correlation function $g^{(2)}$ against the time t. (a) For $n = 1$, $\gamma = -0.25$ and $\mu = -1$ and $\mu = -2$. (b) For $n = 1$, $\mu = -2$ and $\gamma = -0.75$ and $\gamma = -0.5$. (c) For $n = 1$, $\gamma = -0.75$ and $\mu = 2$ and $\mu = 4$. (d) For $\gamma = -0.75, \mu = -2$ and $n = 1$ and $.n = 5$.

behaviour. However it turns to display sub-Poissonian behaviour at the middle of the considered time just for the case in which $\mu = 4$. This means that the function is also affected by the variation in the μ-parameter, see Fig.(2c). Finally we examined the effect of the photon number for the cases in which $n = 1$ and 5. This is displayed in Fig.(2d) in which the function increased its minimum for $n = 5$ (dashed line). Also we can observe from this figure that a decrease in the value of the damping parameter leads to an increase in the function value. This can be realized if one makes a comparison between Fig.(2a) and Fig.(2d) for the case in which $\mu = -2$.

5 Conclusion

In the previous sections of the present paper we have introduced a model for modulated damping or growth of a harmonic oscillator under the effect of an external driving force. The model is totally different to those previously introduced in the literature involving two different parameters. The wave function in the pseudostationary and quasicoherent states are obtained. Furthermore two classes of quadratic invariants are obtained and the eigenfunction as well as the corresponding eigenvalue for each class are given. Finally we have considered the phenomenon of squeezing for which we employed the Perelomov SU$(1,1)$ coherent state to calculate the quadrature variances $F_x(t)$ and $F_y(t)$. The phenomenon is observed in both quadratures. However, it is more pronounced in the first quadrature compared with the second quadrature. Also we have realized that the system is sensitive to variation in the damping factor γ as well as in the μ-parameter. Moreover we have discussed the behaviour of the correlation function and examined the effect of the damping factor as well as the μ-parameter in addition to the photon number on its behaviour. It was shown that the γ factor and the μ-parameter are the most effective parameters on the function behaviour.

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