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The Kumaraswamy Power Function Distribution

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Abstract: In this paper, Kumaraswamy Power function distribution (Kw-PFD) will be obtained. We give some properties for this distribution. The r^{th} traditional moments, TL-moments, L-moments are derived. The maximum likelihood estimation of the unknown parameters is discussed. Real data are used to determine whether the Kw-PFD is better than PFD or not.

Keywords: Kumaraswamy distribution, Power function distribution ,Traditional moments, TL-moments, L-moments.

1 Introduction

The Power function distribution (PFD) is most flexible distribution. It is usually used for the reliability analysis, life time and income distribution data. Meniconi and Barry [15] compare the PFD with Exponential, Lognormal and Weibull distribution to measure the reliability of electrical component. They conclude that the PFD is the best distribution.

The probability density function of PFD type-II (PFD-II) with shape parameter θ and scale parameter λ is as

$$g(x;\theta,\lambda) = \frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1}; \quad \theta > 0,\ 0 \le x \le \lambda. \tag{1}$$

If the scale parameter is one in (1) then PFD-II becomes PFD type-I (PFD-I) with only shape parameter. The cumulative distribution function is

$$G(x;\theta,\lambda) = \left(\frac{x}{\lambda}\right)^{\theta}; \quad \theta > 0,\ 0 \le x \le \lambda. \tag{2}$$

More details on this distribution and its applications can be found in Ahsanullah and Lutful-Kabir [2], Meniconi and Barry [15], Ali et al. [3], Chang [4], Sinha et al. [21] and Tavangar [23].

A random variable X is said to have a Kumaraswamy distribution (KD) if its probability density function is (*pdf*) in the form:

$$f(x;a,b) = abx^{a-1}(1-x^a)^{b-1};\ 0 \le x \le 1, \quad a,b > 0. \tag{3}$$

The cumulative distribution function (*CDF*) is:

$$F(x;a,b) = 1-(1-x^a)^b; \quad 0 \le x \le 1, \quad a,b > 0, \tag{4}$$

More details on this distribution and its applications can be found in Kumaraswamy [13], Sundar and Subbiah [22], Fletcher and Ponnambalam [11], Seifi et al. [20], Ganji et al. [12], Sanchez et al. [18] and Courard-Hauri [8]. Cordeiro and de Castro [5] introduced a new method of adding a parameter into a family of distributions. According to them if $G(x)$ denote the *CDF* of a continuous random variable X, then a generalized class of distributions can be defined by

$$F(x) = 1-[1-(G(x))^a]^b \tag{5}$$

where $a > 0$ and $b > 0$ are two additional shape parameters which govern skewness and tail weights. The *pdf* and hazard rate function (*HRF*) corresponding to $F(x)$are:

$$f(x) = ab(G(x))^{a-1}g(x)[1-(G(x))^a]^{b-1} \tag{6}$$

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and

$$r(x) = \frac{ab(G(x))^{a-1} g(x)}{1-(G(x))^{a}}. \tag{7}$$

2 The Kw-PF distribution

Here, we substituting from (1) and (2) in (6) to obtain a new distribution called Kumaraswamy power function distribution (*Kw-PFD*) as follows

$$f(x) = \frac{ab\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{a\theta-1}\left[1-\left(\frac{x}{\lambda}\right)^{a\theta}\right]^{b-1}; \quad a,b,\theta>0,\ 0\le x\le\lambda \tag{8}$$

The *CDF*, *SF* and *HRF* corresponding to $f(x)$are:

$$F(x) = 1-\left[1-\left(\frac{x}{\lambda}\right)^{a\theta}\right]^{b}; \quad a,b,\theta>0,\ 0\le x\le\lambda, \tag{9}$$

$$\bar{F}(x) = \left[1-\left(\frac{x}{\lambda}\right)^{a\theta}\right]^{b}; \quad a,b,\theta>0,\ 0\le x\le\lambda, \tag{10}$$

and

$$r(x) = \frac{ab\theta x^{a\theta-1}}{\left[\lambda^{a\theta}-x^{a\theta}\right]}; \quad a,b,\theta>0,\ 0\le x\le\lambda \tag{11}$$

Many distributions are proposed using the same technique, the Kw-Weibull, Cordeiro et al. [6], Kw-generalized gamma, Pascoa et al. [17], Kw-Birnbaum-Saunders, Saulo et al. [19] and Kw-Gumbel, Cordeiro et al. [7]. Nadarajah et al. [16], studied general results for the Kumaraswamy-G distribution.

In Figures 1 and 2, we plot the density and failure rate functions of the Kw-PF distribution for selected parameter values, respectively.

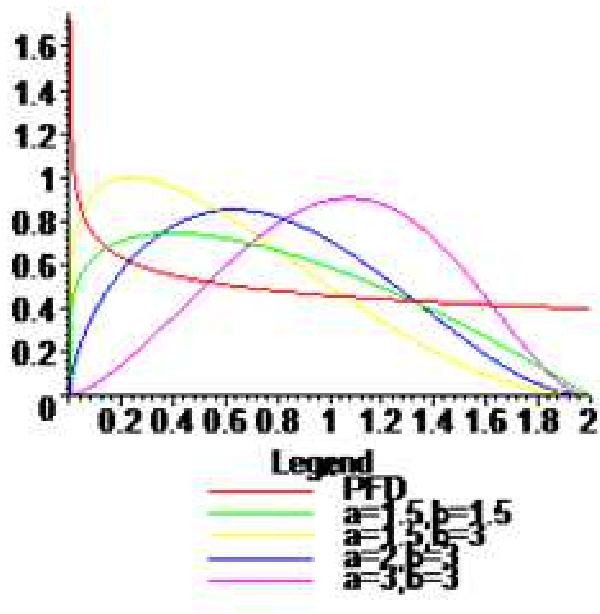

Fig. 1: Plot of the density function at $\theta = 0.8$ and $\lambda = 2$ with different values of a and b.

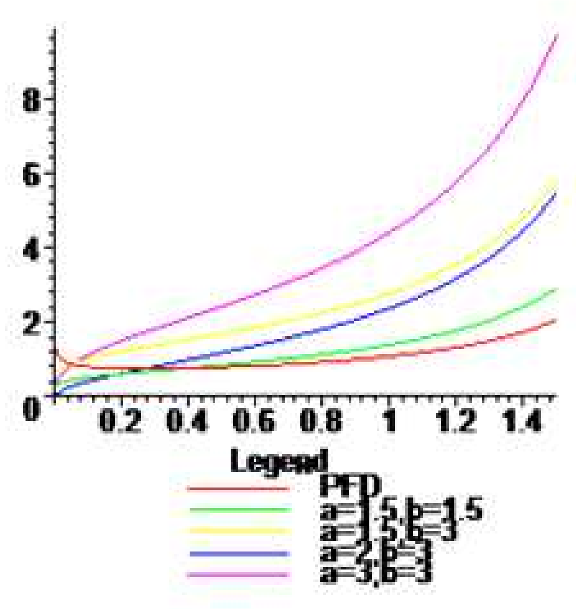

Fig. 2: Plot of the failure rate function at $\theta = 0.8$ and $\lambda = 2$ with different values of a and b.

In view of (8) and (10), we have

$$\bar{F}(x) = \frac{1}{ab\theta}\left[\lambda^{a\theta}x^{1-a\theta} - x\right] f(x). \tag{12}$$

The inverse of the distribution function (9) yields a very simple quantile function

$$Q(y) = \lambda\left[1 - [1-y]^{\frac{1}{b}}\right]^{\frac{1}{a\theta}}; \qquad y \in (0,1), \tag{13}$$

which facilitates ready quantile-based statistical modeling. In addition, $Q(y)$ gives a trivial random variable generation. If $U \sim u(0,1)$, then $X \sim Kw - PFD(a,b,\theta,\lambda)$ is given by

$$X = \lambda\left[1 - [1-U]^{\frac{1}{b}}\right]^{\frac{1}{a\theta}}. \tag{14}$$

The mode for *Kw-PFD* M can be obtained as follows:

$$M = \lambda\left[\frac{a\theta-1}{ab\theta-1}\right]^{\frac{1}{a\theta}}., \tag{15}$$

and the median for *Kw-PFD* is

$$X_{0.5} = \lambda\left[1 - 2^{-\frac{1}{b}}\right]^{\frac{1}{a\theta}}. \tag{16}$$

3 Moments for *Kw-PFD*

3.1 Traditional Moments for Kw-PFD

The r^{th} traditional moments for *Kw-PFD* is

$$\mu_r' = E(X^r) = \frac{ab\theta}{\lambda}\int_0^{\lambda} x^r \left(\frac{x}{\lambda}\right)^{a\theta-1}\left[1-\left(\frac{x}{\lambda}\right)^{a\theta}\right]^{b-1} dx$$

Using the transformation $y=\left(\frac{x}{\lambda}\right)^{a\theta}$, we get

$$\begin{aligned}\mu_r' &= b\int_0^1 \left(\lambda y^{\frac{1}{a\theta}}\right)^r [1-y]^{b-1}\,dy \\ &= b\lambda^r \beta\left(\frac{r}{a\theta}+1,b\right), \qquad r=1,2,\ldots\end{aligned} \tag{17}$$

The first two moments can be obtained by taking r = 1 and 2 in (17) as follows:

$$\mu_1' = b\lambda\beta\left(\frac{1}{a\theta}+1,b\right), \tag{18}$$

and

$$\mu_2' = b\lambda^2\beta\left(\frac{2}{a\theta}+1,b\right). \tag{19}$$

The variance and coefficient of variation (CV) for *Kw-PFD* are

$$\text{Var}\,(\text{X}) = b\lambda^2\beta\left(\frac{2}{a\theta}+1,b\right) - b^2\lambda^2\left[\beta\left(\frac{1}{a\theta}+1,b\right)\right]^2, \tag{20}$$

and

$$\text{CV} = \frac{\sqrt{\beta\left(\frac{2}{a\theta}+1,b\right) - b\left[\beta\left(\frac{1}{a\theta}+1,b\right)\right]^2}}{\sqrt{b}\,\beta\left(\frac{1}{a\theta}+1,b\right)}. \tag{21}$$

3.2 TL-moments for Kw-PFD

In this section, we investigate the population TL-moment of order r for *Kw-PFD*. The r^{th} TL-moments is given in the following formula (Elamir and Seheult [10])

$$L_r^{(s,t)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}E\left(X_{r+s-k:r+s+t}\right), \tag{22}$$

Where r, s and t takes the values 1, 2, 3, ... Maillet and Me'decin [14] introduced an important relation between the first TL-moments $L_1^{(s,t)}$ and the r^{th}TL-moments$L_r^{(s,t)}$ as follows:

$$L_r^{(s,t)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k\binom{r-1}{k}L_1^{(r+s-k-1,t+k)}, r=2,3,\ldots \tag{23}$$

$$\begin{aligned}E\left(X_{r+s-k:r+s+t}\right) &= \frac{(r+s+t)!}{(r+s-k-1)!\,(t+k)!}\int_{-\infty}^{\infty} x\,[F(x)]^{r+s-k-1} f(x)\,[1-F(x)]^{t+k}\,dx \\ &= \frac{\frac{ab\theta}{\lambda}(r+s+t)!}{(r+s-k-1)!\,(t+k)!}\int_0^\lambda x\left[1-\left[1-\left(\frac{x}{\lambda}\right)^{a\theta}\right]^b\right]^{r+s-k-1}\left(\frac{x}{\lambda}\right)^{a\theta-1}\left[1-\left(\frac{x}{\lambda}\right)^{a\theta}\right]^{b\,(t+k+1)-1}dx \\ &= \frac{\frac{ab\theta}{\lambda}(r+s+t)!}{(r+s-k-1)!\,(t+k)!}\sum_{j=0}^{r+s-k-1}\binom{r+s-k-1}{j}(-1)^j\int_0^\lambda x\left(\frac{x}{\lambda}\right)^{a\theta-1}\left[1-\left(\frac{x}{\lambda}\right)^{a\theta}\right]^{b\,(t+k+j+1)-1}dx\end{aligned}$$

Using the transformation $y=\left(\frac{x}{\lambda}\right)^{a\theta}$, we get

$$\begin{aligned}E\left(X_{r+s-k:r+s+t}\right) &= \frac{b\lambda(r+s+t)!}{(r+s-k-1)!(t+k)!}\sum_{j=0}^{r+s-k-1}\binom{r+s-k-1}{j}(-1)^j\int_0^1 y^{\frac{1}{a\theta}}(1-y)^{b^{(t+k+j+1)-1}}dy \\ &= \frac{b\lambda(r+s+t)!}{(r+s-k-1)!(t+k)!}\sum_{j=0}^{r+s-k-1}\binom{r+s-k-1}{j}(-1)^j\beta\left(\frac{1}{a\theta}+1,b\,(t+k+j+1)\right)\end{aligned}$$

Then the r^{th} TL-moments for *Kw-PFD* is

$$L_r^{(s,t)} = \frac{1}{r}\sum_{k=0}^{r-1}\sum_{j=0}^{r+s-k-1}\binom{r+s-k-1}{j}\binom{r-1}{k}\frac{b\lambda\,(r+s+t)!}{(r+s-k-1)!\,(t+k)!}(-1)^{k+j}\beta\left(\frac{1}{a\theta}+1, b\,(t+k+j+1)\right), \quad (24)$$

where r, s, t = 1, 2, 3, ... and s + $t < n$.

The first three TL-moments can be obtained by taking $r=1$ in (24) and using (23) as follows

$$L_1^{(s,t)} = \sum_{j=0}^{s}\binom{s}{j}\frac{b\lambda\,(s+t+1)!}{s!t!}(-1)^{j}\beta\left(\frac{1}{a\theta}+1, b\,(t+j+1)\right), \quad (25)$$

$$\begin{aligned} L_2^{(s,t)} &= \frac{1}{2}\left[L_1^{(s+1,t)} - L_1^{(s,t+1)}\right] \\ &= \frac{b\lambda(s+t+2)!}{2s!t!}\left\{\frac{1}{(s+1)}\sum_{j=0}^{s+1}\binom{+1s}{j}(-1)^j\beta\left(\frac{1}{a\theta}+1, b(t+j+1)\right) - \frac{1}{(t+1)}\sum_{j=0}^{s}\binom{s}{j}(-1)^j\beta\left(\frac{1}{a\theta}+1, b(t+j+2)\right)\right\} \\ &= \frac{b\lambda(s+t+2)!}{2s!t!}\left\{\frac{1}{(s+1)}\beta\left(\frac{1}{a\theta}+1, b(t+1)\right) - \sum_{j=0}^{s}\binom{s}{j}\frac{(t+j+2)(-1)^j}{(t+1)(j+1)}\beta\left(\frac{1}{a\theta}+1, b(t+j+2)\right)\right\}, \end{aligned} \quad (26)$$

and

$$\begin{aligned} L_3^{(s,t)} &= \tfrac{1}{3}\left[L_1^{(s+2,t)} - 2L_1^{(s+1,t+1)} + L_1^{(s,t+2)}\right] \\ &= \tfrac{b\lambda(s+t+3)!}{3s!t!}\left\{\tfrac{1}{(s+1)(s+2)}\textstyle\sum_{j=0}^{s+2}\binom{s+2}{j}(-1)^j\beta\left(\tfrac{1}{a\theta}+1, b\,(t+j+1)\right)\right. \\ &\quad - \tfrac{2}{(s+1)(t+1)}\textstyle\sum_{j=0}^{s+1}\binom{s+1}{j}(-1)^j\beta\left(\tfrac{1}{a\theta}+1, b\,(t+j+2)\right) \\ &\quad \left.+ \tfrac{1}{(t+1)(t+2)}\textstyle\sum_{j=0}^{s}\binom{s}{j}(-1)^j\beta\left(\tfrac{1}{a\theta}+1, b\,(t+j+3)\right)\right\} \end{aligned}$$

$$\begin{aligned} &= \frac{b\lambda(s+t+3)!}{3s!t!}\left\{\frac{1}{(s+1)(s+2)}\beta\left(\frac{1}{a\theta}+1, b(t+1)\right) - \frac{t+3}{(s+1)(t+1)}\beta\left(\frac{1}{a\theta}+1, b(t+2)\right) + \right. \\ &\left. \sum_{j=0}^{s}\binom{s}{j}\frac{(t+j+3)(t+j+4)(-1)^j}{(j+1)(j+2)(t+1)(t+2)}\beta\left(\frac{1}{a\theta}+1, b(t+j+3)\right)\right\} \end{aligned} \quad (27)$$

Special cases

1. Symmetric trimmed$(s=t)$:

In this case the r^{th} and the first three TL-moments are in the forms:

$$L_r^{(s,s)} = \frac{1}{r}\sum_{k=0}^{r-1}\sum_{j=0}^{r+s-k-1}\binom{r+s-k-1}{j}\binom{r-1}{k}\frac{b\lambda\,(r+2s)!}{(r+s-k-1)!\,(s+k)!}(-1)^{k+j}\beta\left(\frac{1}{a\theta}+1, b\,(s+k+j+1)\right), \quad (28)$$

$$L_1^{(s,s)} = \sum_{j=0}^{s}\binom{s}{j}\frac{b\lambda\,(2s+1)!}{(s!)^2}(-1)^{j}\beta\left(\frac{1}{a\theta}+1, b\,(s+j+1)\right), \quad (29)$$

$$\begin{aligned} L_2^{(s,s)} &= \frac{1}{2}\left[L_1^{(s+1,s)} - L_1^{(s,s+1)}\right] \\ &= \frac{b\lambda\,(2s+2)!}{2\,(s!)^2}\left\{\frac{1}{(s+1)}\beta\left(\frac{1}{a\theta}+1, b\,(s+1)\right) - \sum_{j=0}^{s}\binom{s}{j}\frac{(s+j+2)\,(-1)^j}{(s+1)\,(j+1)}\beta\left(\frac{1}{a\theta}+1, b\,(s+j+2)\right)\right\}, \end{aligned} \quad (30)$$

and

$$\begin{aligned} L_3^{(s,s)} =& \frac{1}{3}\left[L_1^{(s+2,s)} - 2L_1^{(s+1,s+1)} + L_1^{(s,s+2)}\right] \\ &= \frac{b\lambda\,(2s+3)!}{3\,(s!)^2}\left\{\frac{1}{(s+1)\,(s+2)}\beta\left(\frac{1}{a\theta}+1, b\,(s+1)\right) - \frac{s+3}{(s+1)^2}\beta\left(\frac{1}{a\theta}+1, b\,(s+2)\right) + \right. \\ &\left. \sum_{j=0}^{s}\binom{s}{j}\frac{(s+j+3)\,(s+j+4)\,(-1)^j}{(j+1)\,(j+2)\,(s+1)\,(s+2)}\beta\left(\frac{1}{a\theta}+1, b\,(s+j+3)\right)\right\} \end{aligned} \tag{31}$$

1.2. Lower trimmed$(t=0)$:

In this case the r^{th} and the first three TL-moments are as follows:

$$L_r^{(s,0)} = \frac{1}{r}\sum_{k=0}^{r-1}\sum_{j=0}^{r+s-k-1}\binom{r+s-k-1}{j}\binom{r-1}{k}\frac{b\lambda\,(r+s)!}{(r+s-k-1)!k!}(-1)^{k+j}\beta\left(\frac{1}{a\theta}+1, b\,(k+j+1)\right), \tag{32}$$

$$L_1^{(s,0)} = \sum_{j=0}^{s}\binom{s}{j}\frac{b\lambda\,(s+1)!}{s!}(-1)^j\beta\left(\frac{1}{a\theta}+1, b\,(j+1)\right), \tag{33}$$

$$\begin{aligned} L_2^{(s,0)} &= \frac{1}{2}\left[L_1^{(s+1,0)} - L_1^{(s,1)}\right] \\ &= \frac{b\lambda\,(s+2)!}{2s!}\left\{\frac{1}{(s+1)}\beta\left(\frac{1}{a\theta}+1, b\right) - \sum_{j=0}^{s}\binom{s}{j}\frac{(j+2)\,(-1)^j}{(j+1)}\beta\left(\frac{1}{a\theta}+1, b\,(j+2)\right)\right\}, \end{aligned} \tag{34}$$

and

$$\begin{aligned} L_3^{(s,0)} &= \frac{1}{3}\left[L_1^{(s+2,0)} - 2L_1^{(s+1,1)} + L_1^{(s,2)}\right] \\ &= \frac{b\lambda\,(s+3)!}{3s!}\left\{\frac{1}{(s+1)\,(s+2)}\beta\left(\frac{1}{a\theta}+1, b\right) - \frac{3}{(s+1)}\beta\left(\frac{1}{a\theta}+1, 2b\right) + \right. \\ &\left. \sum_{j=0}^{s}\binom{s}{j}\frac{(j+3)\,(j+4)\,(-1)^j}{2\,(j+1)\,(j+2)}\beta\left(\frac{1}{a\theta}+1, b\,(j+3)\right)\right\} \end{aligned} \tag{35}$$

1.3. Upper trimmed$(s=0)$:

In this case the r^{th} and the first three TL-moments are as follows:

$$L_r^{(0,t)} = \frac{1}{r}\sum_{k=0}^{r-1}\sum_{j=0}^{r-k-1}\binom{r-k-1}{j}\binom{r-1}{k}\frac{b\lambda\,(r+t)!}{(r-k-1)!\,(t+k)!}(-1)^{k+j}\beta\left(\frac{1}{a\theta}+1, b\,(t+k+j+1)\right), \tag{36}$$

$$L_1^{(0,t)} = b\lambda\,(t+1)\,\beta\left(\frac{1}{a\theta}+1, b\,(t+1)\right), \tag{37}$$

$$L_2^{(0,t)} = \frac{1}{2}\left[L_1^{(1,t)} - L_1^{(0,t+1)}\right] = \frac{b\lambda\,(t+2)!}{2t!}\left\{\beta\left(\frac{1}{a\theta}+1, b\,(t+1)\right) - \frac{(t+2)}{(t+1)}\beta\left(\frac{1}{a\theta}+1, b\,(t+2)\right)\right\}, \tag{38}$$

and

$$\begin{aligned} L_3^{(0,t)} =& \frac{1}{3}\left[L_1^{(2,t)} - 2L_1^{(1,t+1)} + L_1^{(0,t+2)}\right] \\ & \frac{b\lambda(t+3)!}{3t!}\left\{\frac{1}{2}\beta\left(\frac{1}{a\theta}+1, b(t+1)\right) - \frac{t+3}{(t+1)}\beta\left(\frac{1}{a\theta}+1, b(t+2)\right) + \frac{(t+3)\,(t+4)}{2\,(t+1)\,(t+2)}\beta\left(\frac{1}{a\theta}+1, b\,(t+3)\right)\right\} \end{aligned} \tag{39}$$

3.3 L-moments for Kw-PFD

In this section, we discuss the population L-moment of order r for the *Kw-PFD* as a special case of formula (24). Taking $s=t=0$ in (24), the population L-moment of order r for the *Kw-PFD*:

$$L_r = \sum_{k=0}^{r-1}\sum_{j=0}^{r-k-1}\binom{r-k-1}{j}\binom{r-1}{k}\frac{b\lambda\,(r-1)!}{(r-k-1)!k!}(-1)^{k+j}\beta\left(\frac{1}{a\theta}+1,b\,(k+j+1)\right) \tag{40}$$

The first three L-moments can be obtained by taking $r=1,\ 2,\ 3$ in (40) as follows

$$L_1 = b\lambda\beta\left(\frac{1}{a\theta}+1,b\right), \tag{41}$$

$$L_2 = b\lambda\left\{\beta\left(\frac{1}{a\theta}+1,b\right)-\ 2\beta\left(\frac{1}{a\theta}+1,2b\right)\right\}, \tag{42}$$

and

$$L_3 = 2b\lambda\left\{\frac{1}{2}\beta\left(\frac{1}{a\theta}+1,b\right)-3\beta\left(\frac{1}{a\theta}+1,2b\right)+3\beta\left(\frac{1}{a\theta}+1,3b\right)\right\}. \tag{43}$$

L-Moments Coefficient of Variations for the *Kw-PFD* (L-CV) is

$$L-CV = \frac{L_2}{L_1} = \frac{\left\{\beta\left(\frac{1}{a\theta}+1,b\right)-\ 2\beta\left(\frac{1}{a\theta}+1,2b\right)\right\}}{\beta\left(\frac{1}{a\theta}+1,b\right)} = 1-\frac{2\Gamma\left(\frac{1}{a\theta}+b+1\right)\Gamma\,(2b)}{\Gamma\left(\frac{1}{a\theta}+2b+1\right)\Gamma\,(b)}. \tag{44}$$

Abdul-Moniem [1], show that the sample of coefficient of variation (l-cv) estimator of the population L-CV is more accurate than the corresponding CV.

4 Parameters Estimators

In this section, we consider maximum likelihood estimators (MLE) of *Kw-PFD*. Let $x_1,x_2,\ldots,x_n$be a random sample of size n from *Kw-PFD*, then the log-likelihood function $L\,(a,b,\theta,\lambda)$ can be written as

$$L\,(a,b,\theta,\lambda) \propto n\,[\ln(a)+\ln(b)+\ln(\theta)-\ln(\lambda)]+(b-1)\sum_{i=1}^{n}\ln\left[1-\left(\frac{x_i}{\lambda}\right)^{a\theta}\right]+(a\theta-1)\sum_{i=1}^{n}\left[\ln(x_i)-\ln(\lambda)\right].$$

The normal equations become

$$\frac{\partial\ln L}{\partial a} = \frac{n}{a}-\theta\,(b-1)\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\lambda}\right)^{a\theta}\ln\left(\frac{x_i}{\lambda}\right)}{1-\left(\frac{x_i}{\lambda}\right)^{a\theta}}+\theta\sum_{i=1}^{n}\left[\ln(x_i)-\ln(\lambda)\right]=0, \tag{45}$$

$$\frac{\partial\ln L}{\partial b} = \frac{n}{b}+\sum_{i=1}^{n}\ln\left[1-\left(\frac{x_i}{\lambda}\right)^{a\theta}\right]=0, \tag{46}$$

$$\frac{\partial\ln L}{\partial\theta} = \frac{n}{\theta}-a\,(b-1)\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\lambda}\right)^{a\theta}\ln\left(\frac{x_i}{\lambda}\right)}{1-\left(\frac{x_i}{\lambda}\right)^{a\theta}}+a\sum_{i=1}^{n}\ln\left(\frac{x_i}{\lambda}\right)=0, \tag{47}$$

Since $x\le\lambda$, the MLE of λ is the last-order statistic $x_{(n)}$. The MLE of $a,\ b$ and θ can be obtain by solving the equations (45), (46), and (47) .

5 Application of *Kw-PFD*

In this section, we use a real data set to show that the *Kw-PFD* can be a better model than one based on the PFD. We consider a data set of the Plasma concentrations of indomethicin introduced by De Morais [9].

Table 1: Estimated parameters of the Kw-PFD and PFD.

Model	Parameters Estimates	- Log-likelihood
Kw-PFD	$\hat{a} = 0.52196, \hat{b} = 2.5223,\ \hat{\theta} = 1.52408,\ \hat{\lambda} = \max(x) = 2.72$	28.505
PFD	$\hat{\theta} = 0.486,\ \hat{\lambda} = \max(x) = 2.72$	41.833

The LR statistics to test the hypotheses $H_0:\ a = b = 1$ versus $H_1:\ a \neq 1, b \neq 1$ $\omega = 26.66 > 6.635 = \chi_1^2\,(\alpha = 0.01)$, so we reject the null hypotheses.

In order to compare the distributions, we consider some other criterion like K-S (Kolmogorow-Smirnov), -2log(L), AIC (Akaike Information Criterion), AICC (Akaike Information Criterion Corrected) and BIC (Bayesian information criterion) for the real data set. The best distribution corresponds to lower K-S, -2 log(L), AIC, AICC and BIC values:

$$KS = \max_{1 \le i \le n} \left(F(x_i) - \frac{i-1}{n}, \frac{i}{n} - F(x_i) \right), \qquad AIC = 2k - 2\log(L),$$

$$AICC = AIC + \frac{2k(k+1)}{n-k-1}, \quad \text{and} \quad BIC = k\log(n) - 2\log(L),$$

where k is the number of parameters in the statistical model, n the sample size and L is the maximized value of the likelihood function for the estimated model. Also, here for calculating the values of K-S we use the sample estimates of a, b, θ and λ. Table 1 shows parameter MLEs to each one of the two fitted distributions, Table 2 shows the values of K-S, -2 log(L), AIC, AICC and BIC values.

Table 2: Criteria for comparison.

Model	K-S	$-2L$	AIC	AICC	BIC
Kw-PFD	0.124	57.01	65.01	65.67	64.288
PFD	0.192	83.67	87.67	87.86	87.309

The values in Table 2 indicate that the *Kw-PFD* leads to a better fit than the *PFD*.

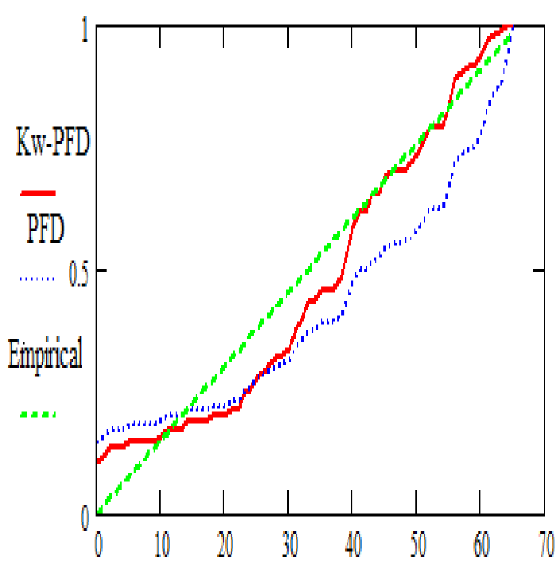

Fig. 3: Empirical, fitted PFD and Kw-PFD CDF of Plasma concentrations of indomethicin

6 Conclusion

In this article, we introduce a new generalization of the Power function distribution called Kumaraswamy Power function distribution and presented its theoretical properties. The estimation of parameters is approached by the method of maximum likelihood. We consider the likelihood ratio statistic to compare the model with its baseline model. An application of the Kumaraswamy Power function distribution to real data show that the new distribution can be used quite effectively to provide better than the Power function distribution.

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