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A New Generalized Burr Family of Distributions for the Lifetime Data

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Abstract: We propose a new family of distributions called the *generalized Burr-G (GBG) family* of distributions motivated mainly for lifetime phenomenon. Some mathematical properties of the new family are obtained such as quantile function, linear representation of the family density, moments and incomplete moments, moment generating function, mean deviations, stochastic ordering, stress-strength reliability parameter and order statistics. The model parameters are estimated by the method of maximum likelihood for complete and censored samples. Four special models are discussed and the properties of one special model, the *generalized Burr-uniform (GBU)*, are obtained. A simulation study is carried out to check the performance of maximum likelihood estimators. The usefulness of GBU model is proved empirically by means of three real lifetime applications to complete and censored samples.

Keywords: Burr XII distribution, G-class of distributions, quantile function, maximum Likelihood estimation, uniform distribution.

1 Introduction

The Burr system of distributions was proposed by Burr (1942) which offers a variety of density shapes. Some well-known standard distributions are limiting forms of the Burr system of distributions. The Burr XII distribution has logistic and Weibull distributions as sub-models which are very popular model for modeling lifetime phenomenon with monotone failure rates. When modeling monotone (increasing or decreasing) hazard rates, the Weibull distribution may be the initial choice because its exhibits increasing and decreasing failure rate shapes. But Weibull model does not provide a reasonable fit for modeling phenomenon with non-monotone failure (or hazard) rates such as the bathtub (BT) or upside-down bathtub (UBT) (or unimodal) which are common in reliability and biological studies. UBT hazard rates can be observed in course of a disease whose mortality reaches a peak after some finite period and then declines gradually.

In the past few decades, several methods for generating new probability distributions have been proposed by using the logit of baseline distribution. Some well-known generalized (G-) classes (or generators) are: beta-G (Eugene et al., 2002; Jones, 2004), Kumaraswamy-G (Cordeiro and de-Castro, 2011), McDonald-G (Alexander et al., 2012), Kummer beta-G (Pescim et al., 2012), gamma-G (Zografos and Balakrishnan, 2009; Ristić and Balakrishnan, 2012; Torabi and Montazari, 2012), log-gamma-G (Amini et al., 2012), logistic-G (Torabi and Montazari, 2014; Tahir et al., 2016a), beta extended Weibull-G (Cordeiro et al., 2012), exponentiated generalized-G (Cordeiro et al., 2013), Transformed-Transformer (T-X) family (Alzaatreh et al., 2013), exponentiated T–X (Alzaghal et al., 2013), Weibull-G (Alzaatreh et al., 2013; Bourguignon et al., 2014; Tahir et al., 2016b), Odd Burr III-G (Jamal et al., 2017), exponentiated half–logistic-G (Cordeiro et al., 2014) and odd generalized-exponential-G (Tahir et al., 2015). A review on well-known G-classes is reported in Tahir and Nadarajah (2015).

Alzaatreh et al. (2013) proposed the *T–X* family of distributions as a general method for generating new family of distributions as follows:

Let $r(t)$ be the probability density function (pdf) and $R(t)$ be the cumulative distribution function (cdf) of a random variable (rv) $T \in [a,b]$ for $-\infty < a < b < \infty$ and let $W[G(x)]$ be a function of the cdf $G(x)$ of some baseline rv X so that

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$W[G(x)]$ satisfies the following conditions:

(i) $W[G(x)] \in [a,b]$,

(ii) $W[G(x)]$ is differentiable and monotonically non-decreasing, and

(iii) $\lim_{x\to-\infty} W[G(x)] = a$ and $\lim_{x\to\infty} W[G(x)] = b$.

The cdf of the *T–X* family is defined by

$$F(x) = \int_a^{W[G(x)]} r(t)\,\mathrm{d}t = R\Big(W[G(x)]\Big), \tag{1.1}$$

where $W[G(x)]$ satisfies the conditions (i)–(iii).

The probability density function (pdf) corresponding to (1.1) reduces to

$$f(x) = r\Big(W[G(x)]\Big)\frac{\mathrm{d}}{\mathrm{d}x}W[G(x)]. \tag{1.2}$$

Let $G(x)$, $g(x)$, $\overline{G}(x) = 1 - G(x)$ and $Q_G(p) = G^{-1}(p)$ be the cdf, pdf, survival function (sf) and quantile function (qf) of any baseline rv X. Then the *generalized Burr-G (GBG) family* of distributions can be defined by using the generator $W[G(x)] = -\log\{1 - G(x;\xi)\}$ which is the quantile function of the standard exponential distribution. If $r(t) = c\,k\,t^{c-1}(1+t^c)^{-k}$ is the pdf of the BXII distribution, then the cdf of the GBG family is defined by

$$F(x;c,k,\boldsymbol{\xi}) = c\,k\int_0^{-\log[\overline{G}(x;\boldsymbol{\xi})]} t^{c-1}(1+t^c)^{-k}dt = 1 - \left(1 + \left\{-\log[\overline{G}(x;\boldsymbol{\xi})]\right\}^c\right)^{-k}. \tag{1.3}$$

The pdf corresponding to Eq. (1.3) is given by

$$f(x;c,k,\boldsymbol{\xi}) = c\,k\,\frac{g(x;\xi)}{1-G(x;\boldsymbol{\xi})}\left\{-\log[\overline{G}(x;\boldsymbol{\xi})]\right\}^{c-1}\left(1 + \left\{-\log[\overline{G}(x;\boldsymbol{\xi})]\right\}^c\right)^{-k-1}. \tag{1.4}$$

Henceforth, a random variable with density (1.4) is denoted by $X \sim \mathrm{GBG}(c,k,\boldsymbol{\xi})$.

The qf has widespread use in general statistics and often find representations in terms of lookup tables for key percentiles. The qf $Q(u)$ can be determined by inverting Eq. (1.3). If $Q_G(u) = X$ is the baseline qf, then

$$Q_x(u) = G^{-1}\left(1 - e^{-[(1-u)^{-\frac{1}{k}}-1]^{\frac{1}{c}}}\right). \tag{1.5}$$

The hazard rate function (hrf) of the GBG family given by

$$h(x) = \frac{c\,k\,g(x;\boldsymbol{\xi})\,\{-\log[1-G(x;\boldsymbol{\xi})]\}^{c-1}}{[1-G(x;\boldsymbol{\xi})]\left[1+\{-\log\,(1-G(x;\boldsymbol{\xi}))\}^c\right]}.$$

The motivation of this family is to obtain more flexible models with less number of parameters and to improve goodness-of-fits of the model by inducting two additional shapes parameters. Furthermore, the basic motivations for proposing GBG family in practice are:

(i) to make the kurtosis more flexible as compared to the baseline model,

(ii) to produce skewness for symmetrical distributions,

(iii) to construct heavy-tailed distributions that are not longer-tailed for modeling real data,

(vi) to generate distributions with symmetric, left-skewed, right-skewed and reversed-J shaped,

(v) to define special models with all types of hazard rates.

The rest of the paper is organized as follows. In Section 2, some general mathematical properties of the GBG family are obtained such as analytical expression for the density and hazard rate shapes, linear representation of GBG density, moments and incomplete moments, moment generating function, stress-strength reliability parameter, stochastic ordering and explicit expression for the density of order statistics. In Section 4, the model parameters are estimated by using the maximum likelihood method for complete and for censored samples. In Section 5, four special models are discussed and the plots of density and hazard rate are displayed to check the flexibility of GBG family in terms of density and hazard rate shapes. The mathematical properties of one special model, the *generalized Burr-uniform* are obtained. In Section 6, a simulation study is carried out to investigate the performance of maximum likelihood estimators. In Section 7, the usefulness of one special model, the *generalized Burr-uniform* is shown to three real-life data sets. Section 8 offers concluding remarks.

2 Mathematical properties

2.1 Shapes

The shapes of the density and hazard rate functions can be described analytically. The critical points of the GBG density are the roots of the equation:

$$\frac{g'(x;c,k,\boldsymbol{\xi})}{g(x;c,k,\boldsymbol{\xi})}+\frac{g(x;c,k,\boldsymbol{\xi})}{1-G(x;c,k,\boldsymbol{\xi})}+\frac{(c-1)\,g(x;c,k,\boldsymbol{\xi})}{\{1-G(x;c,k,\boldsymbol{\xi})\}\left[\log\overline{G}(x;c,k,\boldsymbol{\xi})\right]}-\frac{c\,(k+1)\,g(x;c,k,\boldsymbol{\xi})\,\left[\log\overline{G}(x;c,k,\boldsymbol{\xi})\right]^{c-1}}{\overline{G}(x;c,k,\boldsymbol{\xi})\,\left[1+\{-\log\overline{G}(x;c,k,\boldsymbol{\xi})\}\right]}=0.$$

The critical point of the hazard rate are the roots of the equation:

$$\frac{g'(x;c,k,\boldsymbol{\xi})}{g(x;c,k,\boldsymbol{\xi})}+\frac{g(x;c,k,\boldsymbol{\xi})}{1-G(x;c,k,\boldsymbol{\xi})}+\frac{(c-1)\,g(x;c,k,\boldsymbol{\xi})}{\{1-G(x;c,k,\boldsymbol{\xi})\}\left[\log\overline{G}(x;c,k,\boldsymbol{\xi})\right]}-\frac{c\,g(x;c,k,\boldsymbol{\xi})\,\left[\log\overline{G}(x;c,k,\boldsymbol{\xi})\right]^{c-1}}{\overline{G}(x;c,k,\boldsymbol{\xi})\,\left[1+\{-\log[\overline{G}(x;c,k,\boldsymbol{\xi})]\}\right]}=0.$$

Note that the above equation may have more than one roots.

2.2 Useful representation of GBG family density and cdf

Here, we obtain linear representations of the GBG density and cdf in terms of infinite mixture of exp-G distribution.

Theorem 1. *If $X\sim GBG(c,k,\boldsymbol{\xi})$, then we have the following linear representations*

$$F(x)=\sum_{i=0}^{\infty}w_i\,H_i(x;\boldsymbol{\xi}), \tag{2.1}$$

and

$$f(x)=\sum_{i=0}^{\infty}w_i\,h_{i-1}(x;\boldsymbol{\xi}), \tag{2.2}$$

where $H_i(x;\boldsymbol{\xi})=G^i(x;\boldsymbol{\xi})$ and $h_{i-1}(x;\boldsymbol{\xi})=i\,g(x;\xi)\,G^{i-1}(x;\boldsymbol{\xi})$ represent the exp-G densities of the baseline distributions with i and $i-1$ as power parameters, respectively. The coefficients are

$$a_i=c\binom{i-c}{i}\sum_{j=0}^{i}\frac{(-1)^i}{c-j}\binom{i}{j}P_{j,i}(-1)^{c+i} \tag{2.3}$$

and $w_i=\frac{e_i}{c_0}$, $e_i=d_n-\frac{1}{c_0}\sum\limits_{j=0}^{n}c_{n-j}\,d_j$, c_i are the coefficients of division of two power series (see Gradshteyn and Ryzhik, 2000) with $d_0=c_0-1$ and $d_i=c_i$.

Proof. Consider the following series expansion

$$[\log(1+z)]^a=a\sum_{k=0}^{\infty}\binom{k-a}{k}\sum_{i=0}^{k}\frac{(-1)^k}{a-i}\binom{k}{i}P_{i,k}\,z^k. \tag{2.4}$$

where $P_{j,k}=\frac{1}{k}\sum_{m=1}^{k}(jm-k+m)c_mP_{j,k-m}$, $p_{j,0}=1$ and $c_k=\frac{(-1)^k}{k+1}$.

Using power series division and power series raised to a power (Gradshteyn and Ryzhik, 2000) given in Eq. (2.4), the Eq. (1.3) becomes

$$F(x)=1-\left(1+\sum_{i=0}^{\infty}a_i\,R^i(x)\right)^{-k},$$

where

$$a_i=c\binom{i-c}{i}\sum_{j=0}^{i}\frac{(-1)^i}{c-j}\binom{i}{j}P_{j,i}(-1)^{c+i}.$$

Considering $1+\sum_{i=0}^{\infty} a_i R^i(x) = \sum_{i=0}^{\infty} b_i R^i(x)$, where $b_0 = 1+a_0$ and $b_i = a_i$, the above equation becomes

$$F(x) = \frac{\left(\sum_{i=0}^{\infty} b_i R^i(x)\right)^k - 1}{\left(\sum_{i=0}^{\infty} b_i R^i(x)\right)^k} = \frac{\sum_{i=0}^{\infty} d_i R^i(x)}{\sum_{i=0}^{\infty} c_i R^i(x)}.$$

After some algebra, we have

$$F(x) = \frac{1}{c_0}\sum_{i=0}^{\infty} e_i F^i(x,\boldsymbol{\xi}).$$

The above equation can be expressed as

$$F(x) = \sum_{i=0}^{\infty} w_i H_i(x;\boldsymbol{\xi})$$

and hence by simple differentiation of the above equation, we have

$$f(x) = \sum_{i=0}^{\infty} w_i H_i(x;\boldsymbol{\xi}),$$

where $w_i = \frac{e_i}{c_0}$, $e_i = d_n - \frac{1}{c_0}\sum_{j=0}^{n} c_{n-j}\, d_j$, c_i are the coefficients of division of two power series (see Gradshteyn and Ryzhik, 2000) with $d_0 = c_0 - 1$ and $d_i = c_i$, and $H_i(x;\boldsymbol{\xi})$ is the *exp-R* distribution of the baseline densities with i as power parameter.

2.3 Moments and moment generating function

The rth moment of GBG family can be obtained by using the following expression

$$\mathbb{E}(X^r) = \sum_{i=0}^{\infty} w_i \int_0^{\infty} x^r h_{i-1}(x;\boldsymbol{\xi})\, dx. \tag{2.5}$$

The sth incomplete moment of the GBG family can be obtained by using the following expression

$$\mu^s(x) = \sum_{i=0}^{\infty} w_i T_i'(x), \tag{2.6}$$

where $T_s'(x;\boldsymbol{\xi}) = \int_0^x x^s h_{i-1}(x;\boldsymbol{\xi})\, dx$.
The moment generating function (mgf) of the GBG family can be obtained from

$$M_X(t) = \sum_{i=0}^{\infty} w_i \int_0^{\infty} e^{tx} h_{i-1}(x;\boldsymbol{\xi})\, dx. \tag{2.7}$$

The mean deviations of the GBG family can be obtained by using the following expressions

$$D_\mu = 2\mu F(\mu) - 2\mu^1(\mu), \tag{2.8}$$

and

$$D_M = \mu - 2\mu^1(M), \tag{2.9}$$

where $\mu = \mathbb{E}(X)$ can be obtained from Eq. (2.5), $M = Median(X)$ can be obtained from Eq. (1.5). Here, $F(\mu)$ can be calculated easily from Eq. (1.3) and $\mu^1(.)$ can be obtained from Eq. (2.6). From the above equations, Bonferroni and Lorenz curves can be obtained, for a given probability π

$$B(\pi) = \frac{\mu^1(q)}{\pi\,\mu} \quad \text{and} \quad L(\pi) = \frac{\mu^1(q)}{\mu}, \tag{2.10}$$

where $q = F^{-1}(\pi)$ is the GBG qf at π can be obtained form Eq. (1.5).

2.4 Stress-strength reliability parameter and stochastic ordering

The reliability parameter R is defined by

$$R = \mathbb{P}(X_1 < X_2) = \int_0^\infty f_1(x;\boldsymbol{\xi}_1)\,F_2(x;\boldsymbol{\xi}_2)\,dx,$$

where X_1 and X_2 have independent GBG$(c_1,k_1,\boldsymbol{\xi_1})$ and GBG$(c_2,k_2,\boldsymbol{\xi_2})$ distributions with a common parameter. Using the infinite mixture representations given in Eqs. (2.1) and (2.2), we obtain

$$R = \mathbb{P}(X_1 < X_2) = \sum_{i=0}^{\infty} w_i \sum_{p=0}^{\infty} d_p \int_0^\infty h_{i-1}(x;\boldsymbol{\xi}_1)\,H_p(x;\boldsymbol{\xi}_2)\,dx,$$

where h_{i-1} and $H_p(x)$ are the exp-G densities of the baseline distribution with $i-1$ and p as power parameters.

If $X_1 \sim \text{GBG}(c,k_1,\boldsymbol{\xi})$ and $X_2 \sim \text{GBG}(c,k_2,\boldsymbol{\xi})$ with common parameter c, then the density functions of X_1 and X_2 are, respectively, given by

$$f_1(x) = c\,k_1\,\frac{g(x;\boldsymbol{\xi})}{\overline{G}(x;\boldsymbol{\xi})}\,\{-\log[\overline{G}(x;\boldsymbol{\xi})]\}^{c-1}\,\left[1+\{-\log[\overline{G}(x;\boldsymbol{\xi})]\}^{c}\right]^{k_1-1} \tag{2.11}$$

and

$$f_2(x) = c\,k_2\,\frac{g(x;\boldsymbol{\xi})}{\overline{G}(x;\boldsymbol{\xi})}\,\{-\log[\overline{G}(x;\boldsymbol{\xi})]\}^{c-1}\,\left[1+\{-\log[\overline{G}(x;\boldsymbol{\xi})]\}^{c}\right]^{k_2-1}. \tag{2.12}$$

The ratio of the above two densities is

$$\frac{f_1(x)}{f_2(x)} = \frac{k_1}{k_2}\,\left[1+\{-\log[\overline{G}(x;\boldsymbol{\xi})]\}^{c}\right]^{k_2-k_1}. \tag{2.13}$$

Differentiating the densities ratio, we have

$$\frac{d}{dx}\frac{f_1(x)}{f_2(x)} = \frac{k_1}{k_2}\,(k_2-k_1)\,\frac{c\,g(x;\xi)\,\{-\log[\bar{G}(x;\boldsymbol{\xi})]\}^{c-1}}{1-G(x;\boldsymbol{\xi})}\,\left[1+\{-\log[\bar{G}(x;\boldsymbol{\xi})]\}^{c}\right]^{k_2-k_1-1}. \tag{2.14}$$

From above, we conclude that if $k_1 > k_2$, then $\frac{d}{dx}\frac{f_1(x)}{f_2(x)} < 0$ which implies that $X \leq_{lr} Y$.

3 Order Statistics

Here, we give an expression of the ith order statistics as a linear representation of baseline density.

Theorem 2.*If n be an integer value and $X_1,X_2,\ldots,X_n$, $i=1,2,\ldots,n$, be identically independently distributed random variables, then the density of ith order statistics is given by*

$$f_{i:n}(x) = \sum_{j=0}^{n-i}\sum_{m,r=0}^{\infty} m_j(m,r)\,h_{m+r-1}(x), \tag{3.1}$$

where

$$m_j(m,r) = \frac{n!\,m\,(-1)^j\,w_m\,c_{j+i-1:r}}{(i-1)!\,j!\,(n-i-j)!\,(m+r)} \tag{3.2}$$

and $h_{m+r-1}(x) = (m+r)\,g(x)\,G^{m+r-1}(x)$ are the exp-G densities of the baseline distribution with $m+r-1$ as power parameter.

Proof. Consider the power series expansion (Gradshteyn and Ryzhik, 2000)

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)^n = \sum_{k=0}^{\infty} c_{k:n}\,x^k, \tag{3.3}$$

where $c_0 = a_0^n$ and $c_m = \frac{1}{m\,a_0}\sum_{k=1}^{m}(k\,n-m+k)\,a_k\,c_{n:m-k}$.
The density for ith order statistics is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\,g(x)\,G^{i-1}(x)\,[1-G(x)]^{n-i},$$

which can be written as

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^i f(x)[F(x)]^{i+j-1}.$$

Now using the linear representations defined in Eqs. (2.1), (2.2) and (3.3), the density of ith order statistics can be expressed as

$$f_{i:n}(x) = \sum_{j=0}^{n-i}\sum_{r,m=0}^{\infty} m_j(r,m)\,h_{r+m-1}(x),$$

where the coefficient $m_j(r,m)$ is defined in Eq.(3.2).

4 Estimation of GBG family parameters

Here, we consider the estimation of the unknown parameters of GBG family by the maximum likelihood method for complete and censored samples.. The maximum likelihood estimates (MLEs) enjoy desirable properties that can be used when constructing confidence intervals and deliver simple approximations that work well in finite samples. The resulting approximation for the MLEs in distribution theory is easily handled either analytically or numerically. Let $x_1,\ldots,x_n$ be a sample of size n from GBG family of distributions given in Eq.(1.4).

4.1 Estimation of parameters in case of complete samples

The log-likelihood function of GBG distributions for the vector of parameter $\Theta = (c,k,\xi)^T$ is given by

$$\ell(\Theta) = n\log(ck) + \sum_{i=1}^{n}\log g(x_i;\boldsymbol{\xi}) - \sum_{i=1}^{n}\log\overline{G}(x_i;\boldsymbol{\xi}) + (c-1)\sum_{i=1}^{n}\log\{-\log\overline{G}(x_i;\boldsymbol{\xi})\}$$
$$-(k+1)\sum_{i=1}^{n}\log\left[1+\{-\log\overline{G}(x_i;\boldsymbol{\xi})\}^c\right].$$

The components of the score vector $U = \left(U_k, U_c, U_{\boldsymbol{\xi}}\right)^T$ are

$$U_k = \frac{\partial l}{\partial k} = \frac{n}{k} - \sum_{i=1}^{n}\log\left[1+\{-\log\overline{G}(x_i;\boldsymbol{\xi})\}^c\right],$$

$$U_c = \frac{\partial l}{\partial c} = \frac{n}{c} + \sum_{i=1}^{n}\log\{-\log\overline{G}(x_i;\boldsymbol{\xi})\} - (k+1)\sum_{i=1}^{n}\left[\frac{c\,\{\log\overline{G}(x_i;\boldsymbol{\xi})\}^c\log\{\log\overline{G}(x_i;\boldsymbol{\xi})\}}{1+\left\{\log\overline{G}(x_i;\boldsymbol{\xi})\right\}^c}\right],$$

$$U_{\boldsymbol{\xi}} = \frac{\partial l}{\partial\boldsymbol{\xi}} = \sum_{i=1}^{n}\left[\frac{g^{\boldsymbol{\xi}}(x_i;\boldsymbol{\xi})}{g(x_i;\boldsymbol{\xi})}\right] + \sum_{i=1}^{n}\left[\frac{G^{\boldsymbol{\xi}}(x_i;\boldsymbol{\xi})}{\overline{G}(x_i;\boldsymbol{\xi})}\right] - (c-1)\sum_{i=1}^{n}\left[\frac{G^{\boldsymbol{\xi}}(x_i;\boldsymbol{\xi})}{\left[1+\{-\log\overline{G}(x_i;\boldsymbol{\xi})\}^c\right]\overline{G}(x_i;\boldsymbol{\xi})}\right]$$
$$-c\,(k+1)\sum_{i=1}^{n}\left[\frac{c\,\{-\log\overline{G}(x_i;\boldsymbol{\xi})\}^{c-1}G^{\boldsymbol{\xi}}(x_i;\boldsymbol{\xi})}{\overline{G}(x_i;\boldsymbol{\xi})\left[1+\{-\log\overline{G}(x_i;\boldsymbol{\xi})\}^c\right]}\right],$$

where $G^{\boldsymbol{\xi}}(\cdot)$ and $g^{\boldsymbol{\xi}}(\cdot)$ means the derivative of the function G and g with respect to $\boldsymbol{\xi}$.

Setting U_k, U_c and $U_{\boldsymbol{\xi}}$ equal to zero and solving these equations simultaneously yields the the maximum likelihood estimates. These equations cannot be solved analytically, and analytical softwares are required to solve them numerically. For interval estimation of the parameters, we obtain the 3×3 observed information matrix $J(\Theta) = U_{rs}$ (for $r,s = c,k,\boldsymbol{\xi}$), whose elements are listed in Appendix A. Under standard regularity conditions, the multivariate normal $N_3(0, J(\widehat{\Theta})^{-1})$ distribution is used to construct approximate confidence intervals for the parameters. Here, $J(\widehat{\Theta})$ is the total observed information matrix evaluated at $\widehat{\Theta}$. Then, the $100(1-\alpha)$ confidence intervals for c, k and ξ are given by $\hat{c}\pm z_{\gamma^*/2}\times\sqrt{var(\hat{c})}$, $\hat{k}\pm z_{\gamma^*/2}\times\sqrt{var(\hat{k})}$ and $\hat{\xi}\pm z_{\gamma^*/2}\times\sqrt{var(\hat{\xi})}$, respectively, where the var()s denote the diagonal elements of $J(\widehat{\Theta})^{-1}$ corresponding to the model parameters, and $z_{\gamma^*/2}$ is the quantile $(1-\gamma^*/2)$ of the standard normal distribution.

4.2 Estimation of parameters in case of censored samples

If the lifetime of the first r failed items $x_1, x_2, \ldots, x_r$ have been observed, then the likelihood function for type II censoring is

$$\ell(x_i;\Theta) = A\left[\prod_{i=1}^{r} f(x_i;\Theta)\right] \times \left(\overline{F}(x_{(0)};\Theta)\right)^{n-r}, \tag{4.1}$$

where $f(.)$ and $\overline{F}(.)$ are the pdf and survival function corresponding to $F(.)$, respectively. Here, $X = (x_1, x_2, \ldots, x_r)^T$, $\Theta = (\theta_1, \theta_2, \ldots, \theta_n)^T$ and A is a constant. Inserting Eqs. (1.3) and (1.4) in Eq. (4.1), we obtain

$$\ell(x_i;\Theta) = A\left[\prod_{i=1}^{r} ck\frac{g(x_i;U_{\boldsymbol{\xi}})}{1-G(x_i;\boldsymbol{\xi})}\left\{-\log\overline{G}(x_i;U_{\boldsymbol{\xi}})\right\}^{c-1}\left(1+\left\{-\log\overline{G}(x_i;\boldsymbol{\xi})\right\}^{c}\right)^{-k-1}\right] \times \left[\left[1+\left\{-\log\overline{G}(x_{(0)};\boldsymbol{\xi})\right\}^{c}\right]^{-k}\right]^{n-r}. \tag{4.2}$$

The log-likelihood function for the vector of parameter $\Theta = (c, k, U_{\boldsymbol{\xi}})^T$ is given by

$$\begin{aligned}\ell(x_i,\Theta) =& \log A + n\log(ck) + \sum_{i=1}^{r}\log g(x_i;\boldsymbol{\xi}) - \sum_{i=1}^{r}\log\overline{G}(x_i;\boldsymbol{\xi}) + (c-1)\sum_{i=1}^{r}\log\{-\log\overline{G}(x_i;\boldsymbol{\xi})\} \\ &- (k+1)\sum_{i=1}^{r}\log\left[1+\{-\log\overline{G}(x_i;\boldsymbol{\xi})\}^{c}\right] + k(n-r)\log\left[1+\{-\log\overline{G}(x_{(0)};\boldsymbol{\xi})\}^{c}\right].\end{aligned}$$

The components of the score vector $U = \left(\frac{\partial l}{\partial k}, \frac{\partial l}{\partial c}, \frac{\partial l}{\partial \boldsymbol{\xi}}\right)$ are

$$\begin{aligned}U_k =& \frac{n}{k} - \sum_{i=1}^{r}\log\left[1+\{-\log\overline{G}(x_i;\boldsymbol{\xi})\}^{c}\right] + (n-r)\sum_{i=1}^{r}\log\left[1+\{-\log\overline{G}(x_{(0)};\boldsymbol{\xi})\}^{c}\right], \\ U_c =& \frac{n}{c} + \sum_{i=1}^{r}\log\{-\log\overline{G}(x_i;\boldsymbol{\xi})\} - (k+1)\sum_{i=1}^{r}\left[\frac{\{-\log\overline{G}(x_i;\boldsymbol{\xi})\}^{c}\log\{-\log\overline{G}(x_i;\boldsymbol{\xi})\}}{1+\{-\log\overline{G}(x_i;\boldsymbol{\xi})\}^{c}}\right] \\ &+k(n-r)\left[\frac{\{-\log\overline{G}(x_{(0)};\boldsymbol{\xi})\}^{c}\log\{-\log\overline{G}(x_{(0)};\boldsymbol{\xi})\}}{1+\{-\log\overline{G}(x_{(0)};\boldsymbol{\xi})\}^{c}}\right]. \\ U_{\boldsymbol{\xi}} =& \sum_{i=1}^{r}\left[\frac{g^{\boldsymbol{\xi}}(x_i;\boldsymbol{\xi})}{g(x_i;\boldsymbol{\xi})}\right] + \sum_{i=1}^{r}\left[\frac{G^{\boldsymbol{\xi}}(x_i;\boldsymbol{\xi})}{1-G(x_i;\boldsymbol{\xi})}\right] + (c-1)\sum_{i=1}^{r}\left[\frac{G^{\boldsymbol{\xi}}(x_i;\boldsymbol{\xi})}{\log\overline{G}(x_i;\boldsymbol{\xi})\left[1-G(x_i;\boldsymbol{\xi})\right]}\right] \\ &+(k+1)\sum_{i=1}^{r}\left[\frac{c\{-\log\overline{G}(x_i;\boldsymbol{\xi})\}^{c}G^{\boldsymbol{\xi}}(x_i;\boldsymbol{\xi})}{\left[1+\{-\log\overline{G}(x_i;\boldsymbol{\xi})\}^{c}\right]\left[1-G(x_i;\boldsymbol{\xi})\right]}\right] - k(n-r)\left[\frac{c\{-\log\overline{G}(x_{(0)};\boldsymbol{\xi})\}^{c}G^{\boldsymbol{\xi}}(x_{(0)};\boldsymbol{\xi})}{\left[1+\{-\log\overline{G}(x_{(0)};\boldsymbol{\xi})\}^{c}\right]\left[1-G(x_{(0)};\boldsymbol{\xi})\right]}\right].\end{aligned}$$

Setting U_k, U_c and $U_{\boldsymbol{\xi}}$ equal to zero and solving these equations simultaneously yields the the maximum likelihood estimates.

5 Some special models of GBG family

In this section, we give four special models of the GBG family, viz. generalized Burr-normal (GBN), generalized Burr-Lomax (GBLx), generalized Burr-exponentiated- exponential (GBEE) and generalized Burr-uniform (GBU) distributions. The properties of GBU model is obtained in detail.

5.1 Generalized Burr-normal (GBN) distribution

Let normal distribution be the baseline distribution with the pdf $g(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ and the cdf $\Phi(x) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{x}\mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}dx$, $-\infty < x < \infty$, where μ and σ are location and scale parameters, respectively. Then, the cdf and pdf of the GBN distribution are, respectively, given by

$$F(x) = 1 - \left[1+\{-\log\,(1-\Phi(x))\}^{c}\right]^{-k} \tag{5.1}$$

and

$$f(x)=\frac{c\,k}{\sigma^2}\frac{\mu-x}{1-\Phi(x)}\{-\log(1-\Phi(x))\}^{c-1}\left[1+\{-\log(1-\Phi(x))\}^c\right]^{-k-1}.$$

(i) If $c=1$ in Eq. (5.1), then GBN distribution reduces to the generalized Lomax-normal (GLxN) distribution, (ii) if $k=1$ in Eq. (5.1), then GBN distribution reduces to the generalized log-logistic-normal (GLLN) distribution, and (iii) if $c=k=1$ in Eq. (5.1), then the GBN distribution reduces to normal distribution.

In Figure 1 (a) and (b), the plots for the density and hazard rate functions of the GBN distribution are displayed. From Figure 1(a) and Figure 1(b) we conclude that the density shapes of GBN model are right-skewed, left-skewed and symmetrical and the hazard rate shape is UBT.

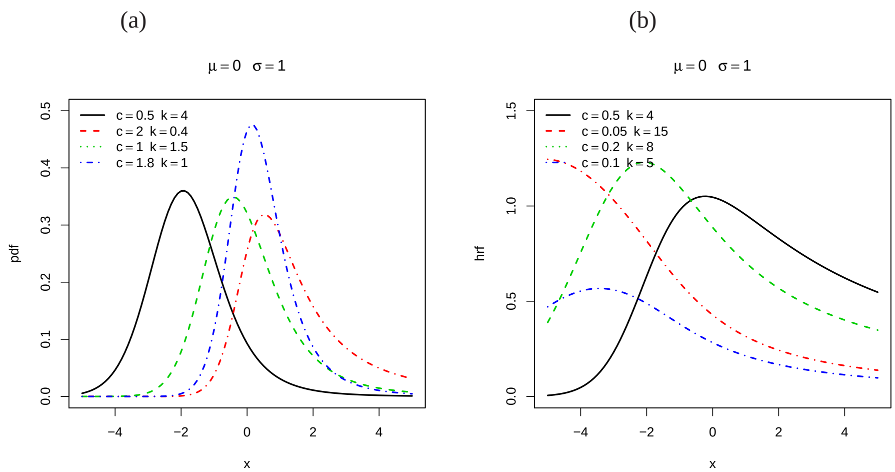

Fig. 1: Plots of (a) density and (b) hazard rate for the GBN distribution for different parameter values.

5.2 Generalized Burr-Lomax (GBLx) distribution

Let Lomax be the baseline distribution having pdf $g(x)=\alpha\,\beta\,(1+\alpha x)^{-\beta-1}, x\geq 0$ and cdf $G(x)=1-(1+\alpha x)^{-\beta}$, where $\alpha>0$ and $\beta>0$ are scale and shape parameters, respectively. Then, the cdf and pdf of the GBLx distribution are, respectively, given by

$$F(x)=1-\left[1+\{\beta\log(1+\alpha x)\}^c\right]^{-k} \tag{5.2}$$

and

$$f(x)=c\,k\,\alpha\,\beta\,(1+\alpha x)^{-1}\{\beta\log(1+\alpha x)\}^{c-1}\left[1+\{\beta\log(1+\alpha x)\}^c\right]^{-k-1}.$$

(i) If $c=1$ in Eq. (5.2), then the GBLx distribution reduces to generalized Lomax-Lomax (GLxLx) distribution, (ii) if $k=1$ in Eq. (5.2), then the GBLx distribution reduces to generalized log-logistic-Lomax (GLLLx) distribution, and (iii) if $c=k=1$ in Eq. (5.2), then the GBLx distribution reduces to the Lomax distribution.

In Figure 2 (a) and (b), the plots for the density and hazard rate functions of the GBLx distribution are displayed. From Figure 2(a) and Figure 2(b) we conclude that the density shapes of GBLx model are right-skewed, left-skewed and decreasing, and the hazard rate shapes are decreasing and UBT.

5.3 Generalized Burr exponentiated-exponential (GBEE) distribution

Let exponentiated-exponential (EE) be the baseline distribution with pdf $g(x)=\alpha\,\beta\,e^{-\alpha x}\,(1-e^{-\alpha x})^{\beta-1}, x>0$ and the cdf $G(x)=(1-e^{-\alpha x})^{\beta}$, where $\alpha>0$ is scale parameter while $\beta>0$ is the shape parameters, respectively. Then the cdf

(a) (b)

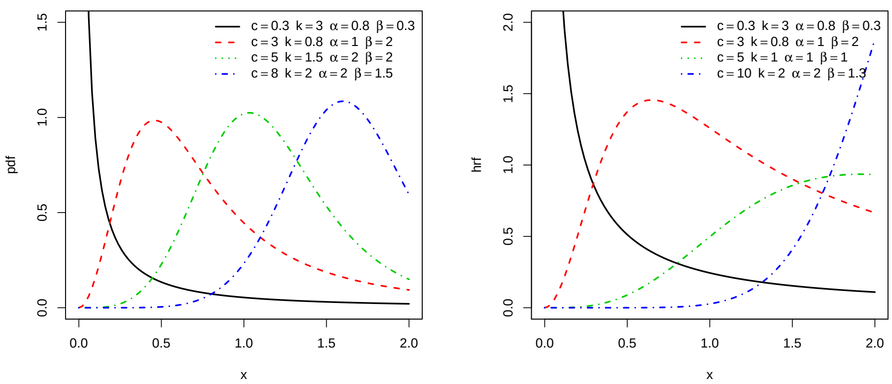

Fig. 2: Plots of (a) density and (b) hazard rate for the GBLx distribution for different parameter values.

and pdf of the GBEE distribution are, respectively, given by

$$F(x) = 1 - \left\{1 + \left[-\log\{1 - \left(1 - e^{-\alpha x}\right)^{\beta}\}\right]^{c}\right\}^{-k} \tag{5.3}$$

and

$$f(x) = \frac{c k \alpha \beta e^{-\alpha x} \left(1 - e^{-\alpha x}\right)^{\beta - 1} \left[-\log\{1 - (1 - e^{-\alpha x})^{\beta}\}\right]^{c-1}}{\left[1 - (1 - e^{-\alpha x})^{\beta}\right] \left\{1 + \left[-\log\{1 - (1 - e^{-\alpha x})^{\beta}\}\right]^{c}\right\}^{k+1}}.$$

(i) If $c = 1$ in Eq. (5.3), then the GBEE distribution reduces to the generalized Lomax-exponentiated exponential (GLxEE) distribution, (ii) if $k = 1$ in Eq. (5.3), the GBEE distribution reduces to generalized exponentiated exponential (GLLEE) distribution, and (iii) if $c = k = 1$ in Eq. (5.3), then the GBEE distribution reduces to the EE distribution.

In Figure 3 (a) and (b), the plots for the density and hazard rate functions of the GBLx distribution are displayed. From Figure 3(a) and Figure 3(b) we conclude that the density shapes of GBLx model are right-skewed, nearly symmetrical and decreasing and the hazard rate shapes are increasing, decreasing and upside-down bathtub.

5.4 Generalized Burr-uniform (GBU) distribution

Let uniform distribution be the baseline distribution having pdf $g(x) = \frac{1}{\theta}$, $\theta > 0$ and cdf $G(x) = \frac{x}{\theta}$, where θ is a scale parameter. Then, the cdf and pdf of the GBU distribution are, respectively, given by

$$F(x) = 1 - \left[1 + \left\{-\ln\left\{1 - \frac{x}{\theta}\right\}\right\}^{c}\right]^{-k} \tag{5.4}$$

and

$$f(x) = \frac{ck}{\theta - x}\left\{-\log\left\{1 - \frac{x}{\theta}\right\}\right\}^{c-1}\left[1 + \left\{-\log\left\{1 - \frac{x}{\theta}\right\}\right\}^{c}\right]^{-k-1}.$$

(i) If $c = 1$ in Eq. (5.4), then the GBU distribution reduces to generalized Lomax-uniform (GLxU) distribution, (ii) if $k = 1$in Eq. (5.4), then GBU distribution reduces to generalized log-logistic uniform (GLLU) distribution, and (iii) if $c = k = 1$ in Eq. (5.4), then the GBU distribution reduces to uniform distribution.

In Figure 4 (a) and (b), the plots for the density and hazard rate functions of the GBU distribution are displayed. From Figure 4(a) and Figure 4(b) we conclude that the density shapes of GBU model are left-skewed, right-skewed, decreasing, U-shape and J, and the hazard rate shape are increasing, decreasing and BT.

(a) (b)

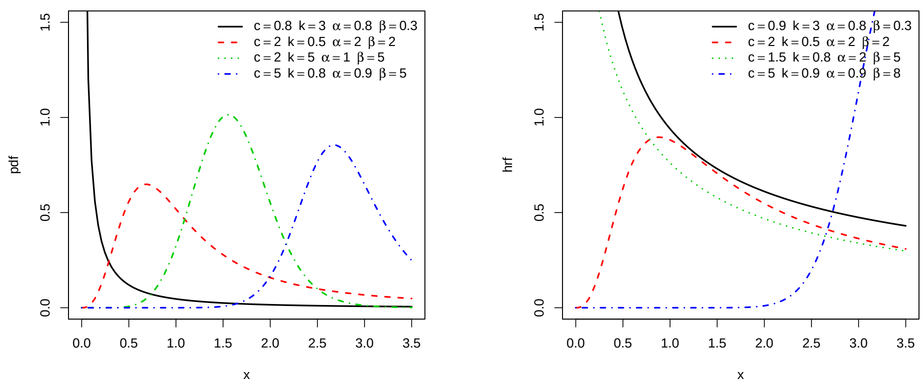

Fig. 3: Plots of (a) density and (b) hazard rate for the GBEE distribution for different parameter values.

(a) (b)

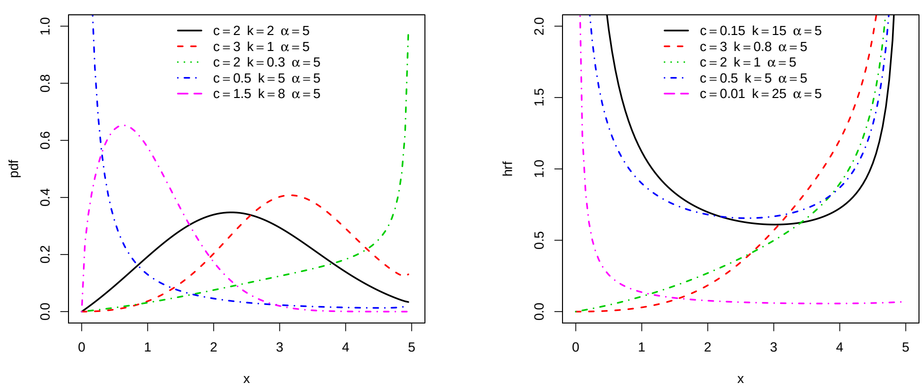

Fig. 4: Plots of (a) density and (b) hazard rate for the GBU distribution for different parameter values.

5.4.1 Properties of GBU distribution

The qf of the GBU distribution is given by

$$Q_x(u) = \theta \left[1 - e^{-\left[(1-u)^{-\frac{1}{k}} - 1 \right]^{\frac{1}{c}}} \right].$$

The rth moment of the GBU distribution is given by

$$\mu_r' = \sum_{i=0}^{\infty} w_i \left(\frac{i}{r+i} \right) \theta^r. \tag{5.5}$$

The sth incomplete moment of the GBU distribution is given by

$$T_s^{(m)}(z) = \sum_{i=0}^{\infty} w_i \left(\frac{i}{\theta^i} \right) \frac{z^{i+s}}{i+s}. \tag{5.6}$$

The moment generating function of the GBU distribution is given by

$$M_X(t) = \sum_{i=0}^{\infty} w_i \frac{i(-1)^i}{\theta^i} \frac{\gamma(i,-tx)}{t^i}.$$

First incomplete moment can be obtained by setting $s = 1$ in Eq. (5.6) as

$$T_1^{(m)}(z) = \sum_{i=0}^{\infty} w_i \left(\frac{i}{\theta^i}\right) \frac{z^{i+1}}{i+1}.$$

The mean deviations about mean and median are

$$D(\mu) = 2\mu F(\mu) - 2\sum_{i=0}^{\infty} w_i \left(\frac{i}{\theta^i}\right) \frac{\mu^{i+1}}{i+1}$$

and

$$D(M) = \mu - 2\sum_{i=0}^{\infty} w_i \left(\frac{i}{\theta^i}\right) \frac{M^{i+1}}{i+1}.$$

The log-likelihood function for the vector of parameter $\Theta = (c,k,\theta)^T$ is

$$l(\Theta) = n\log\left(\frac{ck}{\theta}\right) - \sum_{i=1}^{n} \log\left(1-\frac{x_i}{\theta}\right) + (c-1)\sum_{i=1}^{n} \log\left\{-\log\left(1-\frac{x_i}{\theta}\right)\right\} - (k+1)\sum_{i=1}^{n} \log\left\{1+\left\{-\log\left(1-\frac{x_i}{\theta}\right)\right\}^c\right\}.$$

The components of score vector are:

$$U_k = \frac{n}{k} - \sum_{i=1}^{n} \log(1+z_i), \tag{5.7}$$

$$U_c = \frac{n}{c} + \sum_{i=1}^{n} \log\left\{-\log\left(1-\frac{x_i}{\theta}\right)\right\} - (k+1)\sum_{i=1}^{n} \left(\frac{\acute{z}_{i:c}}{1+z_i}\right), \tag{5.8}$$

where $z_i = \left[-\log\left(1-\frac{x_i}{\theta}\right)\right]^c$ and $\acute{z}_{i:c} = \left[-\log\left(1-\frac{x_i}{\theta}\right)\right]^c \left[\log\left\{-\log\left(1-\frac{x_i}{\theta}\right)\right\}\right]$.
Since the parameter θ involved in the limit of random variable $0 \le x \le \theta$, so we will estimate it by the maximum order statistic $x_{(n)}$. The above equations cannot be solved analytically, rather an analytical software is required to solve them numerically. In this paper the estimates and the standard errors of GBU model are obtained using `R-language`.

6 Simulation study

In this section, a simulation study is carried out to examine the performance of the MLEs of the GBU parameters. We generate 1000 samples of size, n=25, 50, 100 and 500 of the GBU model for fixed $\theta = 0.5$. The evaluation of estimates is based on the mean of the MLEs of the model parameters and the mean squared error (MSE) of the MLEs. The empirical study was conducted with R-software and the results are presented in Table 1. The values in Table 1 indicate that the estimates are quite stable and more importantly the values of the estimates are close to the true values for the these sample sizes. It can be observed from Table 1 that the MSEs decrease as n increases. From this simulation study we conclude that the maximum likelihood method is appropriate for estimating the GBU parameters. In fact, the MSEs of the parameters tend to be closer to the true parameter values when n increases. This fact supports that the asymptotic normal distribution provides an adequate approximation to the finite sample distribution of the MLEs. The normal approximation can be improved by using bias adjustments to these estimators. Approximations to the their biases in simple models can be obtained analytically.

7 Applications of GBG model

In this section three real-life data sets are analyzed as an empirical illustration of the newly proposed family. The first two data sets are based on complete observations (uncensored) while the third one is censored. We tried to show the usefulness of the GBG model in different lifetime phenomenons. In these three applications, the model parameters are estimated by the method of maximum likelihood. The goodness-of-fit criterion: Akaike information criterion (AIC), Bayesian information criterion(BIC) are used to compare the proposed and competitive models. In general, the smaller the values of these statistics, the better the fit to the data. The plots of the fitted pdfs and cdfs of the models are displayed for visual comparison. For all data sets, we use the sub-model GBU to compare it with the Weibull-uniform (WU), Weibull-Burr XII (WBXII), beta-Burr XII (BBXII) and Kumaraswamy-Burr XII (KwBXII) distributions. The required computations are carried out in R-packages.

Table 1: Means and MSEs for the of the MLEs of the parameters of the GBU model.

c	k	n	Mean			MSE		
			c	k	θ	c	k	θ
1	1	25	1.346	1.272	0.699	1.464	1.282	0.406
		50	1.065	1.256	0.682	1.374	1.263	0.399
		100	1.064	1.255	0.645	0.229	1.182	0.214
		500	1.009	1.035	0.509	0.028	0.753	0.040
2	1	25	2.271	0.762	0.518	1.964	1.792	0.093
		50	2.024	0.902	0.509	1.020	1.087	0.062
		100	2.022	0.956	0.507	0.915	0.483	0.026
		500	2.012	0.966	0.501	0.501	0.175	0.006
2	0.5	25	1.606	0.145	0.788	0.776	0.556	0.878
		50	1.783	0.216	0.548	0.539	0.492	0.157
		100	1.849	0.376	0.515	0.256	0.267	0.059
		500	1.977	0.404	0.503	0.123	0.129	0.034
0.5	3	25	0.535	2.764	0.459	0.237	0.111	0.038
		50	0.511	2.915	0.463	0.136	0.061	0.021
		100	0.512	2.965	0.491	0.082	0.026	0.005
		500	0.498	3.007	0.499	0.016	0.015	0.002

7.1 Uncensored (complete) data sets

7.1.1 Data set 1: Birnbaum-Saunders data

The first data set was used by Birnbaum and Saunders (1969) which corresponds to the fatigue time of 101 6061-T6aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second (cps). Bourguignon et al. (2014), and Torabi and Montazeri (2014) studied Birnbaum-Saunders data set and reported AIC values as follows: Weibull-log-logistic (WLL) and logistic-Uniform (LU) distributions = 919.082, beta-generalized exponential (BGE) distribution = 921.285, beta-exponential (BE) distribution = 920.211, beta-normal (BN) distribution = 920.837, beta-Gumbel (BG) distribution = 920.956, Beta-Laplace (BL) distribution = 921.025, beta-generalized half normal (BGHN) distribution = 921.577, beta-Birnbaum-Saunders (BBS) distribution = 921.867, gamma-uniform (GU) distribution =922.745, beta-generalized Pareto (BGP) distribution = 923.081, beta-modified Weibull (BMW) distribution = 923.239, beta-Pareto (BP) distribution = 924.838, respectively.

From Table 2, we see that the proposed model GBU has lower value of AIC and BIC as compared other models. Thus our model provides better fit as compared to other models to this data set. Figure 5(a) and Figure 5(b) show the plots of the estimated pdfs and cdfs of GBU and other competitive models. From this we conclude that GBU estimated density and cdf show good adjustment to the data set 1.

7.1.2 Data set 2: Breast Cancer data

The second real data set represents the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 to 1938 (Lee, 1992). These data were previously studied by Ramos et al. (2013) and Tahir et al. (2014). They reported the AIC values for Kumaraswamy log-logistic (KwLL) distribution = 1189.937, beta log-logistic (BLL) distribution = 1171.861, Zografos-Balakrishnan log-logistic distribution (ZBLL) = 1167.063, exponentiated-Weibull (EW) distribution = 1163.759, exponentiated-log-logistic (ELL) distribution = 1167.341 and McDonald-log-logistic distribution (McLL) = 1164.661, McDonald-Weibull (McW) = 1166.474, log-logistic = 1179.199, gamma distribution = 1166.474, log-Normal (LN) distribution = 1194.067 respectively.

Table 3 indicates that the GBU model gives the best fit among all others competitive model for data set 2. The estimated pdfs and cdfs are presented in Figure 6(a) and Figure 6(b), respectively. Figure 6 also indicates that the GBU distribution provides a better fit to the data as compared to other models.

7.2 Data set 3: Censored data set

In this section, we provide an application of the GBU model to censored data set. The data are related to the times to failure of 20 aluminum reduction cells. and the failure times is in units of 1000 (Lawless, 2003). Table 4 indicates that

(a) (b)

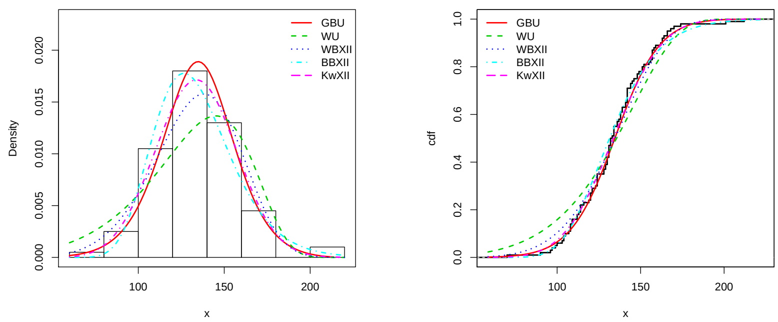

Fig. 5: Plots of the estimated (a) pdfs and (b) cdfs of GBU and other competitive models for data set 1.

(a) (b)

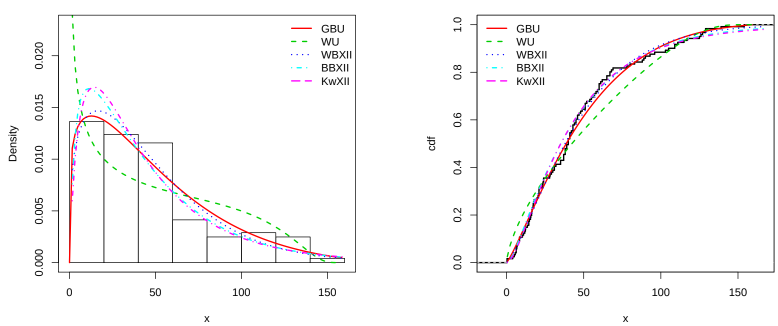

Fig. 6: Plots of the estimated (a) pdfs and (b) cdfs of GBU and other competitive models for data set 2.

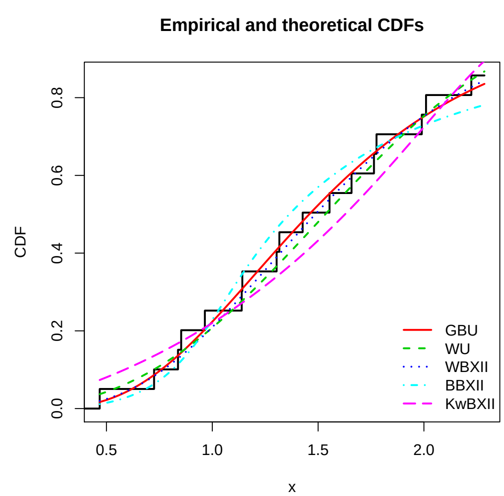

Fig. 7: Plots of estimated cdfs of GBU and other competitive models for data set 3.

Table 2: MLEs and their standard errors (in parentheses) for the data set 1.

Distribution	MLEs				AIC	BIC
GBU(c,k,θ)	5.848	1.075	215	-	910.211	915.421
	(0.526)	(0.116)	-	-		
WU(a,b,θ)	1.184	2.782	215	-	945.090	950.301
	(0.122)	(0.181)	-	-		
WBXII(c,k,α,β)	18.371	22.161	5.073	0.025	923.879	934.323
	(116.891)	(1.496)	(1.048)	(0.002)		
BBXII(c,k,α,β)	66.295	51.750	0.815	31.009	916.923	927.343
	(126.694)	(38.003)	(0.261)	(44.806)		
KwBXII(c,k,α,β)	793.469	588.060	20.290	0.048	915.017	925.438
	(219.976)	(227.223)	(15.357)	(0.002)		

Table 3: MLEs and their standard errors (in parentheses) for the data set 2.

Distribution	MLEs				AIC	BIC
GBU(c,k,θ)	1.183	3.505	160	-	1159.186	1164.777
	(0.077)	(0.340)	-	-		
WU(a,b,θ)	1.405	0.691	160	-	1187.540	1193.132
	(0.128)	(0.042)	-	-		
WBXII(c,k,α,β)	65.022	53.454	0.025	0.880	1166.053	1177.242
	(292.673)	(9.906)	(0. 011)	(0.587)		
BBXII(c,k,α,β)	0.418	159.033	0.366	28.783	1173.254	1184.437
	(0.212)	(104.782)	(0.075)	(10.892)		
KwBXII(c,k,α,β)	32.582	341.059	0.169	1.687	1173.480	1184.663
	(43.880)	(293.180)	(0.115)	(1.602)		

Table 4: MLEs, their standard errors and goodness-of-fit statistics for data set 3.

Model	Parameters	MLE	Standard error	AIC	BIC
GBU	c	2.151	0.407	42.683	44.675
	k	1.750	0.425		
	θ	2.5	-		
WU	a	0.566	0.160	45.708	47.699
	b	1.186	0.232		
	θ	2.5	-		
WBXII	c	5.062	8.841	46.0746	50.057
	k	1.791	1.386		
	α	2.090	2.062		
	β	0.239	0.280		
BBXII	c	0.363	0.551	46.094	50.077
	k	4.372	20.586		
	α	1.739	2.066		
	β	8.431	12.365		
KwBXII	c	16.600	24.992	46.034	50.017
	k	19.361	89.159		
	α	0.553	0.512		
	β	2.116	2.402		

the GBU model gives the best fit among all others competitive model for data set 3. Figure 7 also indicates that the GBU distribution provides a better fit to the censored data as compared to other models.

8 Concluding remarks

In this paper, we proposed a new family of distributions called the *generalized Burr-G family* of distributions. We obtained some of its mathematical properties. Estimation of parameters are dealt for both complete and censored samples. Four special models are considered and the properties of one special model, the *generalized Burr-uniform* are obtained. A simulation study is carried and the MLEs estimates are found quite satisfactory. Three applications to real data sets also reveals that the special model of the proposed family performs better as compared to some other well-known models.

Acknowledgements

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Appendix A

$$
\begin{aligned}
U_{kc} &= -\sum_{i=1}^{n}\left[\frac{\log\left[-\log\overline{G}(x_i;\boldsymbol{\xi})\right]\,\{-\log\overline{G}(x_i;\boldsymbol{\xi})\}^{c}}{\overline{G}(x_i;\boldsymbol{\xi})\,\left\{1+\left[-\log\overline{G}(x_i;\boldsymbol{\xi})\right]^{c}\right\}}\right],\\
U_{k\boldsymbol{\xi}} &= -\sum_{i=1}^{n}\left[\frac{c\,G^{\boldsymbol{\xi}}(x_i;\boldsymbol{\xi})\,\left[-\log\overline{G}(x_i;\boldsymbol{\xi})\right]^{c-1}}{\overline{G}(x_i;\boldsymbol{\xi})\,\left\{1+\left[-\log\overline{G}(x_i;\boldsymbol{\xi})\right]^{c}\right\}}\right],\\
U_{kk} &= -\frac{n}{k^2},\\
U_{c\boldsymbol{\xi}} &= -\sum_{i=1}^{n}\frac{G^{\boldsymbol{\xi}}(x_i;\boldsymbol{\xi})}{\overline{G}(x_i;\boldsymbol{\xi})\,\log\overline{G}(x_i;\boldsymbol{\xi})}\\
&\quad-(k+1)\sum_{i=1}^{n}\left[\frac{G^{\boldsymbol{\xi}}(x_i;\boldsymbol{\xi})\,[-\log\overline{G}(x_i;\boldsymbol{\xi})]^{c-1}}{\overline{G}(x_i;\boldsymbol{\xi})\,\left[-\log\overline{G}(x_i;\boldsymbol{\xi})\right]^{c}}+\frac{c\,\log[-\log\overline{G}(x_i;\boldsymbol{\xi})]\,[-\log\overline{G}(x_i;\boldsymbol{\xi})]^{c-1}}{\overline{G}(x_i;\boldsymbol{\xi})\,\left\{1+\left[-\log\overline{G}(x_i;\boldsymbol{\xi})\right]^{c}\right\}}\right.\\
&\qquad\left.-\frac{G^{\boldsymbol{\xi}}(x_i;\boldsymbol{\xi})\,\log[-\log\overline{G}(x_i;\boldsymbol{\xi})]\,\left[-\log\overline{G}(x_i;\boldsymbol{\xi})\right]^{2c-1}}{\overline{G}(x_i;\boldsymbol{\xi})\,\left\{1+\left[-\log\overline{G}(x_i;\boldsymbol{\xi})\right]^{c}\right\}^{2}}\right],\\
U_{cc} &= -\frac{n}{c^2}-(k+1)\sum_{i=1}^{n}\left[\frac{\left[-\log\overline{G}(x_i;\boldsymbol{\xi})\right]^{c}\,\left[\log\{\log\overline{G}(x_i;\boldsymbol{\xi})\}\right]^{2}}{\left\{1+\left[-\log\overline{G}(x_i;\boldsymbol{\xi})\right]^{c}\right\}}-\frac{\left[-\log\overline{G}(x_i;\boldsymbol{\xi})\right]^{2c}\,\left[\log\{\log\overline{G}(x_i;\boldsymbol{\xi})\}\right]^{2}}{\left\{1+\left[-\log\overline{G}(x_i;\boldsymbol{\xi})\right]^{c}\right\}^{2}}\right],\\
U_{\boldsymbol{\xi}\boldsymbol{\xi}'} &= \sum_{i=1}^{n}\left[\frac{g^{\boldsymbol{\xi}\boldsymbol{\xi}'}(x_i;\boldsymbol{\xi})}{g(x_i;\boldsymbol{\xi})}-\frac{[g^{\boldsymbol{\xi}}(x_i;\boldsymbol{\xi})]^2}{[g(x_i;\boldsymbol{\xi})]^2}\right]+\sum_{i=1}^{n}\left[\frac{[G^{\boldsymbol{\xi}}(x_i;\boldsymbol{\xi})]^2}{[\overline{G}(x_i;\boldsymbol{\xi})]^2}+\frac{G^{\boldsymbol{\xi}\boldsymbol{\xi}'}(x_i;\boldsymbol{\xi})}{\overline{G}(x_i;\boldsymbol{\xi})}\right]\\
&\quad-(k+1)\sum_{i=1}^{n}\left[\frac{(c-1)\,[G^{\boldsymbol{\xi}}(x_i;\boldsymbol{\xi})]^2\,\left[-\log\overline{G}(x_i;\boldsymbol{\xi})\right]^{c-2}}{[\overline{G}(x_i;\boldsymbol{\xi})]^2\,\left\{1+\left[-\log\overline{G}(x_i;\boldsymbol{\xi})\right]^{c}\right\}}+\frac{c\,[G^{\boldsymbol{\xi}}(x_i;\boldsymbol{\xi})]^2\,\left[-\log\overline{G}(x_i;\boldsymbol{\xi})\right]^{c-1}}{[\overline{G}(x_i;\boldsymbol{\xi})]^2\,\left\{1+\left[-\log\overline{G}(x_i;\boldsymbol{\xi})\right]^{c}\right\}}\right.\\
&\qquad\left.-\frac{c^2\,[G^{\boldsymbol{\xi}}(x_i;\boldsymbol{\xi})]^2\,\left[-\log\overline{G}(x_i;\boldsymbol{\xi})\right]^{2c-2}}{[\overline{G}(x_i;\boldsymbol{\xi})]^2\,\left\{1+\left[-\log\overline{G}(x_i;\boldsymbol{\xi})\right]^{c}\right\}^{2}}+\frac{c\,G^{\boldsymbol{\xi}\boldsymbol{\xi}}(x_i;\boldsymbol{\xi})\,\left[-\log\overline{G}(x_i;\boldsymbol{\xi})\right]^{c-1}}{[\overline{G}(x_i;\boldsymbol{\xi})]\,\left\{1+\left[-\log\overline{G}(x_i;\boldsymbol{\xi})\right]^{c}\right\}}\right]\\
&\quad-(c-1)\sum_{i=1}^{n}\left[\frac{[G^{\boldsymbol{\xi}}(x_i;\boldsymbol{\xi})]^2}{[\overline{G}(x_i;\boldsymbol{\xi})]^2[\log\overline{G}(x_i;\boldsymbol{\xi})]^2}+\frac{[G^{\boldsymbol{\xi}}(x_i;\boldsymbol{\xi})]^2}{[\overline{G}(x_i;\boldsymbol{\xi})]^2[\log\overline{G}(x_i;\boldsymbol{\xi})]}+\frac{G^{\boldsymbol{\xi}\boldsymbol{\xi}'}(x_i;\boldsymbol{\xi})}{[\overline{G}(x_i;\boldsymbol{\xi})][\log\overline{G}(x_i;\boldsymbol{\xi})]}\right].
\end{aligned}
$$

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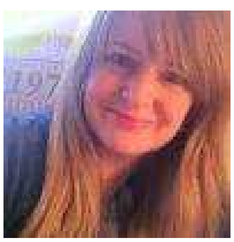

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