Applied Mathematics & Information Sciences **5**(3) (2011), 558-569
– An International Journal

Best Proximity Point on Different Type Contractions

Erdal Karapınar and İnci M. Erhan

[1]Department of Mathematics, Atılım University 06836, İncek, Ankara, Turkey

Email Address: ekarapinar@atilim.edu.tr, erdalkarapinar@yahoo.com

Received Nov. 26, 2010; Revised March 21, 2011; Accepted 8 April 2011

In this manuscript, some proximity points are obtained by using different types cyclic contractions. Also, generalized cyclic Meir Keeler contraction is introduced and a new fixed point theorem for this cyclic mapping is stated.

Keywords: cyclic-contraction, best proximity point, generalized cyclic Meir Keeler contraction

2000 Mathematics Subject Classification: 47H10,54H25

1 Introduction and Preliminaries

Cyclic contraction and best proximity point are among the popular topics in the fixed point theory and have received considerable interest recently. The first result in this area was reported by Kirk-Srinavasan-Veeramani [5] in 2003. Later, many authors continued investigation and more results have been obtained, such as, [1–4, 6–8]. The purpose of this study is to generalize the definition of the cyclic Meir Keeler contraction and give a fixed point theorem for this mapping.

We first define the cyclic map and best proximity point.

Definition 1.1. Let A and B be non-empty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$. T is called cyclic map if $T(A) \subset B$ and $T(B) \subset A$.

A point $x \in A \cup B$ is called a best proximity point if $d(x, Tx) = d(A, B)$ where $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. In 2003, Kirk-Srinavasan-Veeramani [5] proved the following fixed point theorem for a cyclic map.

Theorem 1.1. *Let A and B be non-empty closed subsets of a complete metric space (X, d). Suppose that $T : A\cup B \to A\cup B$ is a map satisfying $T(A) \subset B$ and $T(B) \subset A$ and there exists $k \in (0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x \in A$ and $y \in B$. Then, T has a unique fixed point in $A \cap B$.*

2 Main Results

In this section we introduce different types of cyclic contractions and prove fixed point theorems for these maps.

Definition 2.1. (See [4]) Let A and B be non-empty subsets of a metric space (X,d). A cyclic map $T : A \cup B \to A \cup B$ is said to be a cyclic contraction if there exists $k \in (0,1)$ such that

$$d(Tx,Ty) \leq kd(x,y),\ \forall x \in A,\ \forall y \in B \tag{2.1}$$

We generalize the above definition in the following way:

Definition 2.2. Let A and B be non-empty subsets of a metric space (X,d). A cyclic map $T : A \cup B \to A \cup B$ is said to be a Kannan Type cyclic contraction if there exists $k \in (0, \frac{1}{2})$ such that

$$d(Tx,Ty) \leq k[d(Tx,x) + d(Ty,y)], \forall x \in A,\ \forall y \in B \tag{2.2}$$

The following illustrative examples show that a map can be a cyclic contraction but not a Kannan type cyclic contraction and vise versa.

Example 2.1. Consider the Euclidean ordered space $X = \mathbb{R}$ with the usual metric. Suppose $A = [-1,0]$ and $B = [0,1]$ and let $T : A \cup B \to A \cup B$ be defined by $Tx = -\frac{x}{3}$ for all $x \in A \cup B$. It is clear that for $k \leq \frac{1}{3}$, T is cyclic contraction but not Kannan type cyclic contraction.

Example 2.2. Consider the Euclidean ordered space $X = \mathbb{R}$ with the usual metric. Suppose $A = B = [0,1]$ and define $T : A \cup B \to A \cup B$ by

$$Tx = \begin{cases} \frac{1}{4} & \text{if } x = 1, \\ \frac{1}{2} & \text{if } x \in [0,1). \end{cases}$$

For $x = \frac{15}{16}$ and $y = 1$, cyclic contraction condition fails. However, T is Kannan type cyclic contraction.

Next, we give fixed point theorem for a Kannan type cyclic contraction which can be regarded as a generalization of Theorem 1.1.

Theorem 2.1. *Let A and B be non-empty closed subsets of a complete metric space (X,d) and let $T : A \cup B \to A \cup B$ be a Kannan type cyclic contraction. Then T has a unique fixed point in $A \cap B$.*

Proof. Fix $x \in A$. Due to (2.2) we have

$$d(T^2x, Tx) \leq k[d(T^2x, Tx) + d(Tx, x)], \tag{2.3}$$

which implies $d(T^2x, Tx) \leq td(Tx, x)$, where $t = \frac{k}{1-k}$ and clearly $t \in (0,1)$. Thus, we have $d(T^{n+1}x, T^nx) \leq t^n d(Tx, x)$. Consequently,

$$\sum_{n=1}^{\infty} d(T^{n+1}x, T^nx) \leq \left(\sum_{n=1}^{\infty} t^n\right) d(Tx, x) < \infty.$$

Obviously, $\{T^nx\}$ is a Cauchy sequence. Hence, there exists $z \in A \cup B$ such that $T^nx \to z$. Notice that $\{T^{2n}x\}$ is a sequence in A and $\{T^{2n-1}x\}$ is a sequence in B and that both sequences tend to same limit z. Regarding that A and B are closed, we conclude $z \in A \cap B$. Hence, $A \cap B \neq \emptyset$.

To show that z is a fixed point, we claim that $Tz = z$. Observe that

$$d(Tz, z) = \lim_{n\to\infty} d(Tz, T^{2n}x) \leq k \lim_{n\to\infty} [d(T^{2n}x, T^{2n-1}x) + d(Tz, z)] \leq kd(Tz, z)$$

which is equivalent to $(1-k)d(Tz, z) = 0$. Since $k \in (0, \frac{1}{2})$, then $d(Tz, z) = 0$ which implies $Tz = z$.

To prove the uniqueness of z, assume that there exists $w \in A \cup B$ such that $z \neq w$ and $Tw = w$. Taking into account that T is a cyclic, we get $w \in A \cap B$. From

$$d(z, w) = d(Tz, Tw) = k[d(Tz, z) + d(Tw, w)] = 0,$$

we conclude that $z = w$ and hence z is the unique fixed point of T. $\square$

Remark 2.1. Notice that the point z in the proof of Theorem2.1 is a proximity point. Indeed. $d(A, B) = 0$ since $A \cap B \neq \emptyset$. Regarding that z is the fixed point, $d(Tz, z) = 0$.

Corollary 2.1. *Let T be a self map on a complete metric space (X, d). If for some $x \in X$, there exists a $k \in (0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \leq k[d(Tx, x) + d(Ty, y)],\ \forall x, y \in X \tag{2.4}$$

then, T has a unique fixed point.

We introduce a new cyclic contraction in the following way:

Definition 2.3. Let A and B be non-empty subsets of a metric space (X, d). A cyclic map $T : A \cup B \to A \cup B$ is said to be a Chatterjee Type cyclic contraction if there exists $k \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq k[d(Tx, y) + d(Ty, x)],\ \forall x \in A,\ \ \forall y \in B \tag{2.5}$$

For the map defined in Definition 2.3, we obtain another generalization of Theorem 1.1:

Theorem 2.2. *Let A and B be non-empty closed subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ be a Chatterjee type cyclic contraction. Then T has a unique fixed point in $A \cap B$.*

Proof. Fix $x \in A$. From (2.5), we have

$$\begin{aligned} d(T^2x, Tx) &\leq k[d(T^2x, x) + d(Tx, Tx)] = kd(T^2x, x) \\ &\leq k[d(T^2x, Tx) + d(Tx, x)], \end{aligned} \tag{2.6}$$

and thus

$$d(T^2x, Tx) \leq td(Tx, x), \tag{2.7}$$

where $t = \frac{k}{1-k}$. Since $0 < k < \frac{1}{2}$, then $0 < t < 1$. Inductively, we obtain

$$d(T^{n+1}x, T^n x) \leq t^n d(Tx, x).$$

Then,

$$\sum_{n=1}^{\infty} d(T^{n+1}x, T^n x) \leq \left(\sum_{n=1}^{\infty} t^n\right) d(Tx, x) < \infty.$$

So, $\{T^n x\}$ is a Cauchy sequence. Hence, there exists $z \in A \cup B$ such that $T^n x \to z$. Here $\{T^{2n}x\}$ is a sequence in A and $\{T^{2n-1}x\}$ is a sequence in B and both sequences have the same limit z. Since A and B are closed, $z \in A \cap B$. So, $A \cap B \neq \emptyset$.

Now, we show that $Tz = z$. Notice that

$$d(Tz, z) = \lim_{n\to\infty} d(Tz, T^{2n}x) \leq k \lim_{n\to\infty} [d(Tz, T^{2n-1}x) + d(T^{2n}x, z)] \leq kd(Tz, z)$$

which is equivalent to $(1 - k)d(Tz, z) \leq 0$. Since $k \in (0, \frac{1}{2})$, then $d(Tz, z) = 0$ and thus $Tz = z$.

For the uniqueness of z, assume that there exists $w \in A \cup B$ such that $z \neq w$ and $Tw = w$. However, since T is a cyclic, we get $w \in A \cap B$. Then,

$$d(z, w) = d(Tz, Tw) \leq k[d(Tz, w) + d(Tw, z)] = k[2d(z, w)]$$

yields $(1-2k)d(z, w) \leq 0$ and hence $z = w$, which completes the proof of uniqueness. □

Definition 2.4. Let A and B be non-empty subsets of a metric space (X, d). A cyclic map $T : A \cup B \to A \cup B$ is said to be a Reich type cyclic contraction if there exists $k \in (0, \frac{1}{3})$ such that

$$d(Tx, Ty) \leq k[d(x, y) + d(Tx, x) + d(Ty, y)], \ \forall x \in A, \ \forall y \in B \tag{2.8}$$

In what follows, we state and prove the fixed point theorem for a Reich type cyclic contraction.

Theorem 2.3. *Let A and B be non-empty closed subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ be a Reich type cyclic contraction. Then T has a unique fixed point in $A \cap B$.*

Proof. Take $x \in A$. From (2.8) it follows that

$$d(T^2x, Tx) \leq k[d(x, Tx) + d(T^2x, Tx) + d(Tx, x)] \tag{2.9}$$

and so $d(T^2x, Tx) \leq td(Tx, x)$, where $t = \frac{2k}{1-k}$ and clearly $t \in (0, 1)$. Thus we have $d(T^{n+1}x, T^nx) \leq t^n d(Tx, x)$. Consequently,

$$\sum_{n=1}^{\infty} d(T^{n+1}x, T^nx) \leq \left(\sum_{n=1}^{\infty} t^n\right) d(Tx, x) < \infty.$$

Hence, $\{T^nx\}$ is a Cauchy sequence. Then, there exists a $z \in A \cup B$ such that $T^nx \to z$. Notice that $\{T^{2n}x\}$ is a sequence in A and $\{T^{2n-1}x\}$ is a sequence in B having the same limit z. As A and B are closed, we conclude $z \in A \cap B$, that is, $A \cap B$ is nonempty.

We now show that $Tz = z$. Observe that

$$\begin{aligned} d(Tz, z) &= \lim_{n\to\infty} d(Tz, T^{2n}x) \\ &\leq k \lim_{n\to\infty}[d(z, T^{2n-1}x) + d(T^{2n}x, T^{2n-1}x) + d(Tz, z)] \\ &\leq kd(Tz, z) \end{aligned}$$

which is equivalent to $(1-k)d(Tz, z) \leq 0$. Regarding $k \in (0, \frac{1}{3})$ implies that $d(Tz, z) = 0$ and thus $Tz = z$.

To prove the uniqueness of the fixed point z, assume that there exists $w \in A \cup B$ such that $z \neq w$ and $Tw = w$. Taking into account that T is a cyclic, we get $w \in A \cap B$. It follows from

$$d(z, w) = d(Tz, Tw) = k[d(z, w) + d(Tz, z) + d(Tw, w)]$$

that $(1 - k)d(z, w) \leq 0$ where $k \in (0, \frac{1}{3})$. Thus $z = w$ and hence z is the unique fixed point of T. □

The following corollary is a special case of Theorem 2.3.

Corollary 2.2. *Let T be a self map on a complete metric space (X, d). If for some $x \in X$, there exists a $k \in (0, \frac{1}{3})$ such that*

$$d(Tx, Ty) \leq k[d(x, y) + d(Tx, x) + d(Ty, y)], \ \forall x, y \in X \tag{2.10}$$

then, T has a unique fixed point.

The last cyclic contraction considered in this section is the Ćirić type cyclic contraction defined below.

Definition 2.5. Let A and B be non-empty subsets of a metric space (X, d). A cyclic map $T : A \cup B \to A \cup B$ is said to be a Ćirić type cyclic contraction if there exists a $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kM(x, y), \forall x \in A,\ \forall y \in B \tag{2.11}$$

where $M(x, y) = \max\{d(x, y), d(Tx, x), d(Ty, y)\}$

The fixed point theorem of the Ćirić type cyclic contraction reads as follows.

Theorem 2.4. *Let A and B be non-empty closed subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ be a Ćirić type cyclic contraction. Then T has a unique fixed point in $A \cap B$.*

Proof. Take $x \in A$. Due to (2.11), we have

$$d(Tx, Ty) \leq kM(x, y) \tag{2.12}$$

If $M(x, y) = d(x, y)$, Theorem 1.1 implies the desired result. Consider the case $M(x, y) = d(Tx, x)$, then for $y = Tx$, the expression (2.12) turns into

$$d(Tx, T^2x) \leq kd(Tx, x) \tag{2.13}$$

which implies that $d(T^{n+1}x, T^nx) \leq k^n d(Tx, x)$ and hence

$$\sum_{n=1}^{\infty} d(T^{n+1}x, T^nx) \leq \left(\sum_{n=1}^{\infty} t^n\right) d(Tx, x) < \infty.$$

So, $\{T^nx\}$ is a Cauchy sequence which converges to a limit, say $z \in A \cup B$. The sequence $\{T^{2n}x\}$ is in A and the sequence $\{T^{2n-1}x\}$ is in B and both sequences tend to same limit z. From the fact that A and B are closed, we conclude $z \in A \cap B$. Hence, $A \cap B \neq \emptyset$. Now from,

$$\begin{aligned} d(Tz, z) &= \lim_{n\to\infty} d(Tz, T^{2n}x) \\ &\leq k \lim_{n\to\infty} M(z, T^{2n-1}x) = k \lim_{n\to\infty} d(Tz, z) \\ &\leq kd(Tz, z) \end{aligned}$$

it follows that $(1 - k)d(Tz, z) = 0$, where $k \in (0, 1)$ which yields $d(Tz, z) = 0$ and thus $Tz = z$. For the uniqueness proof of z, assume that there exists $w \in A \cup B$ such that $z \neq w$ and $Tw = w$. The map T is a cyclic, so $w \in A \cap B$. Then

$$d(z, w) = d(Tz, Tw) = kM(z, w) = d(z, Tz) = 0,$$

implying $z = w$ which completes the uniqueness proof.

Consider the last case $M(x,y) = d(Ty,y)$, then for $y = Tx$, the expression (2.12) turns into

$$d(Tx,T^2x) \leq kd(T^2x,Tx) \tag{2.14}$$

which is impossible since $k \in (0,1)$. □

Finally, we state the following corollary.

Corollary 2.3. *Let A and B be non-empty closed subsets of a complete metric space (X,d) and let $T : A \cup B \to A \cup B$ with $T(A) \subset B$ and $T(B) \subset A$. Suppose that there exists $k \in (0,1)$ such that*

$$d(Tx,Ty) \leq ku(x,y), \forall x \in A,\ \forall y \in B \tag{2.15}$$

where $u(x,y) \in \{d(x,y), d(Tx,x), d(Ty,y)\}$. Then T has a unique fixed point in $A \cap B$.

3 Cyclic Meir-Keeler Contractions

In this section we introduce a generalization of cyclic Meir-Keeler contraction and a fixed point theorem for this contraction.

Definition 3.1. (See [4]) Let (X,d) be a metric space, and A and B be nonempty subsets of X. Assume that $T : A \cup B \to A \cup B$ is a cyclic map such that for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x,y) < d(A,B) + \varepsilon + \delta \ \text{ implies } \ d(Tx,Ty) < d(A,B) + \varepsilon, \forall x \in A,\ \forall y \in B \tag{3.1}$$

Then T is said to be a cyclic Meir-Keeler contraction.

The definition of a generalized Reich type cyclic Meir-Keeler contraction reads as:

Definition 3.2. Let (X,d) be a metric space, and A and B be nonempty subsets of X. Assume that $T : A \cup B \to A \cup B$ is a cyclic map such that, for some $x \in A$, and for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\begin{gathered} R(x,y) < d(A,B) + \varepsilon + \delta \\ \text{implies } \ d(Tx,Ty) < d(A,B) + \varepsilon, \forall x \in A,\ \forall y \in B, \end{gathered} \tag{3.2}$$

where $R(x,y) = \frac{1}{3}[d(x,y) + d(Tx,x) + d(Ty,y)]$. Then T is said to be a generalized Reich type cyclic Meir-Keeler contraction.

Next, we prove the following propositions which we need in the proof of the fixed point theorem.

Proposition 3.1. *Let A and B be nonempty and closed subsets of a metric space X and let $T : A \cup B \to A \cup B$ be a generalized Reich type cyclic Meir-Keeler contraction. If $x \in A$ satisfies condition (3.2), then $d(T^{n+1}x, T^n x) \to d(A, B)$, as $n \to \infty$.*

Proof. Suppose that T is generalized Reich type cyclic Meir-Keeler contraction. Take $x \in A$ for which (3.2) holds. Since either n or $n+1$ is even, then for each $x \in A$, we have $\frac{1}{3}[d(T^n x, T^{n-1}x) + d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)] \geq d(A, B)$.

Consider the case

$$\frac{1}{3}[d(T^n x, T^{n-1}x) + d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)] = d(A, B).$$

Then due to (3.2) we have $d(T^{n+1}x, T^n x) < d(A, B) + \varepsilon$ which is equivalent to

$$d(T^{n+1}x, T^n x) < \frac{1}{3}[d(T^n x, T^{n-1}x) + d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)] + \varepsilon.$$

Thus we have

$$d(T^{n+1}x, T^n x) \leq d(T^n x, T^{n-1}x), \text{ as } \varepsilon \to 0.$$

Now, consider the other case, that is,

$$\frac{1}{3}[d(T^n x, T^{n-1}x) + d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)] > d(A, B).$$

Set $\varepsilon_1 = \frac{1}{3}[d(T^n x, T^{n-1}x) + d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)] - d(A, B) > 0$. According to (3.2), for this ε_1, there exists δ_1 such that

$$d(T^{n+1}x, T^n x) < d(A, B) + \varepsilon_1 = \frac{1}{3}[d(T^n x, T^{n-1}x) + d(T^{n+1}x, T^n x) + d(T^n x, T^{n-1}x)].$$

Hence, $d(T^{n+1}x, T^n x) \leq d(T^n x, T^{n-1}x)$ for all $n \in \mathbb{N}$.

Let $s_n = d(T^{n+1}x, T^n x)$. Clearly $\{s_n\}$ is a non-increasing sequence bounded below by $d(A, B)$. Therefore $\{s_n\}$ converges to some s with $s \geq d(A, B)$.

We now show that $s = d(A, B)$ by assuming the contrary, that is, $s > d(A, B)$. Set $\varepsilon = s - d(A, B) > 0$. Then, there exists $\delta > 0$ for which (3.2) holds. Since $\{d(T^{n+1}x, T^n x)\} \to s$, there exist a $n_0 \in \mathbb{N}$ such that

$$s \leq \frac{1}{3}[d(T^n x, T^{n-1}x) + d(T^{n+2}x, T^{n+1}x) + d(T^{n+1}x, T^n x)] < \varepsilon + d(A, B) + \delta, \quad \forall n \geq n_0.$$

Thus,

$$d(T^{n+2}x, T^{n+1}x) < d(A, B) + \varepsilon = s, \ \forall n \geq n_0$$

which is a contradiction. Hence $s = d(A, B)$. □

Proposition 3.2. *Let A and B be nonempty and closed subsets of a metric space X and let $T : A \cup B \to A \cup B$ be a Reich type cyclic Meir-Keeler contraction. Let also $d(A, B) = 0$. Then, for each $\varepsilon > 0$, there exist $n_1 \in \mathbb{N}$ and $\delta > 0$ such that*

$$d(T^p x, T^q x) < \varepsilon + \delta \textit{ implies } d(T^{p+1}x, T^{q+1}x) < \varepsilon \tag{3.3}$$

for $p, q \geq n_1$.

Proof. Take $x \in X$ for which (3.2) is satisfied. Since T is a Reich type cyclic Meir-Keeler contraction, then for a given $\varepsilon > 0$, there exists $\delta > 0$ for which (3.2)holds, that is,

$$\begin{array}{l} \frac{1}{3}[d(x,y) + d(Tx,x) + d(Ty,y)] < \varepsilon + \delta \\ \text{implies } \ d(Tx,Ty) < \varepsilon, \ \forall x \in A, \ \forall\, y \in B \end{array} \tag{3.4}$$

Without loss of generality we can choose $\delta < \varepsilon$. Regarding $d(A,B) = 0$ and making use of Proposition 3.1, one can choose $n_1 \in \mathbb{N}$ in a way that

$$d(T^n x, T^{n+1}x) < \frac{\delta}{2}, \ \text{ for each } \ n \geq n_1. \tag{3.5}$$

We claim that $d(T^p x, T^q x) < \varepsilon + \delta$ implies $d(T^{p+1}x, T^{q+1}x) < \varepsilon$. Take $p, q \in \mathbb{N}$ such that $p, q \geq n_1$. Suppose that $d(T^p x, T^q x) < \varepsilon + \delta$. Without loss of generality we may assume $T^p x \in A$ and $T^q x \in B$ with $p = 2n$ and $q = 2m - 1$. Otherwise, interchange the indices respectively. Thus we have $d(T^p x, T^q x) = d(T^{2n}x, T^{2m-1}x) < \varepsilon + \delta$, for $m \geq n$. Then, from(3.5) we get

$$\frac{1}{3}[d(T^{2m-1}x, T^{2n}x) + d(T^{2m}x, T^{2m-1}x) + d(T^{2n+1}x, T^{2n}x)] \leq \frac{1}{3}[\varepsilon + \delta + \frac{\delta}{2} + \frac{\delta}{2}] < \varepsilon + \delta. \tag{3.6}$$

Consider (3.4) under the assumption $y = T^{2n}x$. The inequality (3.6) yields

$$d(T^{2n+1}x, T^{2m}x) = d(T^{p+1}x, T^{q+1}x) < \varepsilon.$$

Therefore, we conclude that for a given $\varepsilon > 0$, there exist $n_1 \in \mathbb{N}$ and $\delta > 0$ such that

$$d(T^p x, T^q x) < \varepsilon + \delta \ \text{ implies } \ d(T^{p+1}x, T^{q+1}x) < \varepsilon \tag{3.7}$$

where $p, q \geq n_1$. □

Theorem 3.1. *Let X be a complete metric space, and A and B non-empty, closed subsets of X such that $d(A,B) = 0$. Let $T : A \cup B \to A \cup B$ be a Reich type cyclic Meir-Keeler contraction. Then, there exists a unique fixed point, say $z \in A \cap B$, such that for each x satisfying (3.2), the sequence $\{T^{2n}x\}$ converges to z.*

Proof. Take $x \in A$. We will show that $\{T^n x\}$ is a Cauchy sequence. Assume the contrary. Then there exists an $\varepsilon > 0$ and a subsequence $\{T^{n(i)}x\}$ of $\{T^n x\}$ for which

$$d(T^{n(i)}x, T^{n(i+1)}x) > 2\varepsilon. \tag{3.8}$$

For this ε, there exists $\delta > 0$ such that

$$R(x,y) < \varepsilon + \delta \ \text{ implies } \ d(Tx,Ty) < \varepsilon \tag{3.9}$$

where $R(x,y) = \frac{1}{3}[d(x,y) + d(Tx,x) + d(Ty,y)]$. Set $r = \min\{\varepsilon, \delta\}$ and $d_m = d(T^m x, T^{m+1}x)$. Due to Proposition 3.1, one can choose $n_0 \in \mathbb{N}$ such that

$$d_m = d(T^m x, T^{m+1}x) < \frac{r}{4}, \ \text{ for } \ m \geq n_0. \tag{3.10}$$

Now, let $n(i) \geq n_0$. Suppose that $d(T^{n(i)}x, T^{n(i+1)-1}x) \leq \varepsilon + \frac{r}{2}$. From the triangle inequality we have

$$\begin{aligned} d(T^{n(i)}x, T^{n(i+1)}x) &\leq d(T^{n(i)}x, T^{n(i+1)-1}x) + d(T^{n(i+1)-1}x, T^{n(i+1)}x) \\ &\leq \varepsilon + \tfrac{r}{2} + d_{n(i+1)-1} < 2\varepsilon, \end{aligned} \tag{3.11}$$

which contradicts the assumption (3.8). Thus, there are values of k satisfying $n(i) \leq k \leq n(i+1)$ such that $d(T^{n(i)}x, T^k x) > \varepsilon + \frac{r}{2}$. Assume that $d(T^{n(i)}x, T^{n(i)+1}x) \geq \varepsilon + \frac{r}{2}$. Then $d_{n(i)} = d(T^{n(i)}x, T^{n(i)+1}x) \geq \varepsilon + \frac{r}{2} \geq r + \frac{r}{2} > \frac{r}{4}$ which contradicts (3.10). Hence, there are values of k with $n(i) \leq k \leq n(i+1)$ such that $d(T^{n(i)}, T^k x) < \varepsilon + \frac{r}{2}$ where k and $n(i)$ have the opposite parity. Choose the smallest integer k with $k \geq n(i)$ satisfying $d(T^{n(i)}x, T^k x) \geq \varepsilon + \frac{r}{2}$. Then,

$$d(T^{n(i)}x, T^{k-1}x) < \varepsilon + \frac{r}{2}. \tag{3.12}$$

Thus,

$$d(T^{n(i)}x, T^{k-1}x) \leq d(T^{n(i)}x, T^{k-1}x) + d(T^{k-1}x, T^k x) < \varepsilon + \frac{r}{2} + \frac{r}{4} = \varepsilon + \frac{3r}{4}. \tag{3.13}$$

Hence, there exists an integer k satisfying $n(i) \leq k \leq n(i+1)$ such that

$$\varepsilon + \frac{r}{2} \leq d(T^{n(i)}x, T^k x) < \varepsilon + \frac{3r}{4}. \tag{3.14}$$

Making use of the inequalities

$$d(T^{n(i)}x, T^k x) < \varepsilon + \frac{3r}{4} < \varepsilon + r$$

$$d(T^{n(i)}x, T^{n(i)+1}x) = d_{n(i)} < \frac{r}{4} < \varepsilon + r$$

$$d(T^k x, T^{k+1}x) = d_k < \frac{r}{4} < \varepsilon + r$$

we conclude

$$\begin{aligned} R(T^{n(i)}x, T^k x) &= \tfrac{1}{3}[d(T^{n(i)}x, T^k x) + d(T^{n(i)}x, T^{n(i)+1}x) + d(T^{k+1}x, T^k x)] \\ &\leq \tfrac{1}{3}[\varepsilon + r + \varepsilon + r + \varepsilon + r] = \varepsilon + r \end{aligned} \tag{3.15}$$

implying $d(T^{n(i)+1}x, T^{k+1}x) < \varepsilon$. But, on the other hand

$$\begin{aligned} d(T^{n(i)+1}x, T^{k+1}x) &\geq d(T^{n(i)}x, T^k x) - d(T^{n(i)}x, T^{n(i)+1}x) - d(T^k x, T^{k+1}x) \\ &> \varepsilon + \tfrac{r}{2} - \tfrac{r}{4} - \tfrac{r}{4} = \varepsilon \end{aligned}$$

which contradicts the preceding inequality. Therefore, $\{T^n x\}$ is a Cauchy sequence. Hence, $\{T^n x\}$ converges to some $z \in A$. Consider now the sequence $\{d(T^{2n-1}x, z)\}$. From

$$0 \leq d(T^{2n-1}x, z) \leq d(T^{2n-1}x, T^{2n}x) + d(T^{2n}x, z) \tag{3.16}$$

it clearly converges to zero, that is, $\lim_{n\to\infty} d(T^{2n-1}x, z) = 0$. Since $\{T^{2n-1}x\}$ is a sequence in B, it converges to $z \in B$. However, both A and B are closed, so, we get $z \in A \cap B$.

Let us show now that z is a fixed point of T, that is, $Tz = z$. First we observe that

$$d(T^{2n}x, Ty) < R(T^{2n-1}x, y) \text{ if } T^{2n-1}x \neq y. \tag{3.17}$$

Actually, it suffices to show that (3.2) is equivalent to the following condition: For each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{gathered} R(T^{2n-1}x, y) < \varepsilon + \delta \\ \text{implies } d(T^{2n}x, Ty) < \varepsilon, n \in \mathbb{N}, y \in A \end{gathered} \tag{3.18}$$

where $R(T^{2n-1}x, y) = \frac{1}{3}[d(T^{2n-1}x, y) + d(T^{2n}x, T^{2n-1}x) + d(Ty, y)]$ and recall that $d(A, B) = 0$.

If $T^{2n-1}x = y$ then $R(T^{2n-1}x, y) = 0$ and thus (3.17) is satisfied. Suppose that $R(T^{2n-1}x, y) \neq 0$ and fix $\varepsilon \leq R(T^{2n-1}x, y)$. Choose a $\delta > 0$ such that (3.18) holds. Notice that if $R(T^{2n-1}x, y) \leq d(T^{2n}x, Ty)$, we get a contradiction with (3.18). Then clearly, (3.2) implies (3.18). Now let (3.18) hold. Fix $T^{2n-1}x, y \in A \cup B$ and $\varepsilon > 0$. If $R(T^{2n-1}x, y) < \varepsilon$, we have $d(T^{2n}x, Ty) \leq R(T^{2n-1}x, y)$ and consequently $d(T^{2n}x, Ty) < \varepsilon$ because of (3.18). If $R(T^{2n-1}x, y) \geq \varepsilon$, then (3.2) follows immediately. Thus, (3.18) and (3.2) are equivalent provided that $d(A, B) = 0$.

Making use of (3.17) we have,

$$\begin{aligned} d(Tz, z) &= \lim_{n\to\infty} d(T^{2n}x, Tz) < \lim_{n\to\infty} R(T^{2n-1}x, z). \\ &< \lim_{n\to\infty} \tfrac{1}{3}[d(T^{2n-1}x, z) + d(T^{2n}x, T^{2n-1}x) + d(Tz, z)] \end{aligned}$$

which implies that $d(Tz, z) < \frac{1}{3}d(Tz, z)$. But this is impossible, hence $Tz = z$. Lastly, we show z is a unique fixed point of T. Assume that there exists $w \in A \cap B$ such that $z \neq w$ and $Tw = w$. From (3.17) it follows that $d(w, z) < R(z, w) = \frac{1}{3}[d(z, w) + d(Tz, z) + d(Tw, w)] = \frac{1}{3}d(z, w)$ which clearly implies $z = w$. □

References

[1] M.A.Al-Thafai and N.Shahzad: Convergence and existence for best proximity points. *Nonlinear Analysis* **70**(2009)3665-3671.

[2] A.A. Eldered and P. Veeramani: Proximal pointwise contraction. *Topology and its Applications* **156**(2009)2942-2948.

[3] A.A. Eldered and P. Veeramani: Convergence and existence for best proximity points. *J. Math. Anal. Appl.* **323**(2006)1001-1006.

[4] S. Karpagam and Sushama Agrawal: Best proximity points theorems for cyclic Meir-Keeler contraction maps, Nonlinear Analysis, doi:10.1016/j.na.2010.07.026(in press)

[5] W.A. Kirk and P.S. Srinavasan and P. Veeramani: Fixed Points for mapping satisfying cylical contractive conditions *Fixed Point Theory* **4**,(2003)79-89.

[6] G. Petruşhel: Cyclic representations and periodic points, *Studia Univ. Babes-Bolyai Math.*, **50**(2005),107-112.

[7] Păcurar, M., Rus, I.A.: Fixed point theory for cyclic φ-contractions, *Nonlinear Anal.*, 72, (3-4),1181-1187 (2010).

[8] Sh.Rezapour and M. Derafshpour and N.Shahzad: Best proximity point of cyclic φ-contractions in ordered metric spaces, Topological Methods in Nonlinear Analysis,(in press).

Erdal KARAPINAR is an assistant professor in Department Mathematics at the Atılım University. He earned an MS in Mathematics (Point Set Topology) in 1998 and Ph.D. in Mathematics (Functional Analysis) in 2004. He is author of more than 20 articles most of which are published in international ISI journals. His research activity has been developed in the framework of ”linear topological invariants”, ” Orlicz spaces”, ”Fréchet spaces”, ”Köthe spaces”, and ”fixed point theory and its applications”. Current research interests of E. Karapınar is centered on ”best proximity points on cyclic mapping contractions”. He is a member of the Turkish Mathematics Society

Inci MÜFTÜOĞLU ERHAN is an assistant professor in Department Mathematics at the Atılım University. She earned an MS in Mathematics (Applied Mathematics) in 1998 and Ph.D. in Mathematics (Applied Mathematics) in 2003. She is author of more than 10 articles most of which are published in international ISI journals. Her research activity has been developed in the framework of ”Numerical Analysis”, ”Numerical Solutions of ODE”, ”Numerical solutions of PDE” and ”fixed point theory and its applications”. Current research interests of İ. M. Erhan is centered on ”best proximity points on cyclic mapping contractions”. She is a member of the Turkish Mathematics Society