Quantum Physics Letters
An International Journal

http://dx.doi.org/10.18576/qpl/060103

Surface-Induced Spatial-Temporal Structures In Boundary Problems of Hamiltonian Mechanics

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Received: 7 Dec. 2016, Revised: 11 Mar. 2017, Accepted: 17 Mar. 2017
Published online: 1 Apr. 2017

Abstract: An initial value boundary problem for the Liouville equation with nonlinear dynamic boundary conditions which describes velocity of changing on time of the probability of particles at walls that confines the particles. These velocities are nonlinear functions of the density of the probability of particles to occupied the flat walls. The attractor of the problem has been constructed. This attractor contains periodic piecewise constant functions with finite, countable or uncountable points of discontinuities on a period, which propagates along characteristics of the Liouville equation. We call such elements of the attractor by the distributions of relaxation, pre-turbulent and turbulent type, correspondingly — by the classification of Sharkovsky. There are also random distributions of particles, which can be produced by the nonlinear feedback on the walls. The results has been obtained by the reduction of the problem to dynamical system which is described by system of difference equations, depending on coordinates and momenta as of parameters. It is shown that the changing of these parameters leads to period doubling bifurcations of elements of the attractor on 4 - dimensional torus. The problem is solved in class of quasi-periodic functions.

Keywords: Hamiltonian systems • Liouville equation • initial value boundary problem • asymptotic solutions of relaxation, pre-turbulent and turbulent type asymptotic • periodic piecewise constant distributions • system of difference equations • attractor.

1 Introduction

In Hamiltonian systems, there is a separation between of slow and fast degrees of freedom. R. Mackay assumes that the fast variables have the Anosov mixing dynamics. A. Politi and A. Torcini review the concepts of stable chaos, that is, 'the presence of irregular behaviour even though the dynamics is still locally stable' (see, [1],p.8). The transition to chaos in the Hamiltonian systems is different than for dissipative systems. In integrable or non-chaotic Hamiltonian systems the motion is 'quasiperiodic'. A typical example of integrable Hamiltonian systems is harmonic oscillator. Spatial-temporal chaos take place when dynamic behaviour exhibits both spatial disorder and temporal disorder. An attractor of initial boundary value problem contains asymptotic stable waves or pulses, which are called solitons.

In this paper, we consider an initial boundary value problem for the linear Liouville equation with differential dynamic boundary conditions. Such problem describes distributions of probability of free particles in confined medium with process of recombination of particle at a flat walls. It is shown that there are surface induced spatial-temporal distributions of density of particle. We study a structure of attractor of the problem. Similar boundary problem for the transport equation with one spatial variable first has been considered by Sharkovsky [2] by method of reduction of a boundary problem to a difference equation with continuous time. In typical cases, an initial problem for difference equation admits an attractor which contains deterministic piecewise constant periodic functions with finite, countable or uncountable points of discontinuities on a period. For special parameters, there are also random functions which are elements of attractor [3,4,5]. In [2] has been considered also similar problem for two non connected linear transport equations with nonlinear functional boundary conditions and has been shown that elements of attractor can be represented as the functions

$$u(x,y,t) = \varphi(x + a_{11}t, y + a_{12}t), \tag{1}$$

$$v(x,y,t) = \psi(x + a_{21}t, y + a_{22}t) \tag{2}$$

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where $a_{11}, a_{12}, a_{21}, a_{22} \in R$. It is shown that such problem can be reduced to the study of structure of attractor of one difference equation with two arguments, so that

$$w(z+b,\tau+c) = f[w(z,\tau)], \tag{3}$$

where $b,c \in R$, and function f is given by the boundary conditions

$$u = v \quad \text{as} \quad y = 0, \quad v = f[u] \quad \text{as} \quad y = 1. \tag{4}$$

The problem has been considered in the region $-\infty < x < \infty, 0 < y < 1, t > 0$ (see, ([2], p.257). It has been shown that equation (3) by transformation of variable can be reduced to the difference equation

$$w(\sigma,\theta) = f(w(\sigma,\theta-1) \tag{5}$$

with initial condition $w_0(\sigma,\theta)$ on interval $-1,0)$.

If we consider σ as a parameter and equation (5) as an ordinary difference equation, it can be shown that $w(\sigma,\theta)$ tends to N - periodic function $w^*(\sigma,\theta)$, where N is least common multiple of periods of attractive circles of the map f. In origin variables a limit function $u^*(x,y,t) : R \times [0,1] \times R^+ \to 2^N$ has equal values on characteristics

$$x + a_{21}t = \alpha, \quad y + a_{22}t = \beta. \tag{6}$$

In this paper, the above results will be generalised on the bounded region $(x \in [0,1], y \in [0,1])$ for the Liouville equation with Hamiltonians $H(x,p) = \frac{1}{2}p^2$ and $H(x,p) = \frac{1}{2}(x^2 - p^2)$. Introduction of flat walls at point $y = 0$ and $y = 1$ with nonlinear dynamic boundary conditions and depending of the Hamiltonian on x leads to the fact that the boundary problem admits the reduction to the system of nonlinear difference equations, depending on x,p as on 'parameters', where x,p are coordinate and impulse of trajectories of corresponding dynamic systems. Thus the boundary problem is reduced to the study of asymptotic behaviour of trajectories of system of difference equations [6,7]:

$$u(x,p,t+1) = f[u(x,p,t)], \quad u(t) \in C^2(\cdot,\cdot,R^+,R^n), n \geq 2, \tag{7}$$

where $x,p \in [0,l] \times [0,1]$ can be considered as parameters. We confined itself by the study of hyperbolic or Anosov type systems. It means that a set of non-wandering points of the map $f : R^n \to R^n$ is finite and hyperbolic. Then we can consider x,p as parameters so that equation (7) is an ordinary difference equation. As a result, solution of equation (7) tends to N - periodic function $u^*(x,p,t)$ as $t \to \infty$. A limit function $u^*(x,p,t) \in A^+$ almost all points $t \in R^+$, where A^+ is a set of attractive fixed points of the map f, excluding finite, countable or uncountable set of points $t^* \in \Gamma$. Thus the limit function $u^*(x,p,t)$ is asymptotic 2^N - periodic piecewise constant function (see, [2], p. 258).

For one dimensional initial boundary value problem similar asymptotic has been considered for the Cahn-Hilliard equation which describes the evolution of one component of binary mixtures [8] or binary alloys [8, 9]. For $3D$ - the Cahn-Hilliard equation, similar results has been obtained in the paper [10]. A similar boundary problem in Hamiltonian Mechanics has been considered in paper [11].

In Section 1, it will be considered boundary problem with the Hamiltonian $H(p) := \frac{1}{2}p^2$ not depending on x. It is shown that asymptotic solutions have the form $u(x,t) := u(p,t-x/p)$ for every finite p and the function $u(p,\zeta) \Rightarrow P(p,\zeta)$ in C^2 - norm for almost all points $\zeta \in R^+ \Gamma$, where Γ is a set of points of discontinuities. Γ is finite, countable or uncountable. Such limit solutions are typical for higher dimensional Hamiltonian systems with finite or infinite localized vibrations (see, [13], p.15), but we show that such type asymptotic distributions exists also for one-dimensional Hamiltonian systems.The structure of Γ depends on the topological structure of boundary nonlinear functions. The limit function $P(p,\zeta)$ is piecewise constant periodic function. In Section 2, it will be considered boundary problem with the Hamiltonian $H(p) := \frac{1}{2}p^2 + \frac{1}{2}x^2$, depending on x. It is shown that asymptotic solutions have the form $u(x,p,t) := u_1(t-x/p) + u_2(t - \arccos p)$, where $u_1(\zeta)$ and $u_2(\eta)$ are piecewise constant periodic functions. In Section 3, it will be considered some properties of hyperbolic structural stable maps $\Phi : R^2 \to R^2$ to which the considered initial boundary value problems can be reduced. In Section 4, applications for boundary problems of physics of condensed matter has been considered.

2 Formulation of problem

Let us consider a dynamic system with coordinates $q = (q_1,...,q_n)$ (this can be Cartesian coordinates, angles, arc length of a curve and so on) and momentum $p = (p_1,...,p_n)$, where $i = 1,2,...,n$. The Lagranhian is $L = T - U$, where $T := T(q,\dot{q})$ is the kinetic energy and $U := U(q)$ is the potential energy. The Euler equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}. \tag{8}$$

Equation (8) is equivalent to equations $\dot{q} = H_p$ and $\dot{p} = H_q$, where $H := p\dot{p} - L$. This equation is called the variational form of the classical mechanics or the Lagrange form (see, [12], p.7).

The motion of particles of mechanical system can be described by the Liouville equation

Then a function $u(x,p)$ determines a probability $u(x,p)d^nxd^np$, so that a system can be found in a phase space volume d^nxd^np. The Liouville equation equation is

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\dot{x}_i + \frac{\partial u}{\partial p_i}\dot{p}_i\right) = 0, \tag{9}$$

where $u(x,p)$ determines a probability $u(x,p)d^nxd^np$ to find a particles in a region d^nqd^np. The Liouville equation follows from the continuity equation

$$\frac{\partial u}{\partial t}+\frac{\partial J}{\partial x}=0 \tag{10}$$

for a flow of particles $J=vu$, where v is a velocity of particles. Indeed, from (9) and (10) it follows that

$$\frac{\partial u}{\partial t}+\sum_{i=1}^{n}\left(\frac{\partial(u\dot{x}_i)}{\partial x_i}+\frac{\partial(u\dot{p}_i)}{\partial p_i}\right)=0. \tag{11}$$

If we assume that $\dot{x}_i$ is independent from x_i and $\dot{p}_i$ is independent from p_i, then the Liouville equation and the continuity equation are identical.

Thus, the Hamiltonian mechanics use parameters $(q,p)\in R^1_q\times R^1_p$, where p is a generalized momentum, and q is a generalized position. In Newtonian mechanics, the position x should be expressed in rectangular coordinates, but in Hamiltonian mechanics, a position q, generally, does not to be expressed in rectangular coordinates. This gives greater freedom in the choice of coordinates corresponding to the description of the dynamic system. Below we define $q(s):=x(s)$ and $p(s):=p(s)$, where $s\in R^1$ is a parameter. Then the Hamiltonian $H:R^1_x\times R^1_p\to R^1$ generates trajectories:

$$\dot{x}(t)=H_p(x(t),p(t)),\quad \dot{p}(t)=-H_x(x(t),p(t)), \tag{12}$$

where the Hamiltonian is equal to the total energy of the system. Thus this approach use the Hamiltonian canonical equations which are equivalent to the Lagrange form of variational equation. Then from (9) it follows that

$$\frac{du}{dt}(x(t),p(t))\equiv 0. \tag{13}$$

If $u:=u(x,p,t)$, then we obtain that

$$\frac{du}{dt}=\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}\dot{x}_t+\frac{\partial u}{\partial p}\dot{p}_t=\frac{\partial u}{\partial t}+H_p\frac{\partial u}{\partial x}-H_x\frac{\partial u}{\partial p}. \tag{14}$$

Thus, we have the Liouville, or transport equation

$$\frac{\partial u}{\partial t}+H_p\frac{\partial u}{\partial x}-H_x\frac{\partial H}{\partial p}=0. \tag{15}$$

The Liouville equation follows also for the determination of the Poisson brackets

$$[u,H]=\frac{\partial u}{\partial q}\frac{\partial H}{\partial p}-\frac{\partial u}{\partial p}\frac{\partial H}{\partial q}. \tag{16}$$

But if variables q and p satisfies to the Hamiltonian equations, the

$$\frac{du}{dt}=\frac{\partial H}{\partial t}+[u,H], \tag{17}$$

and we again obtain the Liouville equation. But this is true only if q not depends on $\dot{q}$.

Next, we consider the functional boundary conditions

$$u(0,y,t)=\Phi_1[u(l,y,t)],\quad u(x,0,t)=\Phi_2[u(x,l,t)], \tag{18}$$

where $\Phi_1,\Phi_2:R^3\to R^1$ are given functions. We assume that there is an open bounded interval $I\subset R^1$ such that $\Phi_1(I)\in I,\Phi_2(I)\in I$. Then we can prove that solutions exist in the region $\Pi:=\{0<x<l_1,0<y<l_2\}$ for any $t>0$ if $l_1=l_2=l$. (The case $l_1\neq l_2$ will be considered later).

These boundary conditions can be obtained from the dynamic boundary conditions

$$u_t=f_1[u]\quad as\quad \text{x=0},\quad u_t=f_2[u]\quad as\quad \text{x=l}. \tag{19}$$

Indeed, we assume that system of ordinary differential equation (19) has a first integral

$$W[u(0,t),u(l,t)]=\mu, \tag{20}$$

where $\mu=W[u(0,0),u(l,0)]$. Next, we assume that there are $u,v\in I$, where I is an open bounded interval, and $\mu\in R^1$ such that functional relation (20) is globally solvable, so that

$$u(0,t)=\Phi_\mu[v(0,t)],\quad t>0, \tag{21}$$

where $\Phi_\mu:I\to I$ is a given function. As a result, we obtain the functional boundary conditions. But the differential boundary conditions have a simple physical sense, because these conditions describes velocity of a probability to find particles at a given points of boundaries of the dynamic system.

Here, we consider a family of real analytic maps $\Phi_\mu:I\to I$. We are considering all orbits of this map for each fixed $\mu\in R$, and orbits of typical points, and limit sets of these orbits. Let $\omega(u)$ is a set of limit points of the sequences $u,\Phi_\mu[u],\Phi^2_\mu[u],\ldots$. Then, as shown in [2], there are the following types of orbits: (1) $\omega[u]$ is a periodic orbit with multiplier with absolute value ≤ 1; (2) $\omega(x)=\omega(c)$, where c is a critical point of Φ_μ, so that $\Phi'_\mu[c]=0$ with the properties: (i) $\omega(c)$ is the Cantor set, (ii) $\omega(c)$ has zero Lebesque measure, (iii) $\omega(c)$ is equal to a finite union of intervals $I=\bigcup_{k=0}^{n}I_n$, where $n=1,2,\ldots$, which contains a point c, so that Φ_μ is topologically transitive that is there are orbits, which are dense in I.

The simplest case is when Φ_μ is unimodal. It means that Φ_μ has one extremum, and the Schwarzian derivative $\hat{S}$ is:

$$\hat{S}\Phi_\mu(u):=\frac{\Phi'''_\mu(u)}{\Phi'_\mu(u)}-\frac{3}{2}\left(\frac{\Phi'''_\mu(u)}{\Phi'_\mu(u)}\right)^2<0. \tag{22}$$

These maps are called by S - unimodal. A simplest example is the logistic map $\Phi_\mu(u)=\mu u(1-u)$.

It is known that if a region $U \in R^n$, and $A(u) = (a_1(u),...,a_n(u))$ is a smooth vector field in $U, u_0 \in U, A(u_0) \neq 0$, then there is a neighbourhood of the point u_0 such that the system of n antinomic differential equations has $n-1$ functional independent first integrals.

Now we suppose that $-H_p = a_1$, $H_x = a_2$ and let $p := y$, where $a_1, a_2 \in R^1$. Then solutions of equation (15) have the form:

$$H(x,y,t) := \varphi(t - x/a_1, t - y/a_2). \tag{23}$$

Let $\varphi(\zeta,\eta) := \varphi_1(\zeta) + \varphi_2(\eta)$, where $\zeta = t - x/a_1$, $\eta = t - y/a_2$. Note that functions φ_1, φ_2 are constants along lines $t - x/a_1 = c_1, t - y/a_2 = c_2, c_1, c_2 \in R^1$. This allows, using the functional boundary conditions, to obtain the following functional relations:

$$\varphi_1(x,t) + \varphi_2(y,t) = \Phi_1[\varphi_1(x,t - l/a_1) + \varphi_2(y,t)], \tag{24}$$

$$\varphi_1(x,t) + \varphi_2(y,t) = \Phi_2[\varphi_1(x,t) + \varphi_2(y,t - l/a_2)]. \tag{25}$$

Let us define

$$F_1 := Y_1(x,t) + Y_2(y,t) - \Phi_1[X_1(x,t - l/a_1) + Y_2(y,t)], \tag{26}$$

$$F_2 := Y_1(x,t) + Y_2(y,t) - \Phi_2[Y_1(x,t) + X_2(y,t - l/a_2)], \tag{27}$$

where $F := (F_1, F_2)$, so that $F : R^2 \times R^2 \to R^2$. Next, we assume that at a neighbourhood of a point (X_0, Y_0) we have $F \in C^2, F(X_0,Y_0) = 0$, and determinant $T := \det ||\frac{\partial F}{\partial Y}(X_0,Y_0)||$ is equal to

$$T := \Phi_1'[X_1,Y_2]\Phi_2'[Y_1,X_2] - \Phi_1'[X_1,Y_2] - \Phi_2'[Y_1,X_2] \neq 0. \tag{28}$$

Then there are neighbourhoods $U \in R^2 V \in R^2$ of points X_0, Y_0, and a map $f : U \to V$ such that $f \in C^2$, and

$$F(X,Y) = 0 \quad \Leftrightarrow \quad Y = f(X) \tag{29}$$

for each $X \in U$ and $Y \in V$. The map $f(X)$ is determined by the system of difference equations:

$$\varphi_1(x,t) = f_1[\varphi_1(x,t - l/a_1), \varphi_2(y,t - l/a_2)], \tag{30}$$

$$\varphi_2(y,t) = f_2[\varphi_1(x,t - l/a_1), \varphi_2(y,t - l/a_2)], \tag{31}$$

where $f_1, f_2 : R^2 \to R^1$ are known functions. Thus we obtain difference equations with delay arguments if $a_1, a_2 > 0$. It is known [6] conditions on the map $f := (f_1, f_2)$ when components of limit solutions of equations (30), (31) are piecewise constant periodic functions with finite or infinite points of discontinuities on periods.

If $\Phi_2 = \mu Id$, where Id is identical map and $\mu \in R$, then from (41),(42) it follows that this system is decomposed on two difference equations of the form:

$$\varphi_1(x,t) + \varphi_2(y,t) = \Phi_1[\varphi_1(x,t - l/a_1) + \varphi_2(y,t)], \tag{32}$$

$$\varphi_2(y,t) = \mu\varphi_2(y,t - l/a_2). \tag{33}$$

If $|\mu| < 1$ then $\varphi_2(y,t) \to 0$ as $t \to +\infty$, and from (32) it follows with a given accuracy the limit difference equation:

$$\varphi_1(x,t) = \Phi_1[\varphi_1(x,t - l/a_1)]. \tag{34}$$

For unimodal map $\Phi_1 : I \to I$, asymptotic solutions of equation (34) are piecewise constant $2^N l/a_1$ - periodic distributions, where N is least common multiple of periods of attractive circles of Φ_1, with finite or infinite points of discontinuities on periods [6]. If $|\mu| > 1$, then $\varphi_2(y,t) \to \infty$ as $t \to +\infty$. If $|\mu| = 1$, then we have l/a_1 - periodic solutions of equation (34).

This statement will be proved in the next subsection.

3 Quadratic potential

In this section, we consider a quantum oscillator with hamiltonian $H(x,p) = \frac{1}{2}(x^2 + p^2)$, so that

$$\dot{x} = H_p = p, \quad \dot{p} = -H_x = -x \tag{35}$$

with the initial conditions

$$x(t_0) = x_0, \quad p(t_0) = p_0. \tag{36}$$

For the harmonic oscillator, we have

$$\begin{vmatrix} x(t) \\ p(t) \end{vmatrix} = \begin{vmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{vmatrix} \begin{vmatrix} x(t_0) \\ p(t_0) \end{vmatrix}. \tag{37}$$

This is matrix which represents a clockwise rotation through an angle ωt, so that points in the $x - p$ plane move in circle with frequency ω. It means that each initial region rotates around an origin on $x - p$ plane. Hence, this region keeps shape as it rotates in a plane, and areas are conserved [14].

The Hamiltonian on trajectories of equations (35) is constant. It means that the Hamiltonian system has a first integral $dW(x,p) = 0$. Solutions are $x(t) = \rho\sin t$, $p(t) = \rho\cos t$, where $\rho \in R$, and trajectories of the Hamiltonian equations are on the lines

$$\dot{x}^2(t) + \dot{p}^2(t) = p^2(t) + x^2(t) \equiv \rho^2 \tag{38}$$

where $\rho \in R^+$. Below, for simplicity, we assume that $\rho = 1$. Then the Liouville equation equation for the probability of particle can be written as:

$$\frac{\partial u}{\partial t} + p\frac{\partial u}{\partial x} - \sqrt{1-p^2}\frac{\partial u}{\partial p} = 0. \tag{39}$$

It is a particular example of the general scheme. Indeed, projection on the x - space of solutions of the Hamiltonian is called by a characteristic, or extremal. Suppose that there are x_0 and t_0 such that for $t \in [0,t_0)$ there is a neighbourhood of a point p_0 in the p - space $\varpi_t(x_0)$ such that a map $p(t,x_0,p_0) \to x(t,x_0,p_0)$ is a diffeomorphism from $\varpi_t(x_0)$ onto its image. Then this image contains a neighbourhood $D(x_0)$ which not depends on t. For example, the equations $\dot{p} = -x$ and $\dot{x} = p$ can be written as a differential form $xdx - pdp = 0$, where $x,p \neq 0$. Then the Hamiltonian $H := \frac{1}{2}(x^2+p^2$ is constant on the circles $x^2+p^2 = \rho^2$ and we have the diffeomorphism $p \to \pm\sqrt{1-p^2}$.

This observation has important application for the construction of local field of extremals, so that it is a main technique for the construction of WKB - type asymptotic for solutions of equations [15,16]:

$$\frac{\partial u}{\partial t} = H\left(x, \frac{\partial}{\partial x}\right)u \tag{40}$$

where H is the Hamiltonian.

Solutions of this equation have the form $u(x,p,t) := u_(\zeta) + u_2(\eta)$, where $\zeta = t - x/p$ and $\eta = t - \arccos p$. where impulse $|p| < 1$ and $0 < \arccos p < \pi$. As in Section 1, substituting this representation of a solution in the functional boundary conditions, as we obtain the following functional relations:

$$\varphi_1(x,t) + \varphi_2(p,t) = \Phi_1[\varphi_1(x,t-l_1/p) + \varphi_2(p,t)], \tag{41}$$

$$\varphi_1(x,t) + \varphi_2(p,t) = \Phi_2[\varphi_1(x,t) + \varphi_2(p,t-\arccos l_2)], \tag{42}$$

where $l_1^2 + l_2^2 = 1$.

The structure of an attractor for this system is studied as well as in Section 1. But in this section it will be done the more concrete prove of a scenario of reduction of these functional relations to a system of nonlinear difference equations. Indeed, if $\Phi_1 = \Phi_2 := Id : z \to z, z \in R^1$, then from (42) it follows that

$$\varphi_1(x,t) = \varphi_1(x,t-l_1/p), \quad \varphi_2(p,t) = \varphi_2(p,t-\arccos l_2), \tag{43}$$

and from (43) it follows that a solution of the problem is sum of l_1/p and $\arcsin l_2$ - periodic functions.

Next, we assume that $\Phi_1 \neq Id$, but $\Phi_2 := Id$. then we get that

$$\varphi_1(x,t) + \varphi_2(p,t) = \Phi_1[\varphi_1(x,t-l/p) + \varphi_2(p,t-\arccos l]. \tag{44}$$

If $l_1/p = \arcsin l_2 = \Delta$, where $0 < p < 1$, then solutions of equation (44 can be determined step by step, iterating an initial function $h(t) = \varphi_1(x,t-\Delta) + \varphi_2(p,t-\Delta)$ on interval $[-\Delta, 0)$. We define $w(x,p,t) = \varphi_1(x,t) + \varphi_2(p,t)$. Then equation (44) can be written as autonomic difference equation

$$w(x,p,t) = \Phi_1[w(x,t-l/p)], \quad t \in R^+, \tag{45}$$

where $\Phi_1 : I \to I$ is a given function. We assume that $\Phi_1 \in C^2(I,I)$ and the initial function $h(t) \in C^2(I,I)$, and

$$w(x,p,0) = \Phi_1[w(x,-l/p)], \tag{46}$$

$$w'(x,p,0) = \Phi_1'[w(x,-l/p)]w'(x,-l/p), \tag{47}$$

$$w''(x,p,0) = \Phi_1''[w(x,-l/p)]w'(x,-l/p)^2 + \Phi_1''[w(x,-l/p)]w'(x,-l/p) \tag{48}$$

for each $x,p \in (0,1)$.

Let us define a separator $D := \bigcup_{n\geq 0} \bar{P}^-$, where $\bar{P}^-$ is the closure of a set of attractive points of the map Φ. The separator determines the structure of pre-images of repelling fixed points of a map $\Phi : I \to I$ on interval I. For example, if Φ is monotone with two attractive fixed points and one repelling fixed point a^-, then $D = \Phi^{-1}(a^-)$ is unique point on I. If $\Phi(u)$ is non-monotone on $I_1 \subset I$, where $I_1 := [a_+, a_-]$, so that $\Phi''(u) < 0$ as $u \in I_1$, and a^+ is an attractive fixed point. Then D is countable set with limit point a^+. The union of pre-images of repelling fixed points on I determines the set of points of discontinuities Γ of the corresponding limit function $u(t)$ of the difference equation as $t \to +\infty$ (see, Fig.1).

Now we assume that the map Φ_1 is structural stable. Then there is a finite number P^+ of attractive fixed points of the map. If Φ_1 has an attractive circle of period 1, then a solution $w(t)$ of the difference equation tends to a unique attractive fixed point. If there is an attractive circle of period 1, which is formed by points a_1, a_2, then a solution tends to a piecewise constant periodic distribution as $t \to +\infty$ in C^2 - metric for almost all points $t \in R^+$. If the separator D has only attractive circles of periods 2^i, $i = 0,1,...$, then the set D is countable, and a solution tends to an asymptotic 2^N - periodic piecewise constant function with countable set of points of discontinuities on a period, where N is least common multiple of periods of attractive circles of the map Φ_1. If $l/p \neq \arcsin l$, then there are $m,n \in Z^+$ such that $ml/p = n \arcsin l = q$, where $q \in Z^+$. As a result, there are asymptotic $2^N q$ - periodic piecewise constant functions with countable set of points of discontinuities on a period.

If $\Phi_2 \neq Id$, then we can apply the results of previews Section 2. The difference is only in the delay arguments in the corresponding systems of difference equations — between form of the characteristics $t - x/p = Const$ and $t - \arccos p = const$. As a result, we obtain the system of independent difference equations with continuous time. If $\Phi_2 = Id$, then we obtain the previous result. In general, solutions of these equations are asymptotic 2^{N_1} and 2^{N_2} - periodic piecewise constant functions with finite or countable set of points of discontinuities on a period, where N_k is least common multiple of periods of attractive circles of the map $\tilde{\Phi}_k$, $k = 1,2$.

4 Asymptotic behaviour of system of difference equations

In this section, it will be considered hyperbolic dynamics of structural stable dynamic systems (Fig.1), which appears as a source of impredicative behaviour [13], but we consider only a case when two-dimensional map, describing behavior of solutions of initial boundary value problem, produce asymptotic solutions of relaxation type (Fig.2). Solutions of difference equations are generated by initial functions $h(x,p,t)$ where $x,p \in R$ can be considered as parameters. In two-dimensional case, a separator is determined as $D(x,p) = \bigcup_{n\geq 0} f^{-n}A^{\pm}(x,p)$, where $f : (u_1,u_2) \to ((f_1[u_1], f_2[u_2]))$ and $A\pm(x,p)$ is a set of saddle points of codimensional one. Then a set of points of discontinuities is determines as $\Gamma(x,p) = f^{-1}(D[x,p])$. (Below parameters x,p will be omitted). The curve $h(t)$ can be determined by the initial data of the boundary problem by method of characteristics. An initial curve $h(x,p,t) = (h_1(x,p,t), h_2(x,p,t))$ on interval $[-l/p,0)$ satisfies to the transversal condition $dh(t)/dt \neq 0$ as $t \in \Gamma$. The separator D and the set Γ are closed nowhere dense sets on $[-l/p,0)$. Iterating the map $f : R^2 \to R^2$, we obtain that

$$u(x,p,t) = f^n[h(x,p,t-np/l)], \quad x,p \in R, \quad f : R^2 \to R^2. \tag{49}$$

We assume that [6,7]: 1) there is $G \subset R^2$ such that $f(G) \subset G$; 2) the differential $Df[u]$ is continuous on G; 3) a set $f^{-1}[u]$ is finite for each $u \in G$; 4) a set of non-wandering points of the map f is finite and hyperbolic; 4) no trajectories going from saddle to saddle. A point $u \in R^2$ is non-wandering if for a neighbourhood $U(u)$ exists $m > 0$ such that $f^m(U)\bigcap U \neq 0$. A map is hyperbolic if a spectra $\sigma(Df^r[u])\bigcap\{|u|\} \neq 1$, where ϕ is empty set. We suppose that $u(x,p,t) \in C^2(R^+,G)$ for each $x \in X$ and $p \in P$ if $f \in C^2$, and consider a set of initial functions

$$\check{H} = \{h(t) \in C^2([-l/p),G)\,|\,h(-l/p) = f[h(0)]\}. \tag{50}$$

Then a set of non-wandering points is $\Omega[f] = Per[f] = Fix[f^N]$ for each integer N, and $\Omega[f] = A^+\bigcup A^-\bigcup A^{\pm}$, where $A^+, A^-, A^{\pm}$ are sets of attractive, repelling and saddle type points of the map f^N. Then G can be represented as $G = \bigcup_{a\in\Omega[f]} W^s(a)$ where $W^s(a) = \{u \in G\,|\,\lim_{m\to+\infty} f^{mN}[u] = a\}$ is a stable manifold of fixed point a of the map f^N. Indeed, since $\Omega[f]$ is finite, then each point $u \in G$ is attracted by finite set, according to the Sharkovsky theorem [2], each point $u \in G$ is attracted by a circle of the map f. From this it follows existence of the finite limit [6]

$$\lim_{j\to+\infty} f^{Nj}[u] := f^*[u] \tag{51}$$

where $f^*[u] \in \Omega[f]$. For each $u(x,p,t) \in \check{H}$, and each $x \in X, p \in P$ this limit exists. Then each solution of the system of difference equations tends to the function

$$\lim_{j\to+\infty} ||u(x,p,tNj) - u^*(x,p,t)||_{R^2} = 0 \tag{52}$$

for each fixed $t \in R^+, x \in X, p \in P$. Relation (52) is not uniform on t and

$$\lim_{j\to+\infty} \sup_{t\in[0,N)} ||u(x,p,tNj) - u^*(x,p,t)||_{R^2} \neq 0. \tag{53}$$

The behavior of limit solutions in points of discontinuities can be characterized in the Hausdorff or Schorohod metrics (see, [2], but for our goals it is enough to know that in a neighbourhood of the points of discontinuities Γ the convergence to a limit solution is not uniform.

5 Example 1

In this section, we consider a difference equation which depends on the spatial variable x as on 'parameter'. For usual difference equation (non-depending on parameter), there are piecewise constant asymptotic periodic solutions. But for this type of equations, initial data $u_0(x,t)$ for $t \in [-\Delta,0)$ must be done in R^2. For example, it can be paraboloid. If in $1D$ - case a set Γ of points of discontinuities contains points, that in $2D$ - case a set Δ we obtain a closed curve. As a result, there are $2D$ - structures as 'white-black' spots with the boundary $v \in R^2$.

Let us consider the difference equation

$$u(x,t) = f[u(x,t-\Delta)] \tag{54}$$

where $\Delta > 0$, and $f \in C^2(I,I)$. Here, $u(x,t) : [0,l] \times [-\Delta,0) \to I$. Asymptotic properties of solutions of equation (54) can be determined by asymptotic properties of trajectories of dynamic system which are produced by the map f. Indeed, let f has two attractive fixed points a_1, a_3 and one repelling fixed point a_2, so that $\leq a_1 < a_2 < a_3$ and $[a_1,a_2] \subset I$. Let us define an initial function $h(x,t) : [0,l] \times [-\Delta, 0) \to I$. Then for each fixed x we can use the above theory of difference equations for one argument.

For two argument, the corresponding approval must be modified, but this modification is simple. Indeed, let us consider as above the separator of the map f, so that $D(f) := \bigcup_{n\geq 0} P^-$, where P^- is a set of repelling points of the map f. Then the set of curves of discontinuities is $\Gamma := f^{-1}(D)$. Let the initial data $h(x,t) \in \Gamma$ on rectangle $[0,l] \times [-\Delta, 0)$. Then the set Γ consists finite, countable or uncountable 'manifolds' of discontinuities. It means that structure of the set Γ can be very complex. It depends on the topological forms of the initial function h and the map f.

Let us consider simple example. If $h(x,t)$ is a plane and f is monotone, then $\Gamma := h(x,t) \in a_2 \in P^-$ is a strait line. As a result, a limit solution of difference equation tends to the function: $f*(x,t) = a_1 \bigcup a_3$ if $h(x,t) \neq a_2$, and $f*(x,t)$ is interval $[a_1, a_3]$ if $h(x,t) \neq a_2$ (see, Fig.2). In this case, Γ is a straight line. Next, if $h(x,t) = x^2 + t^2$ on $[0,l] \times [-\Delta, 0)$, that is paraboloid, than a set Γ is a circle $S = \{x,t : x^2 + t^2 = a_2\}$. In this case, we obtain limit oscillations of relaxation type. If f is nonmonotone, then there is a countable set of pre-images $f^{-n}[S]$ of the circle S with limit circle S^*. In this case, we obtain limit oscillations of pre-turbulent type (by terminology of Sharkovsky [2]. Similarly, we can consider a system of difference equations which produce a hyperbolic map Φ on a space of smooth functions. We can also consider a 'parameter' x as vector in R^n. Of course, there are not of general theory for such difference equations, but solutions can be found with help of computer, applying method of iteration of initial data $h(x_1, ..., x_n, p_1, ..., p_n, t)$.

This example shows that for the initial boundary problem appears new $2D$ dimensional type of asymptotic solutions. But in reality the problem is reduced to the functional dependent systems of equations, depending on x and p as on parameters. The theory of such equations is not developed. Indeed, such equations produce a map $\tilde{\Phi} : R^2 \to R^2$ with parameters x, p. But in particular case, for the Henon map, an attractor of problem is one dimensional. The corresponding example will be done in the next section.

5.1 Example 2

Let us consider the system

$$u(x,t) = u^2(x, t-\Delta) + w(x,t) + \lambda, \tag{55}$$

$$w(t+\Delta) = bw(t), \tag{56}$$

where $b = e^a$, $a < 0$ and $\lambda \in R^1$. Solutions of system (55) can be find step by step if initial data $(u_0(x,t), w_0(x,t)$ are known on interval $[-\Delta, 0)$ for each $x \in J$, where J is interval.

The system produce a map $\Phi_{\lambda,b} : R^2 \to R^2$, so that

$$\Phi_{\lambda,b} : (u,w) \to (u^2 + w + \lambda, bw). \tag{57}$$

A set of non-wandering points of the map $\Phi_{\lambda,b}$ is $\Omega(\Phi_{\lambda,b}) = Fix\,\Phi_{\lambda,b} = (u^*_\lambda, 0)$, where u^*_λ is a fixed point of the map $\varphi_\lambda : u \to u^2 + \lambda$.

Let us define an initial curve

$$\gamma(h(t)) := \{(u,w) \in R^2 : u(t) = (h(t), w \in R^1, t \in [0, 2l/V)\}, \tag{58}$$

where a vector-function $h(t)$ is determined by the initial data of the initial value boundary problem.

If $\lambda > 1/4$ the map φ_λ has not fixed points and, hence, for each initial curve $h(t)$ in R^2 given in the interval $0 < t < \Delta)$, solutions of the problem is such that $(u(t), w(t)) \to \infty$ as $t \to \infty$. For $\lambda(x) < -2$ each point $u_\lambda \in I_\lambda \ \Omega(\varphi_{\lambda(x)})$ go out from the interval $I_{\lambda(x)}$ under an action of iterations of the map φ_λ. Here, $I_\lambda = (-\beta_0, \beta_0)$, where $\beta_0 = 1/2 + \sqrt{1/4 - \lambda}$ is the repelling fixed point of the map φ_λ. It means that each component of the solution tends to infinity as $t \to \infty$.

Solutions are bounded if and only if $-2 < \lambda \leq 1/4$. For $\lambda = -2$ fixed points are $\beta_0 = 2$ and $\beta_1 = 1$. Indeed, if $|u_0| < 2$, then there is θ_0 such that $u_0 = \pm 2\cos\theta_0$. Then $u_n = \pm 2\cos 2^n \theta_0$. If θ_0 is commensurate with $\pi - \theta_0 = \frac{m}{n}\pi$, $((m,n) = 1$ (that is m/n is irreducible fraction). In this case, there are numbers k and i such that $2^i(2^k - 1) \equiv 0 (mod\, n)$. Then, beginning from some number, we obtain a circle. For almost all (with respect of the Lebesque measure), this sequence is uniformly distributed in the interval.

There is a set Λ such that for almost $u \in \Lambda$ trajectories $\{\varphi^i_{-2}\}^\infty_{i=0}$ are placed on Λ everywhere dense. Trajectories on Λ are unstable, but the set Λ are stable generally. It means that Λ attracts almost all trajectories from its neighbourhoods. For $\lambda = -2$ the set Λ is the interval $I^{-2} = [-2, 2]$. It means that any solution tends as $t \to \infty$ to a function $p_1(\zeta, p_1\eta)$, where $\zeta = t - x/V$ and $\eta = t + x/V$. This function is equal $[-2, 2]$ on the interval $(\zeta + d, \eta + d)$ for each given $d > 0$. The number of oscillations increase infinitely as $t \to \infty$. Such behaviour of trajectories exists not only for $\lambda = -2$, but for continuum values of λ.

If $-3/4 < \lambda < 1/4$, then $\bar{\varphi}(I_\lambda) \subset I_\lambda$ and there is the fixed attractive point β_1 on this interval. It means that $(u(t), w(t) \to \infty)$ as $t \to \infty$. If $-5/4 < \lambda < 3/4$, then the fixed point β_1 become repelling, but instead on Λ appears an attractive circle of the period 2 which consists from the two points $\beta_{2,3} = -1/2 \pm \sqrt{-3/4 - \lambda}$. For the two-dimensional map $\Phi_{\lambda,b}$ it means that the set of attractive fixed points is $P^+ = \{(2\beta_2, 0), (2\beta_3, 0)\}$. The set of saddle fixed points consists from the unique point $P^{\pm} = \{(2\beta_1, 0)\}$. Then vectors, corresponding to these eigenvalues, are $(1,0)$ and $(0,1)$ (Fig.1). If $-3/4 < \lambda < 5/4$, then for $\lambda_n < \lambda < \lambda_{n+1}$, $n = 0, 1, 2, ...$ the map φ_λ has an attractive circle of the period 2^n, but all another circles are repelling. For the system of difference

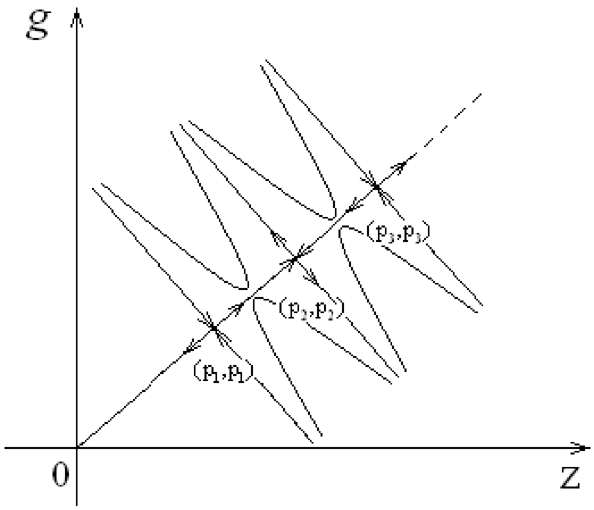

Fig. 1: Typical distributions of trajectories for a hyperbolic map.

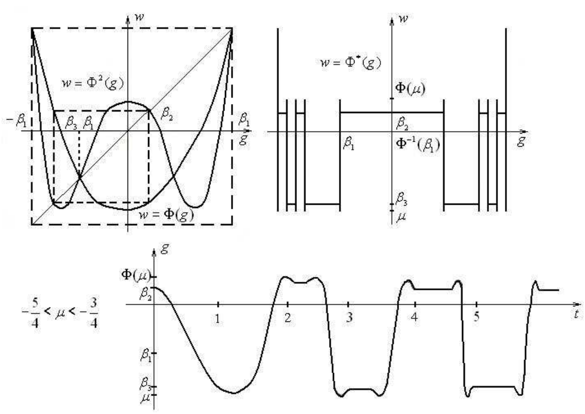

Fig. 2: Computer modelling of attractor of relaxation type in x, p, t - space.

equations it means that $u(t)$ tends to a $2^{2n\Delta}$ - periodic piecewise constant function and $w(x,t)$ tends to zero.

6 Physical interpretation of problem

The problem can be generalised on the case of n - quantum oscillators. Indeed, let us consider the system of the Hamiltonian equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, i = 1, 2, ..., n. \tag{59}$$

Now we consider a torus T^n, so that $T^n = S^1 \times ...S^1$, and let there is a map $\pi : R^n \to T^n$, $\pi(\varphi) = \varphi \, mod \, 2\varpi$, where φ is an angular variable on the i - circle. Then the map π is 2π - periodic. Thus, we can represent torus as n - dimensional cube $[0, 2\pi]^n \subset R^n(\varphi_1, ..., \varphi_n)$ with identified opposite sites on the cube. Then a quasi-periodic motion on T^n is a projection of a line with the map π, so that

$$\varphi_i(t) = \varphi_i(t_0) + \omega_i t. \tag{60}$$

Trajectories, which are produced by equalities (60), we call by a winding of the torus. If the Hamiltonian is $H = \frac{1}{2m}p^2 + \frac{a}{2}q^2$, where $p, q \in R^1$, then along trajectories of the Hamiltonian system $dH[q(t), p(t)]/dt = 0$ or $H(q, p) = E$, where E is an energy of system. Thus, the energy E is defined on ellipses M, so that $\frac{1}{2m}p^2 + \frac{a}{2}q^2 = E$, where $E \in R^+$.

Let us consider a variable φ, so that

$$q(t) = \sqrt{\frac{2E}{a}} \cos\varphi(t), \quad p(t) = \sqrt{2Em} \sin\varphi(t). \tag{61}$$

Then from the Hamilton equations it follows that $\varphi(t) = \sqrt{\frac{a}{m}}t + \varphi(t_0)$, where $\varphi(t_0) \in [0, 2\pi)$ is determined by initial conditions $q(t_0), p(t_0)$. For simplicity, in the above sections has been considered a case when $a = m = 1$, $\varphi(t_0) = 0$ and $\rho = \sqrt{2E} = 1$. In this case, the iso-energetic surface is a circle, and we get a first integral $f_1 = q^2 + p^2 = 1$. In general case, we obtain a sphere S^{2n-1}.

For oscillator with n - degree of freedom, we have independent integrals

$$f_i(q, p) = \frac{a_1}{2}q_i^2 + \frac{1}{2m_i}p_i^2, \tag{62}$$

where f_i is an energy of i - th oscillator. Then a phase space of trajectories of the Hamilton system is a product of ellipses or circles. If we consider a space $q, p, \dot{q}, \dot{p}$, then trajectories foliates the product of circles $S^1 \times S^2$, where $S^1 = \{q, p \,|\, q^2 + p^2 = 1\}$ and $S^2 = \{\dot{q}, \dot{p} \,|\, \dot{q}^2 + \dot{p}^2 = 1\}$. Thus, we have 4 - dimensional $(q, p, \dot{q}, \dot{p})$ - space. A projection of this space on $(\dot{q}, \dot{p})$ - space determines characteristic of the hyperbolic transport equation. Thus, the energy of the Hamilton system is conserved along the characteristic. The energy changes only at the boundaries or flat walls which confides particles of a physical system. The law of changing between 'numbers' of particles is determined by functional two-points boundary conditions as q is fixed at the walls, or p is fixed at the walls (see, Sections 2,3). The dynamic boundary conditions describes probabilities of 'particle production' or 'particle annihilation' at the walls confined the physical systems.

Similar problems, in applications to the radio-physics, has been studied in works of Kuznetsov et all. [17,18] by method of reduction of the boundary problem to the system of difference equations in real or complex spaces. In applications to the polymer mixtures, the same boundary condition are introduced by Binder et all. [19, 20]. For polymers and binary alloys method of reduction to the difference equations has been applied in papers [8, 9].

7 Conclusion

Thus, for the main equation of Hamiltonian mechanics with dynamic boundary conditions and general class of the initial conditions, the attractor of relaxation, pre-turbulent and turbulent type has been constructed. The attractor contains piecewise constant periodic spatial temporal wave functions with finite, countable or uncountable 'points' of discontinuities on a period. The problem has been reduced to the study of asymptotic behavior of trajectories of $1D$ and $2D$ dynamic hyperbolic structural stable systems. It is shown that in $1D$ - case the solutions of the problem satisfies to the famous Sharkovsky ordering. In $1D$ - case, period-doubling bifurcations of solutions take place. In $2D$ - case situation is more complex because there are not of the Sharkovsky ordering, but there are distributions of relaxation type for which a computer simulation (see, Fig.2) has been done.

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