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Perturbation-Iteration Algorithm for Systems of Fractional Differential Equations and Convergence Analysis

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Abstract: In this study, a perturbation-iteration algorithm, namely PIA, is applied to solve some types of systems of fractional differential equations (FDEs) and also the convergence analysis of the method is presented for the first time. To illustrate the efficiency of the method, numerical solutions are compared with the results existing in the literature by considering a systems of FDEs. The results confirm that the PIA is robust, simple and reliable method for solving systems of nonlinear fractional differential equations.

Keywords: Fractional differential equations, Caputo fractional derivative, initial value problems, perturbation-iteration algorithm.

1 Introduction

As an important mathematical branch investigating the properties of derivatives and integrals, the fractional calculus and its history is nearly as old as classical integer order analysis. Practical applications of fractional calculus could not be found for many years from its genesis time. However in last decades, it has found an important place in various areas such as control theory [1], viscoelasticity [2], electrochemistry [3] and electromagnetic [4].

The evolution of the symbolic computation programs such as Matlab and Mathematica is one of the driving forces behind this increased usage. The most important descriptions of fundamentals of fractional calculus have been studied by [5], [6] and [7]. Existence and uniqueness of the solutions has also been studied by [8] and the references therein.

Parallel to the studies in applied sciences, systems of fractional differential equations (FDEs) allowed scientists to describe and formulate various important and useful physical problems.

The number of differential equations whose solution can not be found analytically. Those situations appear in FDEs more than the other types of differential equations. In this case, the study of algorithms in numerical analysis is used for finding favorite approximate solutions of FDEs. In recent years, a significant effort has been extended to propose numerical methods for this purpose. These methods include, fractional variational iteration method [9,10], homotopy perturbation method [11,12,13] and fractional differential transform method [14,15,16].

In this study, we have applied the previously developed method PIA to obtain approximate solutions for some systems of FDEs. Our method is suitable for a broad class of equations and does not require special assumptions and restrictions.

Our results show that only a few terms are required to obtain an approximate solution, which is more accurate and efficient than many other methods in the literature. In general, PIA method has been classified with respect to the number of correction terms (n) and with respect to the degrees of derivatives in the Taylor expansions (m). Briefly, this process is represented as PIA(n,m) that will be explained completely in the introduction of the method.

In this paper, at first we present basic definitions of fractional calculus and investigation of its mathematical space, then we express in details special case of Perturbation-Iteration Algorithm PIA(1,1) to find its iteration formula. In the sequel, we prove convergence of PIA method for the first time and when $m = 1$ and $n = 1$ to find special conditions that convergence of mentioned model holds. For the convergence stop criteria appointed for PIA(1,1) and same process holds

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for PIA(1,2), PIA(2,2) and so on. In order to show effectiveness, efficiency and reliability of the method, we investigate an applied example. At last, we summarize obtained results of the method in the section of conclusion.

2 Basic Definitions

In the literature, there exists a few fractional derivative definitions of an arbitrary order. Two mostly used of them are the Riemann-Liouville and Caputo fractional derivatives. The two definitions are quite similar but have different order of evaluation of derivation[7, **?**].

Definition 1. *A real function $f(t)$, $t>0$ is said to be in the space C_μ, $(\mu>0)$ if there exists a real number $p(>\mu)$, such that $f(t)=t^p f_1(t)$ where $f_1\in C[0,\infty)$, and it is said to be in the space C_μ^m if $f^{(m)}\in C_\mu$ [17].*

Definition 2. *The Riemann-Liouville fractional integral operator (J^α) of order $\alpha\geq 0$, of a function $f\in C_\mu$, $\mu\geq -1$ is defined as [5]:*

$$J^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau, \quad \alpha,t>0 \tag{1}$$

and $J^0 f(t)=f(t)$, where Γ is the well-known gamma function. For $f\in C_\mu$, $\mu\geq -1$, $\alpha,\beta\geq 0$ and $\lambda>-1$, the following properties hold.

– $J^\alpha J^\beta f(t)=J^{\alpha+\beta}f(t)$,
– $J^\alpha J^\beta f(t)=J^\beta J^\alpha f(t)$,
– $J^\alpha t^\lambda=\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+\alpha)}t^{\lambda+\alpha}$.

Definition 3. *The Caputo fractional derivative of f of order α, $f\in C_{-1}^m$ is defined as [6]:*

$$D^\alpha f(t)=J^{m-\alpha}f^{(m)}(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}f^{(m)}(\tau)d\tau, \quad \alpha,t>0, \tag{2}$$

where $m-1<\alpha<m$ with the following properties:

– $D^\alpha(af(t)+bg(t))=aD^\alpha f(t)+bD^\alpha g(t)$, $a,b\in\Re$,
– $D^\alpha J^\alpha f(t)=f(t)$,
– $J^\alpha D^\alpha f(t)=f(t)-\sum_{j=0}^{k-1}f^{(j)}(0)\frac{t^j}{j!}$, $t>0$.

3 Overview of the Perturbation-Iteration Algorithm PIA(1,1)

As one of the most practical subjects of physics and mathematics, differential equations create models for a number of problems in science and engineering to give an explanation for a better understanding of the events. Perturbation methods have been used for this purpose for over a century [18,19,20].

However, the main disadvantage of using the perturbation methods is the necessity of a small parameter or to install a small auxiliary parameter in the equation. For this reason, the obtained solutions are restricted by a validity range of physical parameters. Therefore, to overcome the disadvantages come with the perturbation techniques, some methods have been suggested by several authors [21,22,23,24,25,11,26,27,28].

Parallel to these studies, a new approach, perturbation-iteration algorithm, has been proposed by Aksoy, Pakdemirli and their co-workers [29,30,31]. In the new technique, an iterative algorithm is established on the perturbation expansion. The method has been applied to first order equations [30] and Bratu type second order equations [29] to obtain approximate solutions. Then the algorithms were tested on some nonlinear heat equations also [31]. The solutions of the Volterra and Fredholm type integral equations [32], ordinary differential equation and systems [33] and the solutions of ordinary fractional differential equations [34] have been given by the present method, finally.

In this study, the previously developed technique is applied to systems of nonlinear fractional differential equations for the first time. To obtain the approximate solutions of equations, taking one correction term in the perturbation expansion and Taylor series expansion up to the first derivatives, i.e. $n=1, m=1$, generates PIA(1,1) iteration algorithm.

Consider the following initial value problem.

$$F_k(D^{\alpha_k}u_k,u_j,\varepsilon,t)=0, \tag{3}$$

$$k = 1,2,\ldots,K,$$

$$j = 1,2,\ldots,K,$$

$$u^{(k)}(0) = c_k, \quad m-1 < \alpha \le m, \tag{4}$$

where k is the number of the equations in the system and D^{α} is the Caputo fractional derivative of order α. As more clearly the system could be expressed by:

$$\begin{aligned} F_1 &= F_1\left(u_1^{(\alpha)}, u_1, u_2, \ldots, u_k, \varepsilon, t\right) = 0, \\ F_2 &= F_2\left(u_2^{(\alpha)}, u_1, u_2, \ldots, u_k, \varepsilon, t\right) = 0, \\ &\vdots \\ F_K &= F_K\left(u_K^{(\alpha)}, u_1, u_2, \ldots, u_k, \varepsilon, t\right) = 0. \end{aligned} \tag{5}$$

In this method, ε is the auxiliary perturbation parameter and (u_c) is the only correction term in the perturbation expansion.

$$\begin{aligned} u_{k,n+1} &= u_{k,n} + \varepsilon (u_c)_{k,n}, \\ u'_{k,n+1} &= u'_{k,n} + \varepsilon (u'_c)_{k,n}, \end{aligned} \tag{6}$$

where subscript n represents the $n-th$ iteration.

Replacing (6) into (3) and writing in the Taylor Series expansion for $m-th$ order derivatives in the neighborhood of $\varepsilon = 0$ gives

$$F_K = \sum_{m=0}^{M} \frac{1}{m!} \left[\left(\frac{d}{d\varepsilon} \right)^m F_K \right]_{\varepsilon=0}^{m} \varepsilon^m, \quad k = 1,2,\ldots,K, \tag{7}$$

for

$$\frac{d}{d\varepsilon} = \frac{\partial u_{k,n+1}^{(\alpha)}}{\partial \varepsilon} \frac{\partial}{\partial u_{k,n+1}^{(\alpha)}} + \sum_{j=1}^{K} \left(\frac{\partial u_{j,n+1}}{\partial \varepsilon} \frac{\partial}{\partial u_{j,n+1}} \right) + \frac{\partial}{\partial \varepsilon}, \tag{8}$$

where m represents the $m-th$ term in the power series expansion and n represents the $n-th$ term in the Taylor series expansion.This equation is defined for the $(n+1)-th$ iteration equation as follows:

$$F_k\left(u_{k,n+1}^{(\alpha)}, u_{j,n+1}, \varepsilon, t\right) = 0. \tag{9}$$

Replacing (8) in (7) yields our iteration equation:

$$F_k = \sum_{m=0}^{M} \frac{1}{m!} \left[\left(\frac{((u_c)_{k,n})^{(\alpha)}}{\partial \varepsilon} \frac{\partial}{\partial u_{k,n+1}^{(\alpha)}} + \sum_{j=1}^{K} (u_c)_{j,n} \frac{\partial}{\partial u_{j,n+1}^{(\alpha)}} + \frac{\partial}{\partial \varepsilon} \right)^m F_{k-1} \right]_{\varepsilon=0}^{m} \varepsilon^m = 0, \tag{10}$$

where $k = 1,2,\ldots,K$.

All derivatives are calculated at $\varepsilon = 0$. Therefore in the procedure of computations, each term is obtained when ε tends to zero. The method converges in few iterations and in fact we have a saturated solution after doing computations even in the initial steps to find favorite approximate solution.

Beginning with an initial function u_0, first $(u_c)_{k,n}$'s has been determined by the help of (10). Then using (6), $(n+1)$. iteration solution could be found. Iteration process is repeated using (10) and (6) until achieving an acceptable result. The ability of the method is so high that it can become convergent in just a few of computational iterations. The reliability and effectiveness of the method is shown by an example after presenting the convergence of PIA method.

4 Convergence Analysis of the PIA

In this section we give a convergence analysis of the method.

Theorem 1.*PIA*$(1,1)$ *converges for Eq.*(3) *when* $\|u_{k+1}-u_k\| \leq \varepsilon^{'}$ *and* $\varepsilon^{'} \to 0$.

*Proof.*Suppose that the functions $F_{u^{'}_{k+1}}$, $F_{u_{k+1}}$, F_{u_k} and $F_{u^{'}_k} \in C^m_\mu$, $\mu \geq -1$. Meanwhile, suppose that the function F_k is m times continuous and differentiable on $[a,b]$. The general iteration formula of $PIA(m,n)$ is converted to $PIA(1,1)$ in recursive relation (10) by substituting $m=1$ and $n=1$ that can be stated as follows:

$$u^{'}_k(t)+\frac{F_{u_k}}{F_{u^{'}_k}}u_k(t)=-\frac{F_\varepsilon+\frac{F}{\varepsilon}}{F_{u^{'}_k}}. \tag{11}$$

By changing k to $k+1$ in Eq.(11) to obtain a relation with respect to $u^{'}_{k+1}(t)$ and $u_k(t)$ and imposing norm 2 on both sides of equations, we have:

$$\left\|u^{'}_k\right\| \leq \left\|\frac{F_\varepsilon+\frac{F}{\varepsilon}}{F_{u^{'}_k}}\right\|+\left\|\frac{F_{u_k}}{F_{u^{'}_k}}\right\|.\|u_k\|, \tag{12}$$

$$\left\|u^{'}_{k+1}\right\| \leq \left\|\frac{F_\varepsilon+\frac{F}{\varepsilon}}{F_{u^{'}_{k+1}}}\right\|+\left\|\frac{F_{u_{k+1}}}{F_{u^{'}_{k+1}}}\right\|.\|u_{k+1}\|. \tag{13}$$

Now, we need to obtain $\|u_{k+1}-u_k\|$ from $\left\|u^{'}_{k+1}-u^{'}_k\right\|$. By rewriting inequalities with respect to $\|u_k\|$ and $\|u_{k+1}\|$ and by using the magnitude rules in calculus, we get:

$$\begin{aligned}&\|u_{k+1}-u_k\| \geq \|u_{k+1}\|-\|u_k\|,\\ &=\left\|\frac{F_{u^{'}_{k+1}}}{F_{u_{k+1}}}\right\|.\left(\left\|u^{'}_{k+1}\right\|-\left\|\frac{F_\varepsilon+\frac{F}{\varepsilon}}{F_{u^{'}_{k+1}}}\right\|\right)-\left\|\frac{F_{u^{'}_k}}{F_{u_k}}\right\|.\left(\left\|u^{'}_k\right\|-\left\|\frac{F_\varepsilon+\frac{F}{\varepsilon}}{F_{u^{'}_k}}\right\|\right).\end{aligned} \tag{14}$$

So we need to obtain bound for $\|u_{k+1}-u_k\|$. As a result, we need to prove that $\{u_n\}$ is a Cauchy sequence that needs to be convergent in the defined space. All elements of $\|u_{k+1}-u_k\|$ in right hand side of inequality are known except $\left\|u^{'}_k\right\|$ and $\left\|u^{'}_{k+1}\right\|$. Therefore, we have :

$$\begin{aligned}\|u_{k+1}-u_k\| \geq &\left\|\frac{F_{u^{'}_{k+1}}}{F_{u_{k+1}}}\right\|.\left\|u^{'}_{k+1}\right\|-\left\|\frac{F_{u^{'}_{k+1}}}{F_{u_{k+1}}}\right\|.\left\|\frac{F_\varepsilon+\frac{F}{\varepsilon}}{F_{u^{'}_{k+1}}}\right\|\\ &-\left\|\frac{F_{u^{'}_k}}{F_{u_k}}\right\|.\left\|u^{'}_k\right\|+\left\|\frac{F_{u^{'}_k}}{F_{u_k}}\right\|.\left\|\frac{F_\varepsilon+\frac{F}{\varepsilon}}{F_{u^{'}_k}}\right\|.\end{aligned} \tag{15}$$

The function $F_k=F\left(u^{'}_k,u_k,\varepsilon\right)$ is a chaotic functional with respect to ε and it can be written in general state as follows:

$$F_k=F\left(u^{'}_k,u_k,\varepsilon\right)=\frac{1}{\Gamma(m-\alpha)}\varepsilon\int_0^t\frac{u^{'}_{k,n}(s)}{(t-s)^\alpha} \tag{16}$$

$$\pm\eta_1u^{'}_{k,n}(t)\pm\eta_2\varepsilon u^{'}_{k,n}(t)\pm\eta_3\varepsilon u^{(k)}_{k,n}(t). \tag{17}$$

in which $\eta_i, i=1,2,3$ are constants. Therefore, the functions of $F_{u^{'}_k}$ and F_{u_k} can be written based on F_k. We also suppose that $F_{u^{'}_{k+1}}$, $F_{u_{k+1}}$, F_{u_k} and $F_{u^{'}_k} \neq 0$. Since $F_{u^{'}_{k+1}}$, $F_{u_{k+1}}$, F_{u_k} and $F_{u^{'}_k} \in C^m_\mu$, $\mu \geq -1$, then they are bounded and we have:

$$\left\|\frac{F_{u^{'}_{k+1}}}{F_{u_{k+1}}}\right\| \leq M_1,\ \left\|\frac{F_{u^{'}_k}}{F_{u_k}}\right\| \leq M_2. \tag{18}$$

$$\left\| \frac{F_\varepsilon + \frac{F}{\varepsilon}}{F_{u'_{k+1}}} \right\| \leq M_1' \,, \; \left\| \frac{F_\varepsilon + \frac{F}{\varepsilon}}{F_{u'_k}} \right\| \leq M_2'. \tag{19}$$

Therefore, we obtain:

$$\|u_{k+1} - u_k\| \geq M_1 \left\| u'_{k+1} \right\| - M_1.M_1' - M_2. \left\| u'_k \right\| + M_2.M_2'. \tag{20}$$

Now we consider:

$$u'_k = L\left[u_k\right] u'_{k+1} = L\left[u_{k+1}\right], \tag{21}$$

where L is a linear operator that is defined as $L = \frac{d}{d()}[]$ that is defined as an operator in infinite dimensional space. Since any linear operator is bounded in infinite dimensional space, then we can define

$$\left\| u'_k \right\| = \left\| L\left[u_k\right] \right\| \leq N_1, \left\| u'_{k+1} \right\| = \left\| L\left[u_{k+1}\right] \right\| \leq N_2. \tag{22}$$

Then, we obtain:

$$\|u_{k+1} - u_k\| \geq M_1.N_2 - M_1.M_1' - M_2.N_1 + M_2.M_2' = M_1\left(N_2 - M_1'\right) + M_2\left(M_2' - N_1\right). \tag{23}$$

If $\|u_{k+1} - u_k\| \to 0$ then, we have

$$\lim_{\varepsilon' \to 0} \left(M_1\left(N_2 - M_1'\right) + M_2\left(M_2' - N_1\right) \right) = 0. \tag{24}$$

If $M_1\left(N_2 - M_1'\right) = 0$ then $M_1 = 0$ or and if $M_2\left(M_2' - N_1\right) = 0$ then $M_2 = 0$ or $M_2' = N_1$. The proof is complete. The proof can be done in similar manner for $PIA(1,2)$, $PIA(2,2)$ and so on. Therefore, we can find such these conditions for $PIA(1,2)$, $PIA(2,2)$ states and so on. In fact, we have found the condition of stop process for PIA method by all involved expressions in the iteration algorithm. Therefore, all the governing conditions need to be imposed on PIA method to become convergent just in a few of computational iterations.

5 Application

It is noticed that all computations have been done by Package Mathematica 10.

For an example consider the following system of nonlinear fractional differential equations [35]:

$$\begin{aligned} D^{\alpha_1} u_1(t) \; &= \frac{1}{2} u_1(t) \\ D^{\alpha_2} u_2(t) \; &= u_2(t) + u_1^2(t), \end{aligned} \tag{25}$$

where

$$0 < \alpha_1, \alpha_2 \leq 1$$

with the initial conditions $u_1(0) = 1$ and $u_2(0) = 0$. The exact solutions, when $\alpha_1 = \alpha_2 = 1$, are

$$u_1(t) = e^{\frac{t}{2}} \tag{26}$$

and

$$u_2(t) = te^t. \tag{27}$$

In the system, if add and subtract $u'_{1,n}(t)$ and $u'_{2,n}(t)$ respectively, the system can be rewritten in the following form:

$$\varepsilon D^{\alpha_1} u_1(t) \; + u'_{1,n}(t) - \varepsilon u'_{1,n}(t) - \frac{1}{2} \varepsilon u_{1,n}(t) = 0, \tag{28}$$

$$\varepsilon D^{\alpha_2} u_2(t) \; + u'_{2,n}(t) - \varepsilon u'_{2,n}(t) - \varepsilon u_{2,n}(t) - \varepsilon u^2_{1,n}(t) \; = 0, \tag{29}$$

where ε is a small parameter. For

$$F\left(u_1^{'},u_1,\varepsilon\right)=\frac{1}{\Gamma(1-\alpha_1)}\varepsilon\int_0^t\frac{u_{1,n}'(s)}{(t-s)^{\alpha_1}}ds+u_{1,n}'(t)-\varepsilon u_{1,n}'(t)-\frac{1}{2}\varepsilon u_{1,n}(t), \tag{30}$$

$$F\left(u_2^{'},u_2,\varepsilon\right)=\frac{1}{\Gamma(1-\alpha_2)}\varepsilon\int_0^t\frac{u_{2,n}'(s)}{(t-s)^{\alpha_2}}ds+u_{2,n}'(t)-\varepsilon u_{2,n}'(t)-\varepsilon u_{2,n}(t)-\varepsilon u_{1,n}^2(t) \tag{31}$$

and terms in equation (25) become

$$\begin{aligned}&F=u_{1,n}'(t),\quad F_{u_1}=0,\quad F_{u_1'}=1,\\&F_\varepsilon=-u_{1,n}'(t)-\frac{u_{1,n}(t)}{2}+\frac{1}{\Gamma(1-\alpha_1)}\int_0^t\frac{u_{1,n}'(t)}{(t-s)^{\alpha_1}}ds\end{aligned} \tag{32}$$

and

$$\begin{aligned}&F=u_{2,n}'(t),\quad F_{u_2}=0,\quad F_{u_2'}=1,\\&F_\varepsilon=-u_{2,n}'(t)-u_{2,n}(t)-u_{1,n}^2(t)+\frac{1}{\Gamma(1-\alpha_2)}\int_0^t\frac{u_{2,n}'(s)}{(t-s)^{\alpha_2}}ds.\end{aligned} \tag{33}$$

After writing these terms in the iteration formula, we obtain the following differential equations:

$$2\left(\frac{\int_0^t(-s+t)^{-\alpha_1}u_{1,n}'(s)ds}{\Gamma(1-\alpha_1)}+(u_c)_{1,n}'(t)\;-\frac{(-1+\varepsilon)(u_{1,n})^{'}(t)}{\varepsilon}\right)=\qquad u_{1,n}(t) \tag{34}$$

and

$$u_{2,n}(t)+u_{1,n}^2(t)+\frac{(-1+\varepsilon)(u_{2,n})^{'}(t)}{\varepsilon}=\qquad\frac{\int_0^t(-s+t)^{-\alpha_2}(u_{2,n})^{'}s)ds}{\Gamma(1-\alpha_2)}+(u_c)_{2,n}'(t). \tag{35}$$

Beginning with the initial functions

$$u_{1,0}(t)=1 \text{ and } u_{2,0}(t)=0$$

and using the iteration formula, the following successive approximate solutions are obtained for $n=0,1,2,\ldots$

$$\begin{aligned}
&u_{1,1}(t)=1+\frac{t}{2},\\
&u_{2,1}(t)=t,\\
&u_{1,2}(t)=\frac{1}{8}\left(8+8t+t^2+\frac{4t^{2-\alpha_1}}{\Gamma(2-\alpha_1)(-2+\alpha_1)}\right),\\
&u_{2,2}(t)=2t+t^2+\frac{t^3}{12}+\frac{t^{2-\alpha_2}}{\Gamma(2-\alpha_2)(-2+\alpha_2)},\\
&u_{1,3}(t)=\frac{1}{48}(48+72t+18t^2+t^3+\frac{24t^{3-2\alpha_1}}{\Gamma(4-2\alpha_1)}-\frac{24t^{2-\alpha_1}(9+t-3\alpha_1)}{\Gamma(4-\alpha_1)}),\\
&u_{2,3}(t)=3t+3t^2+\frac{5t^3}{6}+\frac{t^4}{12}+\frac{t^5}{320}+\frac{t^{3-2\alpha_2}}{\Gamma(4-2\alpha_2)}-\frac{t^{5-2\alpha_1}}{4\Gamma(3-\alpha_1)^2(-5+2\alpha_1)},\\
&\qquad-\frac{t^{3-\alpha_1}(4(40+3t(10+t)))}{8\Gamma(6-\alpha_1)}-\frac{\alpha_1(-72-t(64+7t)+(8+t(8+t))\alpha_1)}{8\Gamma(6-\alpha_1)},\\
&\qquad-\frac{t^{2-\alpha_2}(72+t(24+t)+6\alpha_2(-7-t+\alpha_2))}{2\Gamma(5-\alpha_2)}
\end{aligned} \tag{36}$$

and so on. In the same manner, the fourth iteration solutions $u_{1,4}(t)$ and $u_{2,4}(t)$ are calculated. Again we compared our results in Figures 1, 2, 3 and 4 as well as in Tables 1 and 2 with the exact solutions.

Table 1: Numerical results for some values of $u_1(t)$.

	$\alpha_1 = 0.5$, $\alpha_2 = 0.8$		$\alpha_1 = \alpha_2 = 1$			
t	$u_{1,3}(t)$	$u_{1,4}(t)$	$u_{1,3}(t)$	$u_{1,4}(t)$	*Exact Solution*	*Absolute Error*
0.0	1.00000	1.00000	1.00000	1.00000	1.00000	0.0000000
0.1	1.12011	1.14395	1.05127	1.05127	1.05127	2.6260E-9
0.2	1.22155	1.25625	1.10516	1.10517	1.10517	8.4742E-8
0.3	1.31398	1.35476	1.16181	1.16183	1.16183	6.4897E-7
0.4	1.40065	1.44505	1.22133	1.22140	1.22140	2.7581E-6
0.5	1.48332	1.52999	1.28385	1.28401	1.28402	8.4896E-6
0.6	1.56312	1.61133	1.34950	1.34983	1.34985	0.0000213
0.7	1.64087	1.69024	1.41839	1.41990	1.41906	0.0000464
0.8	1.71714	1.76755	1.49066	1.49173	1.49182	0.0000913
0.9	1.79240	1.84388	1.56643	1.56814	1.56831	0.0001166
1.0	1.86700	1.91969	1.64583	1.64843	1.64872	0.0002837

Table 2: Numerical results for some values of $u_2(t)$.

	$\alpha_1 = 0.5$, $\alpha_2 = 0.8$		$\alpha_1 = \alpha_2 = 1$			
t	$u_{2,3}(t)$	$u_{2,4}(t)$	$u_{2,3}(t)$	$u_{2,4}(t)$	*Exact Solution*	*Absolute Error*
0.0	0.00000	0.00000	0.00000	0.00000	0.00000	0.0000000
0.1	0.18222	0.19983	0.110505	0.110516	0.110517	2.4666E-7
0.2	0.37432	0.41674	0.244084	0.244272	0.244280	8.1286E-6
0.3	0.59361	0.67238	0.403929	0.404894	0.404957	0.0000635
0.4	0.84506	0.97264	0.593365	0.596453	0.596729	0.0002760
0.5	1.13208	1.32203	0.815852	0.823492	0.824360	0.0008683
0.6	1.45755	1.72484	1.074993	1.091043	1.0932712	0.0022277
0.7	1.82421	2.18551	1.374530	1.404661	1.409626	0.0049654
0.8	2.23477	2.70869	1.718357	1.770446	1.780432	0.0099863
0.9	2.69196	3.29932	2.110517	2.195074	2.213642	0.0185684
1.0	3.19856	3.96263	2.555208	2.685825	2.718281	0.0324559

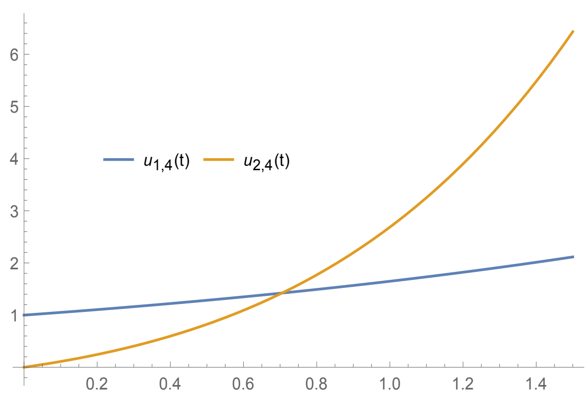

Fig. 1: $u_{1,4}(t)$ and $u_{2,4}(t)$ for $\alpha_1 = \alpha_2 = 1$.

6 Conclusion

In this paper, we have applied a previously developed numerical method so-called Perturbation-Iteration Algorithm (PIA) to find approximate solutions of systems of nonlinear Fractional Differential Equations for the first time. We also give the convergence analysis of the method which has not been done before. The numerical results obtained in this study show that PIA method is a remarkably successful numerical technique for solving systems of FDEs. Only a few iterations are required to reach to favorite solution. We expect that the present method can be used to calculate the approximate solutions of the other types of fractional differential equations such as fractional integro-differential equations and fractional partial differential equations. Our next study will be about these types of equations.

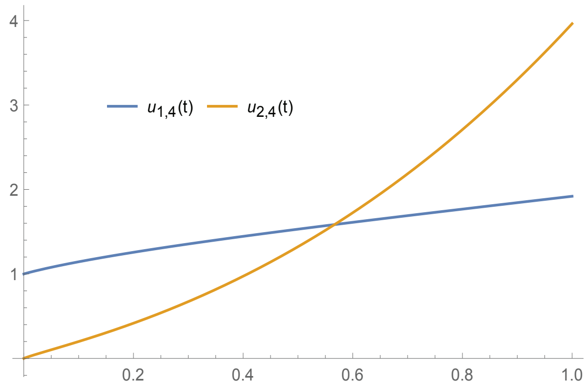

Fig. 2: $u_{1,4}(t)$ and $u_{2,4}(t)$ for $\alpha_1 = 0.5$ and $\alpha_2 = 0.8$.

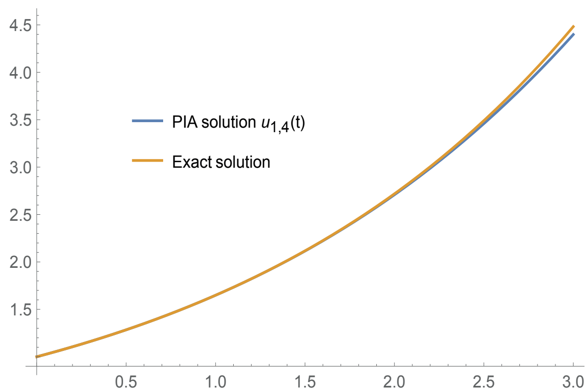

Fig. 3: The PIA $u_{1,4}(t)$ and the exact solutions for $\alpha_1 = \alpha_2 = 1$.

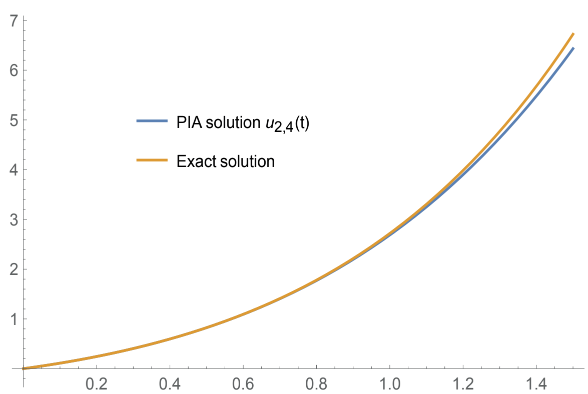

Fig. 4: The PIA $u_{2,4}(t)$ and the exact solutions for $\alpha_1 = \alpha_2 = 1$.

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