*Applied Mathematics & Information Sciences* **5**(3) (2011), 525-546
– An International Journal


# On Differential Sequences

K. Andriopoulos[1], P.G.L. Leach[1] and A. Maharaj[2]

[1] School of Mathematical Sciences, University of KwaZulu-Natal,
Private Bag X54001 Durban 4001, Republic of South Africa

[2] Department of Mathematics, Statistics and Physics, Durban University of Technology,
P O Box 953, Steve Biko Campus, Durban 4000, Republic of South Africa

*adhirm@dut.ac.za, kand@aegean.gr*

Received Nov. 02, 2010; Revised March 13, 2011; Accepted May 26, 2011

We introduce the notion of Differential Sequences of ordinary differential equations. This is motivated by related studies based on hierarchies of evolution partial differential equations. We discuss the Riccati Sequence in terms of symmetry analysis, singularity analysis and identification of the complete symmetry group for each member of the Sequence. The singularity analysis reveals considerable structure for the values of the coefficients of the leading-order terms and resonances of the different principal branches. Full proofs of the symmetry properties are performed for differential equations defined by their recursion properties and not given in explicit form.
**Keywords**: Recursion operator; Lie symmetry;complete symmetry group; singularity analysis.

## 1 Introduction

In the study and application of differential equations there are certain equations which are prominent by means of their utility of application or their ubiquity of occurrence. In the particular case of ordinary differential equations examples of members of this prominent class are the Riccati equation, the linear second-order equation, the linear third-order equation of maximal symmetry, the Ermakov-Pinney equation, the Kummer-Schwarz equation and its generalisation. These equations are not independent. As Conte observed [10], the study of any of the Riccati equation, the linear second-order

---

The work reported in this paper is part of a project funded under the Swedish-South African Agreement, Grant Number 60935. AM was supported by bursaries from the National Research Foundation of South Africa and the University of KwaZulu-Natal. KA thanks the National Research Foundation of the Republic of South Africa for its support. PGLL thanks the University of the Aegean and the Università di Perugia for the provision of facilities while this work was being prepared and the National Research Foundation of South Africa and the University of KwaZulu-Natal for their continued support. Any opinions expressed in this paper should not be construed as being those of either institution.

equation or the Kummer-Schwartz equation is equivalent to a study of the other two. More recently the closely connected linear third-order of maximal symmetry and the Ermakov-Pinney equation have been added to the list given by Conte. Since these equations are of different orders and have differing symmetry properties, it is evident that the connections among them are nonlocal.

A considerable part of the motivation for this study is found in the works of Peterssen *et al* [27] and Euler *et al* [15] in which the authors obtain recursion operators for linearisable $1+1$ evolution equations. The specific equation of relevance to this work is the eighth of their classification[1], *videlicet*

$$u_t = u_{xx} + \lambda_8 u_x + h_8 u_x^2,$$

where $\lambda_8$ is an arbitrary constant and $h_8$ is an arbitrary function of the dependent variable $u$. The corresponding recursion operator is

$$\mathbf{VIII} \qquad R_8[u] = D_x + h_8 u_x.$$

The usual area of application of recursion operators has been that of partial differential equations and in particular evolution equations of which the equation above is a sample. Euler *et al* [16] applied the idea of a recursion operator to ordinary differential equations. They did not present a treatment of all possible classes, but concentrated upon two representative equations, *videlicet* the Riccati equation and the Ermakov-Pinney equation.

The recursion operator which generates the Riccati Sequence[2] is

$$\text{R.O.} = D + y, \tag{1.1}$$

where $D$ denotes total differentiation with respect to $x$. The recursion operator essentially originates from Case VIII, $R_8[u]$, for partial differential equations when the dependence on $t$ is removed, *ie* $(u_t = 0)$, and one writes $u_x = y$. We note that in applications to ordinary differential equations the notation usually used in the context of partial differential equations can be and is simplified.

When (1.1) acts upon[3] $y$ (which can be denoted as $R_0$) it generates, in succession, all the mem-

---

[1] Note that the arbitrary function is misleading as an indicator of generalisation which one can readily see by looking at the corresponding equation for $U(t,x)$, where $U = H(u)$ and $h_8(u) = H''(u)/H'(u)$.

[2] In [16] the term 'Riccati Hierarchy' is used. For a reason which becomes apparent in what follows we believe that the term 'sequence' is more appropriate. The 'correct' term to be used to describe this sequence has been a matter of perception of the language.

[3] This is a matter of notation. In this paper we have chosen the simplest. In another context, especially when one wishes to be 'physically' correct, one would choose $\exp[\int y \mathrm{d}x]$ since, if $y$ is replaced by $-x$ one obtains the generating function for the solution of the time-independent Schrödinger equation for the simple harmonic oscillator. An alternative interpretation based upon generalised symmetries of the potential Burgers' equation is given in [6].

bers of the Sequence which we denote by $R_n$, *videlicet*

$$\begin{aligned} y' + y^2 &= 0 \quad (R_1) && (1.2)\\ y'' + 3yy' + y^3 &= 0 \quad (R_2) && (1.3)\\ y''' + 4yy'' + 3y'^2 + 6y^2y' + y^4 &= 0 \quad (R_3) && (1.4)\\ y'''' + 5yy''' + 10y'y'' + 10y^2y'' + 15yy'^2 + 10y^3y' + y^5 &= 0 \quad (R_4) && (1.5)\\ &\vdots \\ (D+y)^n y &= 0 \quad (R_n). && (1.6) \end{aligned}$$

As an aid in our discussion we distinguish between an element of the Sequence, denoted as indicated above by $R_n$, and its left side by writing the latter as $\tilde{R}_n$.

In addition to the formal definition given in (1.6) there is a differential recurrence relation.

**Lemma 1:** *The members of the Riccati Sequence satisfy the differential recurrence relation*

$$\frac{\partial}{\partial y}\left(\tilde{R}_{m+1}\right) = (m+2)\tilde{R}_m. \tag{1.7}$$

**Proof:** By inspection of (1.2), (1.3) and (1.4) it is evident that (1.7) is true for the initial members of the sequence. We assume that

$$\frac{\partial}{\partial y}\left(\tilde{R}_{k+1}\right) = (k+2)\tilde{R}_k \tag{1.8}$$

for some $k \geq 2$. Then

$$\begin{aligned} \frac{\partial}{\partial y}\left(\tilde{R}_{k+2}\right) &= \frac{\partial}{\partial y}\left[(D+y)\tilde{R}_{k+1}\right] \\ &= \left[(D+y)\partial_y + 1\right]\tilde{R}_{k+1} \\ &= (D+y)(k+2)\tilde{R}_k + \tilde{R}_{k+1} \\ &= (k+3)\tilde{R}_{k+1} \end{aligned}$$

as required.

**Remark 1:** This recurrence relation is a little intriguing bearing in mind the connection to the generating function for the Hermite polynomials. One recalls the differential recurrence relation for the Hermite polynomials, *videlicet*

$$\frac{\mathrm{d}H_n(x)}{\mathrm{d}x} = 2nH_{n-1}(x),$$

and notes the similarity of structure to that given in (1.7).

The Riccati Sequence contains as its two lower members the Riccati equation and the Painlevé-Ince equation. The Riccati equation [29] has an history now of almost three centuries and is one of the few examples known for which there exists a nonlinear superposition principle. Its close relationship to the linear second-order differential equation and the Kummer-Schwartz equation has already been noted [10]. The Painlevé-Ince equation [19, 26] is a notable example of a nonlinear second-order differential equation of maximal symmetry possessing both Left and Right Painlevé Series [18] and arises in a remarkable number of applications[4].

[4]For a listing see [3, 9, 18, 24]. The authors of the third paper place (1.3) in a broader class of equations which they describe as being of modified Emden type.

In this paper we report the notable properties of the members of the Riccati Sequence. We concentrate upon the number of Lie point symmetries, singularity properties, first integrals, explicit integrability and complete symmetry groups. In Section 2 we present the symmetry analysis of the members of the Sequence. In terms of the explicit integrability of the members of the Sequence the lack of point symmetry in the higher-order members is remarkable and indicates the necessity to consider nonlocal symmetries. We note that this was the case in the treatment of a pair of equations of Ermakov-Pinney type [22]. In Section 3 we elaborate upon the results of the singularity analysis of the lower members of the Riccati Sequence presented by Euler *et al* [16] by considering the pattern of the values of the resonances for the general member of the Sequence. Section 4 is devoted to a consideration of the invariants and first integrals of the members of the Sequence. A short discussion on the symmetries of the integrals is presented. In Section 5 we present the complete symmetry group of the general member of the Sequence after firstly considering the results for $R_4$ to give a concrete basis for the theoretical discussion. In Section 6 we recall the solution of $R_n$ and extend our discussion to an equation containing combinations of the $R_i$. In the case of the second-order equation so formed the properties have been known for a long time [19,20,26]. Among our concluding remarks in Section 6 we summarise the remarkable properties found for the Riccati Sequence which is based upon a rather elementary recursion operator. We indicate that the route to complexification presented in Section 6 broadens the class of differential equations which may be subsumed into a more general concept of a Riccati Sequence.

One of the aspects considered in this paper is the calculation of the Lie point symmetries of the nonlinear ordinary differential equations of second and higher order which constitute the elements of the differential sequence. Although we eventually present a formal proof of the possession of a representation of the algebra $sl(2,R)$ for the equations of order greater than two, one needs to have some clue as to the forms of the symmetries to be found. The calculation of those symmetries by hand is a formidable task for the higher-order equations when starting from a *tabula rasa* and one does well to make use of one of the various symbolic codes which have been developed over the past thirty-plus years to make those tedious, lengthy and error-prone calculations more easily. We make use of SYM which is a Mathematica add-on developed by Dimas [11, 12]. It quickly provided results for $R_2$ and several of the higher-order members of the sequence and provided the Ansatz which lead to the Satz of Proposition 1 in §2. We remark that SYM has far wider capabilities than we needed for the purposes of this paper in terms of both ordinary and partial differential equations, symmetries more exotic than point and the calculation of algebraic structures.

## 2 Symmetry analysis

As a first-order ordinary differential equation (1.2) possesses an infinite number of Lie point symmetries[5]. Equation (1.3) was examined in [24] for its Lie point symmetries which were found to be the following eight, indeed the maximal number for a second-order ordinary differential equation,

[5] A Lie point symmetry is the generator of an infinitesimal transformation which leaves some object unchanged. Specifically it has the form $\Gamma = \xi(x,y)\partial_x + \eta(x,y)\partial_y$ in which the coefficient functions $\xi$ and $\eta$ are functions of $x$ and $y$ only. The infinitesimal transformation is $\bar{x} = x + \varepsilon\xi$ and $\bar{y} = y + \varepsilon\eta$, where $\varepsilon$ is an infinitesimal parameter. In the case of a differential equation, $E = 0$, the condition of invariance is $\Gamma^{[n]}E_{|_{E=0}=0}$, *ie*, the differential equation is taken into account in the calculation of the symmetry. The superscript $[n]$ indicates that

here written in a manner more elegant than in [24] to match the results of the succeeding members of the sequence,

$$\begin{aligned}
\Gamma_1 &= \partial_x \\
\Gamma_2 &= x\partial_x - y\partial_y \\
\Gamma_3 &= x^2\partial_x + (2-2xy)\partial_y \\
\Gamma_4 &= y\partial_x - y^3\partial_y \\
\Gamma_5 &= xy\partial_x + y^2(1-xy)\partial_y \\
\Gamma_6 &= x^2y\partial_x - y[2+xy(xy-2)]\partial_y \\
\Gamma_7 &= -x^2(xy-2)\partial_x + xy(xy-2)(xy-1)\partial_y \\
\Gamma_8 &= x^3(xy-2)\partial_x - x(xy-2)[2+xy(xy-2)]\partial_y
\end{aligned}$$

with the algebra $sl(3,R)$.

We proceed with $R_3$ to find an unexpected three, *ie*

$$\begin{aligned}
\Gamma_1 &= \partial_x \\
\Gamma_2 &= x\partial_x - y\partial_y \\
\Gamma_3 &= x^2\partial_x + (3-2xy)\partial_y
\end{aligned}$$

with the algebra $sl(2,R)$.

**Proposition 1:** *The general member of the Riccati Sequence possesses the symmetries*

$$\begin{aligned}
\Gamma_1 &= \partial_x \\
\Gamma_2 &= x\partial_x - y\partial_y \\
\Gamma_3 &= x^2\partial_x + (n-2xy)\partial_y
\end{aligned}$$

*with the algebra* $sl(2,R)$.

**Remark 2:** For $R_2$ there are an additional five symmetries and for $R_1$ an additional infinity.

**Proof:** See Appendix A.

## 3 Singularity analysis

We use singularity analysis as a tool to determine whether a given differential equation be integrable in terms of functions almost everywhere analytic. For equations which pass the Painlevé Test we are then encouraged to seek closed-form solutions. We find that all members of the Riccati Sequence possess the Painlevé Property. As we see below, explicit integrability follows. Independently of this integrability the results of the singularity analysis provide some very interesting patterns in terms of the parameters, that is the $p$, $\alpha$ and $r$ of the standard analysis.

Euler *et al* [16] presented the singularity analysis of the Riccati Sequence. For all elements the leading-order behaviour is $\alpha\chi^{-1}$, where $\chi = x - x_0$ and $x_0$ is the location of the putative singularity, with the possible values of $\alpha$ being listed in the Tables. We summarise their results.

---

the differential operator, $\Gamma$, is extended to give the infinitesimal transformation induced in the $n$th derivative by $\Gamma$, where $n$ is the order of the highest derivative in the equation.

Table I be here

In Table II we advance from [16] and present the properties in terms of the singularity analysis for the general member of the Riccati Sequence. In that way we are able to comment upon the integrability or otherwise of all members of the Sequence in the sense of Painlevé.

Table II be here

The pattern of the resonances is as follows: The set of resonances for $\alpha = j$ is obtained from the set for $\alpha = j - 1$ by subtracting the number $n + 1$ from the largest positive resonance of the latter set.

It follows from Tables I and II that all members of the Riccati Sequence pass the Painlevé Test and therefore each is integrable in terms of analytic functions.

When we apply the Riccati transformation

$$y = \alpha \frac{w'}{w} \tag{3.1}$$

to the $n$th member of the Riccati Sequence, we observe that the most simplified equation, *ie* $w^{(n+1)} = 0$, arises when we chose $\alpha = 1$ which is a consequence of the singularity analysis itself [4]. Therefore (3.1) may be written as

$$x = x, \quad w = \exp\left[\int y \mathrm{d}x\right].$$

It is a matter of simple calculation to verify the following proposition.

**Proposition 2:** *The general solution of the $n$th member of the Riccati Sequence, $n \geq 1$, is given by*

$$y_n = \frac{\left(\sum_{i=0}^{n} A_i x^i\right)'}{\sum_{i=0}^{n} A_i x^i}, \tag{3.2}$$

*where the $A_i$, $i = 0, n$, are constants of integration.*

**Remark 3:** As it is evident, the Riccati tranformation, (3.1), enables one to write the solution for any member of the Riccati Sequence, (1.2) to (1.6), in closed form. It could be thought that the existence of such a transformation makes the Riccati Sequence trivial and not worth studying. However, there are some points for such an argument: Firstly the equations may be trivially integrated, but remain particularly important in applications[6]. Secondly this almost immediate integration enables one to make direct comparisons with symmetry or singularity results. If the closed-form solution or the indication of integrability did not exist, then one could not claim that the interpretation of the results of the singularity analysis, for example, are correct. Thirdly this triviality is essential for the general considerations we report in this manuscript in terms of the symmetry analysis and complete symmetry groups. This inherent structure is vital when one wishes to prove the existence of these symmetries and the complete specification of the equations in the general case when the equations themselves are unknown, but the recursion operator provides the route to the $n$th-order member of the Sequence, (1.6).

[6]This reminds one of many equations in the literature, such as the Black-Scholes-Merton equation in Financial Mathematics. Although the equation has been found to be linked via a coordinate transformation to the heat equation and Black and Scholes themselves provided the closed-form solution, the equation continues to attract the attention of many researchers in Financial Mathematics because of the fundamental position it holds in Finance nowadays [13].

## 4 Invariants and First Integrals

For the purposes of this paper we commence with some definitions.

**Definition 1:** An invariant $I$ of an ordinary differential equation is said to be any nontrivial function which satisfies

$$\frac{\mathrm{d}I}{\mathrm{d}x} = 0 \tag{4.1}$$

on solution curves of the differential equation.

**Definition 2:** A first integral of an $n$th-order ordinary differential equation is any function, $I = I(x, y, y', \ldots, y^{(n-1)})$, which satisfies (4.1).

Note that we do depart from some standard definitions for reasons which become evident below. There are various conventions concerning the meanings of the expressions 'first integral', 'invariant' and 'conserved quantity'. We are not concerned by the third in this paper since we do not use that expression. Some writers use the first two expressions interchangeably. Others prefer to distinguish between the two by insisting that the former be autonomous whereas the latter is allowed to depend upon the independent variable. This is a sensible distinction under appropriate circumstances. Indeed in terms of the integration of an ordinary differential equation the distinction can be quite critical. In this paper we have varied the definitions to suit the very precise purpose of distinguishing between two classes of function both of which have the property of having a zero total derivative on solution curves of the differential equation.

We wish to identify all invariants and first integrals, as defined above, of each member of the Riccati Sequence. In order to do so we take the $n$th member of that Sequence, perform an increase of order by using the Riccati transformation to obtain $w^{(n+1)} = 0$. We compute the fundamental integrals and invariants of that equation and by reverting the transformation one can deduce the whole set of invariants for the $n$th member of the Riccati Sequence, *videlicet*

$$I_j = \left(\sum_{i=1}^{j} \frac{(-1)^{i+1}}{(j-i)!} x^{j-i} \tilde{R}_{(n-i)}\right) \exp\left[\int y \mathrm{d}x\right], \quad j = 1, n+1, \tag{4.2}$$

where $\tilde{R}_{-1} = 1$ by convention.

Note that for the invariants of $w^{(n)} = 0$ we would have the above invariants (4.2) except that instead of $\tilde{R}_{(n-i)}$ we would have $w^{(n+1-i)}$ and there would be no exponential.

We now turn our attention to first integrals. One takes the ratio of two separate invariants to obtain a first integral. An independent set for $R_n$ can be variously defined. The set $\{\mathcal{F}_{ij}\}$, defined by

$$\mathcal{F}_{ij} = \frac{I_j}{I_i}, \quad j = 1, i-1, i+1, n, \tag{4.3}$$

is an independent set of first integrals for $R_n$. Such a simple formula is not available if one wishes to describe a set of autonomous first integrals.

An interesting aspect arises in the symmetry properties of the first integrals which we briefly note. If one computes the symmetries of all first integrals of (1.3), one finds that they all share the algrebra $A_1 \oplus_s A_2$. By a curious misfortune this feature does not persist and therefore one is confronted with an absolute zero for contact (not to mention point) symmetries for the first integrals of (1.4)[7].

[7] When equation (1.4) is related to $w^{(iv)} = 0$ by an increase of order, the symmetries of the first integrals of the latter become, by inverting the transformation, nonlocal symmetries for the invariants of the former.

## 5 Complete symmetry groups

The concept of a complete symmetry group of a differential equation was introduced by Krause [21] as the group associated with the set of symmetries, be they point, contact, generalised or nonlocal, required to specify the equation or system completely [7].

We start with $R_4$ to give a flavour of the procedure and then we prove the general result for any member of the Riccati Sequence.

**Proposition 3:** *The complete symmetry group of $R_4$ is given by the symmetries*

$$\begin{aligned}
\Delta_1 &= -\exp\left[-\int y\mathrm{d}x\right]\{y\}\partial_y \\
\Delta_2 &= -\exp\left[-\int y\mathrm{d}x\right]\{xy-1\}\partial_y \\
\Delta_3 &= -\exp\left[-\int y\mathrm{d}x\right]\{x^2y-2x\}\partial_y \\
\Delta_4 &= -\exp\left[-\int y\mathrm{d}x\right]\{x^3y-3x^2\}\partial_y \\
\Delta_5 &= -\exp\left[-\int y\mathrm{d}x\right]\{x^4y-4x^3\}\partial_y.
\end{aligned}$$

**Proof:** See Appendix B.

**Proposition 4:** *The complete symmetry group of $R_n$ is given by the $(n+1)$ symmetries*[8]

$$\Delta_i = -\exp\left[-\int y\mathrm{d}x\right]\{x^{i-1}y-(i-1)x^{i-2}\}\partial_y,\ i=1,n+1.$$

**Proof:** See Appendix C.

## 6 Concluding remarks

We are familiar with the use of sequences of numbers and functions in the mathematical and wider scientific literature. The last several decades have seen intensive study of hierarchies based upon nonlinear evolution equations which have their bases in the mathematical modelling of physical phenomena. The extension to ordinary differential equations has only been made recently [16, 17]. We have chosen to term these related equations as differential sequences to reflect the dual nature of a definition which combines the idea of an operation which generates the elements of the sequence and also that the elements of the sequence are composed of derivatives. It seems to us that the word sequence is more appropriate in this context than hierarchy and is more in keeping with the tradition of mathematical terminology.

The Riccati Sequence, which has been the subject of the present study, illustrates with a degree of excellence, which one hopes can be surpassed, the type of mathematical properties which are likely to make such sequences an object of fond study. The Riccati equation has a long and distinguished history in both the theory of differential equations and the application to divers phenomena. The

[8] The correct number of symmetries required to specify an equation completely has been discussed in [1], [2] and [5].

Riccati Sequence and its adjoint are based on recursion operators closely related in form to the Dirac operators of the quantum mechanical simple harmonic oscillator which in themselves are simply autonomous versions of two of the Lie point symmetries of the classical simple harmonic oscillator. With such a lavish heritage one should not be too surprised that the Riccati Sequence[9] exhibits such properties in terms of symmetry and integrability.

Each element of the sequence can be linearised by means of the nonlocal transformation – often called a Riccati transformation – and so is trivially integrable. Although $R_1$ and $R_2$ – the Riccati and Painlevé-Ince equations – display exceptional symmetry in the sense of Lie point symmetries, the remaining elements of the sequence possess just the three-element algebra $sl(2,R)$ of Lie point symmetries. The distinguishing feature of the Lie symmetries of the elements of the sequence is the possession of $n+1$ (in the case of $R_n$) exponential nonlocal symmetries which completely specify the elements. The algebra of these symmetries is $(n+1)A_1$.

As a consequence of the ability to linearise the equations of the Riccati Sequence each element possesses an invariant derived from the so-called fundamental integrals of the parent linear equation[10]. These invariants are not integrals in the conventional sense as they contain the integral of the dependent variable. However, the integral of the dependent variable appears as an exponential term and so functions of these invariants which are homogeneous of degree zero in the dependent variable are first integrals. It was for this reason that we introduced specialised meanings for the two expressions, ‘invariant’ and ‘first integral’, for the purposes of this paper. Since there are $(n+1)$ linearly independent (exponential nonlocal) invariants, there are $n$ functionally independent integrals which reflect most adequately the integrability of each member of the Riccati Sequence.

In terms of singularity analysis the Riccati Sequence possesses properties which may even be regarded to outshine the symmetry properties. Not only is each element of the Sequence explicitly integrable in terms of analytic functions apart from isolated polelike singularities – a property which has been used to illustrate certain subtle and not widely appreciated implications of singularity analysis [3, 8] – but also there are patterns to the possible values of the coefficients of the leading-order terms and the values of the resonances for each of the principal branches. In passing one recalls that each branch is principal and the need to be concerned with the implications of branches with unfortunate properties is obviated. The patterns of the resonances deserve further, separate treatment, particularly in respect of other possible sequences.

The Riccati Sequence has provided an excellent vehicle for the introduction of the notion of differential sequences of ordinary differential equations. Its Recursion Operator is simple. Its generating function is elementary. Two of the elements, $R_1$ and $R_2$, are well-known in the literature as well as in applications. Already Euler *et al* [16] have made a brief excursion into the properties of the Ermakov-Pinney Sequence, as the name suggests, based upon the Ermakov-Pinney equation [14, 28] which has been found in many varied applications. Note also a variation of the Riccati Sequence [23].

In the Introduction we have recalled the classical association of the Riccati equation, the linear

[9] Equally the adjoint sequence which is generated by the operator $D-y$, the adjoint to (1.1). In the following discussion the properties of the adjoint sequence may be inferred *mutatis mutandis* from those of the Riccati Sequence.

[10] These integrals have been termed fundamental since an $n$th-order linear ordinary differential equation possesses $n$ linearly independent integrals which are linear in the dependent variable and its first $(n-1)$ derivatives. The integrals are known for their interesting algebraic properties.

second-order ordinary differential equation and the third-order Kummer-Schwarz equation. Since these equations are related by means of nonlocal transformations, there is no reason to exclude from this select group other equations which are similarly related[11]. One looks forward to further revelations of fascinating properties of other differential sequences.

As a final remark we recall that the evolution equations and associated Recursion Operators [27] which are the source material for this work contain arbitrary parameters and unspecified functions. In this work we have deliberately kept to the minimum required to make a sensible sequence. It is a little like treating the autonomous linear oscillator rather than a time-dependent, damped and forced oscillator. The latter can be transformed to the former by means of well-defined transformations and a study of the former is easier due to the simplicity possible in the presentation. Despite our personal preference for simplicity we do nevertheless accept that there are those who wish to see the treatment of the more general systems.

Consider

$$E_n = \sum_{i=0}^{n} f_i(t)\tilde{R}_i = 0. \tag{6.1}$$

Equation (6.1) is linearisable by means of the Riccati transformation (3.1), with $\alpha = 1$, to

$$\sum_{i=0}^{n} f_i(t)w^{(i+1)} = 0. \tag{6.2}$$

Equation (6.2) possesses $(n+1)$ solution symmetries and the homogeneity symmetry.

Whilst there can be no dispute that the solution of (6.2) is more difficult than that of $w^{(n+1)} = 0$, for our purposes there is no difference. Subject to some mild conditions on the functions $f_i(t)$ – basically that they be continuous [19] [p 45] – the solutions of (6.2) exist and that is all that is required for inclusion into the general framework of the treatment above. An $n$th-order linear ordinary differential equation can have $n+1$, $n+2$ or $n+4$ Lie point symmetries which represent the $n$ solution symmetries, linearity, autonomy (in the right variables) and the general $sl(2,R)$ symmetry of equations of maximal symmetry [25]. There is no gainsaying that the additional symmetries make the process of solution in closed form just that much easier. However, all we need is the existence of the $n$ solution symmetries to provide the symmetries necessary for our purposes. Consequently we have taken the clearer part so that the ideas and results contained in this paper be as evident as can be possible in such matters.

We look forward to a development of the subject of differential sequences of ordinary differential equations to match that of the related areas in partial differential equations.

## Appendix A

The possession of $\Gamma_1$ is straightforward since (1.6) is autonomous. We proceed with $\Gamma_2$. The $n$th extension of $\Gamma_2$ is

$$\Gamma_2^{[n]} = x\partial_x - \sum_{j=0}^{n}(j+1)y^{(j)}\partial_{y^{(j)}} \tag{6.3}$$

[11] One is well aware that all properties may not travel well through the process of nonlocal transformation, but that obstacle is already encountered under far less exotic transformations.

and it is a simple calculation to demonstrate that

$$\begin{aligned}\Gamma_2^{[1]}\tilde{R}_1 &= -2\tilde{R}_1\\ \Gamma_2^{[2]}\tilde{R}_2 &= -3\tilde{R}_2.\end{aligned}$$

We assume that the statement

$$\Gamma_2^{[m]}\tilde{R}_m = -(m+1)\tilde{R}_m \tag{6.4}$$

is correct and we prove that

$$\Gamma_2^{[m+1]}\tilde{R}_{m+1} = -(m+2)\tilde{R}_{m+1}.$$

We commence with the statement

$$\Gamma_2^{[m+1]}\tilde{R}_{m+1} = \Gamma_2^{[m+1]}(D+y)\tilde{R}_m. \tag{6.5}$$

We break the calculation of the right side of (6.5) into some parts. Firstly we have

$$\Gamma_2^{[m+1]}D = \left[\Gamma_2^{[m+1]}, D\right]_{LB} + D\Gamma_2^{[m+1]}. \tag{6.6}$$

The Lie Bracket is

$$\begin{aligned}&\left[x\partial_x - \sum_{j=0}^{m+1}(j+1)y^{(j)}\partial_{y^{(j)}}, \partial_x + \sum_{k=0}^{\cdot} y^{(k+1)}\partial_{y^{(k)}}\right]_{LB}\\
&= -\partial_x - \left[\sum_{j=0}^{m+1}(j+1)y^{(j)}\partial_{y^{(j)}}, \sum_{k=0}^{\cdot} y^{(k+1)}\partial_{y^{(k)}}\right]_{LB}\\
&= -\partial_x - \sum_{j=0}^{m+1}(j+1)\sum_{k=0}^{\cdot}\left[y^{(j)}\partial_{y^{(j)}}, y^{(k+1)}\partial_{y^{(k)}}\right]_{LB}\\
&= -\partial_x - \sum_{j=0}^{m+1}(j+1)\sum_{k=0}^{\cdot}\left\{y^{(j)}\delta_{j,k+1}\partial_{y^{(k)}} - y^{(k+1)}\delta_{k,j}\partial_{y^{(j)}}\right\}\\
&= -\partial_x - \sum_{j=0}^{m+1}(j+1)\left\{y^{(j)}\partial_{y^{(j-1)}} - y^{(j+1)}\partial_{y^{(j)}}\right\}\\
&= -\partial_x - \sum_{j=0}^{m}(j+2)y^{(j+1)}\partial_{y^{(j)}} + \sum_{j=0}^{m+1}(j+1)y^{(j+1)}\partial_{y^{(j)}}\\
&= -\partial_x - \sum_{j=0}^{m} y^{(j+1)}\partial_{y^{(j)}} + (m+2)y^{(m+2)}\partial_{y^{(m+1)}}\\
&= -D + (m+2)y^{(m+2)}\partial_{y^{(m+1)}},\end{aligned}$$

where the overdot on the summation means that the sum is taken to whatever order of derivative required.

In a much simpler calculation we have

$$\begin{aligned}\Gamma_2^{[m+1]}y\tilde{R}_m &= \left(\Gamma_2^{[m+1]}y\right)\tilde{R}_m + y\left(\Gamma_2^{[m+1]}\tilde{R}_m\right)\\ &= -y\tilde{R}_m + y\Gamma_2^{[m+1]}\tilde{R}_m. \end{aligned}\tag{6.7}$$

We substitute the results of both the Lie Bracket and (6.7) into (6.5). We have

$$\begin{aligned}
\Gamma_2^{[m+1]}\tilde{R}_{m+1} &= \Big\{-D+(m+2)y^{(m+2)}\partial_{y^{(m+1)}}+D\Gamma_2^{[m+1]} \\
&\quad -y+y\Gamma_2^{[m+1]}\Big\}\tilde{R}_m \\
&= -(D+y)\tilde{R}_m+(D+y)\Gamma_2^{[m+1]}\tilde{R}_m \\
&\stackrel{(6.4)}{=} -(D+y)\tilde{R}_m-(D+y)(m+1)\tilde{R}_m \\
&= -(m+2)(D+y)\tilde{R}_m \\
&= -(m+2)\tilde{R}_{m+1}
\end{aligned} \tag{6.8}$$

which proves the result.

Note that in the proof we have made use of the fact that $\tilde{R}_m$ contains derivatives only up to $y^{(m)}$.

We turn now to the proof in the case of $\Gamma_3$.

It is a simple matter to show that

$$\begin{aligned}
\Gamma_3^{[1]}\tilde{R}_1 &= -2x(1+1)\tilde{R}_1 \\
\Gamma_3^{[2]}\tilde{R}_2 &= -2x(2+1)\tilde{R}_2 \\
\Gamma_3^{[3]}\tilde{R}_3 &= -2x(3+1)\tilde{R}_3
\end{aligned}$$

and one can easily assume that

$$\Gamma_3^{[m]}\tilde{R}_m = -2x(m+1)\tilde{R}_m \tag{6.9}$$

so that the task is now to demonstrate that from this property it follows that

$$\Gamma_3^{[m+1]}\tilde{R}_{m+1} = -2x(m+2)\tilde{R}_{m+1}. \tag{6.10}$$

We commence with the definition and so we have

$$\begin{aligned}
\Gamma_3^{[m+1]}\tilde{R}_{m+1} &= \Gamma_3^{[m+1]}(D+y)\tilde{R}_m \\
&= \Big[\Gamma_3^{[m+1]},(D+y)\Big]_{LB}\tilde{R}_m+(D+y)\Gamma_3^{[m+1]}\tilde{R}_m \\
&= \Big[\Gamma_3^{[m+1]},(D+y)\Big]_{LB}\tilde{R}_m+(D+y)\left(\Gamma_3^{[m]}+\partial_y\right)\tilde{R}_m,
\end{aligned}$$

where the last line is a consequence of remembering the definition of $\Gamma_3$ and that the $(m+1)th$ derivative in $\Gamma_3^{[m+1]}$ does not act on $\tilde{R}_m$.

The first task is to compute the Lie Bracket. We have

$$\Bigg[x^2\partial_x + (m+1-2xy)\partial_y - 2x\sum_{j=1}^{m+1}(j+1)y^{(j)}\partial_{y^{(j)}} - \sum_{j=1}^{m+1} j(j+1)y^{(j-1)}\partial_{y^{(j)}},$$
$$\partial_x + \sum_{k=0}^{\cdot} y^{(k+1)}\partial_{y^{(k)}} + y\Bigg]_{LB}$$
$$\begin{aligned} &= -2x\partial_x + 2y\partial_y + 2\sum_{j=1}^{m+1}(j+1)y^{(j)}\partial_{y^{(j)}} + 2xy'\partial_y + (m+1-2xy) \\ &\quad +2x\sum_{j=1}^{m+1}(j+1)\sum_{k=0}^{\cdot}\left[y^{(k+1)}\partial_{y^{(k)}},\, y^{(j)}\partial_{y^{(j)}}\right]_{LB} \\ &\quad +\sum_{j=1}^{m+1} j(j+1)\sum_{k=0}^{\cdot}\left[y^{(k+1)}\partial_{y^{(k)}}, y^{(j-1)}\partial_{y^{(j)}}\right]_{LB}. \end{aligned}$$

We compute the two subsidiary Lie Brackets separately.

$$\begin{aligned} \mathrm{LB}_1 &= 2x\sum_{j=1}^{m+1}(j+1)\sum_{k=0}^{\cdot}\left\{y^{(k+1)}\delta_{k,j}\partial_{y^{(j)}} - y^{(j)}\delta_{j,k+1}\partial_{y^{(k)}}\right\} \\ &= 2x\sum_{j=1}^{m+1}(j+1)\left\{y^{(j+1)}\partial_{y^{(j)}} - y^{(j)}\partial_{y^{(j-1)}}\right\} \\ &= 2x\left\{(m+2)y^{(m+2)}\partial_{y^{(m+1)}} - 2y'\partial_y - \sum_{j=1}^{m} y^{(j+1)}\partial_{y^{(j)}}\right\}. \end{aligned}$$

In a similar way we find that

$$\mathrm{LB}_2 = (m+1)(m+2)y^{(m+1)}\partial_{y^{(m+1)}} - 2y\partial_y - 2\sum_{j=1}^{m}(j+1)y^{(j)}\partial_{y^{(j)}}.$$

We return to the main calculation and after some simplification of terms which cancel we continue as below.

$$\begin{aligned} \Gamma_3^{[m+1]}\tilde{R}_{m+1} &= \Bigg\{-2x\left(\partial_x + y'\partial_y + \sum_{j=1}^{m} y^{(j+1)}\partial_{y^{(j)}}\right) + 2(m+2)y^{(m+1)}\partial_{y^{(m+1)}} \\ &\quad +m+1-2xy + (m+1)(m+2)y^{(m+1)}\partial_{y^{(m+1)}} \\ &\quad +2x(m+2)y^{(m+2)}\partial_{y^{(m+1)}} + (D+y)\left(-2x(m+1)+\partial_y\right)\Bigg\}\tilde{R}_m \\ &= \{-2x(D+y) + (m+1) + (D+y)\partial_y - 2(m+1)(D+y)x\}\tilde{R}_m \\ &= -2x(m+2)(D+y)\tilde{R}_m + [(D+y)\partial_y - (m+1)]\tilde{R}_m \\ &= -2x(m+2)\tilde{R}_{m+1} \end{aligned}$$

since

$$[(D+y)\partial_y - (m+1)]\tilde{R}_m = 0.$$

This last statement is equivalent to the result of Lemma 1, *ie*

$$\frac{\partial}{\partial y}\left(\tilde{R}_{m+1}\right) = (m+2)\tilde{R}_m$$

and so the result is proven.

Remark: Note that in the proof above we have again made use of the fact that $\tilde{R}_m$ contains derivatives only up to $y^{(m)}$.

# Appendix B

We calculate the fourth extensions of $\Delta_1 - \Delta_5$.

$$
\begin{aligned}
\Delta_1^{[4]} &= -\exp\left[-\int y\mathrm{d}x\right]\{y\partial_y + (y' - y^2)\partial_{y'} + (y'' - 3yy' + y^3)\partial_{y''} \\
&\quad +(y''' - 4yy'' - 3y'^2 + 6y^2y' - y^4)\partial_{y'''} \\
&\quad +(y^{(iv)} - 5yy''' - 10y'y'' + 10y^2y'' + 15yy'^2 - 10y^3y' + y^5)\partial_{y^{(iv)}}\} \\
\Delta_2^{[4]} &= -\exp\left[-\int y\mathrm{d}x\right]\{-\partial_y + 2y\partial_{y'} + 3(y' - y^2)\partial_{y''} + 4(y'' - 3yy' + y^3)\partial_{y'''} \\
&\quad +5(y''' - 4yy'' - 3y'^2 + 6y^2y' - y^4)\partial_{y^{(iv)}}\} + x\Delta_1^{[4]} \\
\Delta_3^{[4]} &= -2\exp\left[-\int y\mathrm{d}x\right]\{-\partial_{y'} + 3y\partial_{y''} + 6(y' - y^2)\partial_{y'''} + 10(y'' - 3yy' + y^3)\partial_{y^{(iv)}}\} \\
&\quad +2x\Delta_{2eff}^{[4]} + x^2\Delta_1^{[4]} \\
\Delta_4^{[4]} &= -6\exp\left[-\int y\mathrm{d}x\right]\{-\partial_{y''} + 4y\partial_{y'''} + 10(y' - y^2)\partial_{y^{(iv)}}\} \\
&\quad +3x\Delta_{3eff}^{[4]} + 3x^2\Delta_{2eff}^{[4]} + x^3\Delta_1^{[4]} \\
\Delta_5^{[4]} &= -24\exp\left[-\int y\mathrm{d}x\right]\{-\partial_{y'''} + 5y\partial_{y^{(iv)}}\} \\
&\quad +4x\Delta_{4eff}^{[4]} + 6x^2\Delta_{3eff}^{[4]} + 4x^3\Delta_{2eff}^{[4]} + x^4\Delta_1^{[4]},
\end{aligned}
$$

where the subscript *eff* stands for the effective part of that symmetry.

When we act all the above extensions on the general fourth-order ordinary differential equation, *videlicet*

$$y^{(iv)} = f(x, y, y', y'', y'''),$$

we obtain the system of five equations

$$
\begin{aligned}
y^{(iv)} - 5yy''' - 10y'y'' + 10y^2y'' + 15yy'^2 - 10y^3y' + y^5 &= y\frac{\partial f}{\partial y} + (y' - y^2)\frac{\partial f}{\partial y'} \\
&+(y'' - 3yy' + y^3)\frac{\partial f}{\partial y''} + (y''' - 4yy'' - 3y'^2 + 6y^2y' - y^4)\frac{\partial f}{\partial y'''} \qquad (6.11) \\
5(y''' - 4yy'' - 3y'^2 + 6y^2y' - y^4) &= -\frac{\partial f}{\partial y} + 2y\frac{\partial f}{\partial y'} \\
&+3(y' - y^2)\frac{\partial f}{\partial y''} + 4(y'' - 3yy' + y^3)\frac{\partial f}{\partial y'''} \\
10(y'' - 3yy' + y^3) &= -\frac{f}{\partial y'} + 3y\frac{\partial f}{\partial y''} + 6(y' - y^2)\frac{\partial f}{\partial y'''} \\
10(y' - y^2) &= -\frac{\partial f}{\partial y''} + 4y\frac{\partial f}{\partial y'''} \\
5y &= -\frac{\partial f}{\partial y'''},
\end{aligned}
$$

which can be solved backwardsly to give the following expressions for the derivatives of all arguments in $f$

$$
\begin{aligned}
\frac{\partial f}{\partial y'''} &= -5y \\
\frac{\partial f}{\partial y''} &= -10(y' + y^2) \\
\frac{\partial f}{\partial y'} &= -10(y'' + 3yy' + y^3) \\
\frac{\partial f}{\partial y} &= -5(y''' + 4yy'' + 3y'^2 + 6y^2y' + y^4).
\end{aligned}
$$

When these expressions are substituted into (6.11), we recover $R_4$.

## Appendix C

We remind the reader that $R_n$ stands for the $n$th member of the Riccati Sequence and $\tilde{R}_n$ for the left hand side of $R_n$. It is essential for the notation required in this proof to introduce the adjoint recursion operator, $R.O.^\alpha = D - y$, which generates the adjoint Riccati Sequence, *videlicet*

$$\begin{aligned}
y' - y^2 &= 0 \quad (R_1^\alpha) && (6.12)\\
y'' - 3yy' + y^3 &= 0 \quad (R_2^\alpha) && (6.13)\\
y''' - 4yy'' - 3y'^2 + 6y^2y' - y^4 &= 0 \quad (R_3^\alpha) && (6.14)\\
y^{(iv)} - 5yy''' - 10y'y'' + 10y^2y'' + 15yy'^2 - 10y^3y' + y^5 &= 0 \quad (R_4^\alpha) && (6.15)\\
y^{(v)} - 6yy^{(iv)} - 15y'y''' + 15y^2y''' + 60yy'y'' - 10y''^2 - 20y^3y'' & \\
+15y'^3 - 45y^2y'^2 + 15y^4y' - y^6 &= 0 \quad (R_5^\alpha) && (6.16)\\
&\vdots \\
(D-y)^n y &= 0 \quad (R_n^\alpha). && (6.17)
\end{aligned}$$

The left hand side of the above members is denoted by $\tilde{R}_n^\alpha$.

Remark: We note that there is a reflection here between the Riccati Sequence and its adjoint in that the formulæ for the Sequence are reflected in the formulæ for the adjoint Sequence.

**Lemma 2:** *The general members of the Riccati and the adjoint Riccati Sequences satisfy the recurrence relation*

$$\tilde{R}_n = \tilde{R}_n^\alpha + \sum_{i=1}^{n} \binom{n+1}{i} \tilde{R}_{i-1}^\alpha \tilde{R}_{n-i}. \tag{6.18}$$

**Proof:** It is easy to show that (6.18) holds for $n = 2$ and $n = 3$. We assume that (6.18) is true for $n = k$ and we prove that it is true for $n = k + 1$.

$$\begin{aligned}
\tilde{R}_{k+1} &= (D+y)\tilde{R}_k = (D+y)\left[\tilde{R}_k^\alpha + \sum_{i=1}^{k}\binom{k+1}{i}\tilde{R}_{i-1}^\alpha \tilde{R}_{k-i}\right] \\
&= \tilde{R}_{k+1}^\alpha + 2y\tilde{R}_k^\alpha + \sum_{i=1}^{k}\binom{k+1}{i}\left\{D\left[\tilde{R}_{i-1}^\alpha \tilde{R}_{k-i}\right] + y\tilde{R}_{i-1}^\alpha \tilde{R}_{k-i}\right\} \\
&= \tilde{R}_{k+1}^\alpha + 2y\tilde{R}_k^\alpha + \sum_{i=1}^{k}\binom{k+1}{i}\left(\tilde{R}_i^\alpha + y\tilde{R}_{i-1}^\alpha\right)\tilde{R}_{k-i} \\
&\quad + \sum_{i=1}^{k}\binom{k+1}{i}\tilde{R}_{i-1}^\alpha\left(\tilde{R}_{k-i+1} - y\tilde{R}_{k-i}\right) + y\sum_{i=1}^{k}\binom{k+1}{i}\tilde{R}_{i-1}^\alpha \tilde{R}_{k-i} \\
&\overset{(n=k)}{=} \tilde{R}_{k+1}^\alpha + 2y\tilde{R}_k^\alpha + \sum_{i=1}^{k}\binom{k+1}{i}\tilde{R}_i^\alpha \tilde{R}_{k-i} + \sum_{i=1}^{k}\binom{k+1}{i}\tilde{R}_{i-1}^\alpha \tilde{R}_{k-i+1} + y(\tilde{R}_k - \tilde{R}_k^\alpha) \\
&= \tilde{R}_{k+1}^\alpha + y\tilde{R}_k^\alpha + y\tilde{R}_k + \sum_{i=2}^{k+1}\binom{k+1}{i-1}\tilde{R}_{i-1}^\alpha \tilde{R}_{k-i+1} + \sum_{i=1}^{k}\binom{k+1}{i}\tilde{R}_{i-1}^\alpha \tilde{R}_{k-i+1} \\
&= \tilde{R}_{k+1}^\alpha + y\tilde{R}_k^\alpha + y\tilde{R}_k + \sum_{i=1}^{k+1}\left\{\binom{k+1}{i-1} + \binom{k+1}{i}\right\}\tilde{R}_{i-1}^\alpha \tilde{R}_{k-i+1} - \tilde{R}_k^\alpha \tilde{R}_0 - \tilde{R}_0^\alpha \tilde{R}_k \\
&= \tilde{R}_{k+1}^\alpha + \sum_{i=1}^{k+1}\binom{k+2}{i}\tilde{R}_{i-1}^\alpha \tilde{R}_{k-i+1},
\end{aligned}$$

which proves the result.

The $n$th extension of $\Delta_i$ is

$$\begin{aligned}\Delta_i^{[n]} \quad = \quad & -\exp\left[-\int y\mathrm{d}x\right]\left\{-(i-1)!\partial_{y^{(i-2)}}+\frac{i!}{1!}y\partial_{y^{(i-1)}}+\frac{(i+1)!}{2!}\tilde{R}_1^\alpha\partial_{y^{(i)}}\right.\\ & \left.+\frac{(i+2)!}{3!}\tilde{R}_2^\alpha\partial_{y^{(i+1)}}+\ldots+\frac{(n+1)!}{(n-i+2)!}\tilde{R}_{n-i+1}^\alpha\partial_{y^{(n)}}\right\}\\ & +\sum_{k=0}^{i-2}\frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}x^k}(x^{i-1})\Delta_{(k+1)eff}^{[n]}, \quad i=1,n+1,\end{aligned}$$

where

$$\Delta_{(1)eff}^{[n]}=\Delta_1^{[n]},\partial_{y^{(-1)}}=0,\ \partial_{y^{(0)}}=\partial_y,\ \partial_{y^{(1)}}=\partial_{y'}\ \text{etc},\ \frac{\mathrm{d}^0}{\mathrm{d}x^0}=1,\ \frac{\mathrm{d}^1}{\mathrm{d}x^1}=\frac{\mathrm{d}}{\mathrm{d}x}\ \text{etc.}$$

The action of the effective part of $\Delta_i^{[n]}$,

$$\Delta_{(i)eff}^{[n]}=-\partial_{y^{(i-2)}}+\sum_{j=0}^{n-i+1}\binom{i+j}{j+1}\tilde{R}_j^\alpha\partial_{y^{(i-1+j)}},$$

on the general $n$th-order ordinary differential equation, *videlicet*

$$y^{(n)}=f(x,y,y',y'',y''',\ldots,y^{(n-1)}),$$

gives

$$\binom{n+1}{n-i+2}\tilde{R}_{n-i+1}^\alpha=-\frac{\partial f}{\partial y^{(i-2)}}+\sum_{j=0}^{n-i}\binom{i+j}{j+1}\tilde{R}_j^\alpha\frac{\partial f}{\partial y^{(i-1+j)}},\ i=1,n+1. \tag{6.19}$$

For $i=1$ (6.19) gives

$$\tilde{R}_n^\alpha=\sum_{j=0}^{n-1}\tilde{R}_j^\alpha\frac{\partial f}{\partial y^{(j)}}. \tag{6.20}$$

The remaining $n$ of (6.19) are equivalent to

$$\binom{n+1}{n-i+1}\tilde{R}_{n-i}^\alpha=-\frac{\partial f}{\partial y^{(i-1)}}+\sum_{j=i+1}^{n}\binom{j}{j-i}\tilde{R}_{j-i-1}^\alpha\frac{\partial f}{\partial y^{(j-1)}},\ i=1,n. \tag{6.21}$$

System (6.21) may be conveniently represented as the system of equations

$$\mathbf{L}_n^\alpha=\mathbf{Q}_n^\alpha\mathbf{F}_n, \tag{6.22}$$

where

$$\begin{aligned}(\mathbf{L}_n^\alpha)_i \quad &= \quad \binom{n+1}{i}\tilde{R}_{n-i}^\alpha, \qquad i=1,n,\\ (\mathbf{F}_n)_i \quad &= \quad \frac{\partial f}{\partial y^{(i-1)}}, \quad i=1,n,\\ (\mathbf{Q}_n^\alpha)_{ij} \quad &= \quad \begin{cases}\binom{j}{i}\tilde{R}_{j-i-1}^\alpha, & i<j,\\ -1, & i=j,\\ 0, & i>j.\end{cases}\end{aligned} \tag{6.23}$$

We define

$$(\mathbf{Q}_n)_{ij} = \begin{cases} -\binom{j}{i}\tilde{R}_{j-i-1}, & i<j, \\ -1, & i=j, \\ 0, & i>j. \end{cases} \tag{6.24}$$

and

$$(\mathbf{L}_n)_i = \binom{n+1}{i}\tilde{R}_{n-i}, \qquad i=1,n.$$

We prove the following two Lemmas.

**Lemma 3:** *The matrices $\mathbf{Q}_n^\alpha$ and $\mathbf{Q}_n$ defined in (6.23) and (6.24) are the inverses of each other, ie they satisfy*

$$\mathbf{Q}_n^\alpha \mathbf{Q}_n = \mathbf{I}_n. \tag{6.25}$$

**Proof:** That (6.25) is true for $n=1$ is obvious. We assume that it is true for $n=k$, *ie*

$$\mathbf{Q}_k^\alpha \mathbf{Q}_k = \mathbf{I}_k. \tag{6.26}$$

Then

$$\begin{aligned} \mathbf{Q}_{k+1}^\alpha \mathbf{Q}_{k+1} &= \begin{bmatrix} \mathbf{Q}_k^\alpha & \mathbf{L}_k^\alpha \\ \mathbf{O}_k^T & -1 \end{bmatrix} \begin{bmatrix} \mathbf{Q}_k & -\mathbf{L}_k \\ \mathbf{O}_k^T & -1 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{Q}_k^\alpha \mathbf{Q}_k, & -\mathbf{Q}_k^\alpha \mathbf{L}_k - \mathbf{L}_k^\alpha \\ \mathbf{O}_k^T & 1 \end{bmatrix} \\ &\overset{(6.26)}{=} \mathbf{I}_{k+1} \end{aligned}$$

provided $\mathbf{Q}_k^\alpha \mathbf{L}_k + \mathbf{L}_k^\alpha = 0$. We consider the $i$th element of this term.

$$\begin{aligned} (\mathbf{Q}_k^\alpha \mathbf{L}_k + \mathbf{L}_k^\alpha)_i &= \sum_{j=1}^{k} (\mathbf{Q}_k^\alpha)_{ij} (\mathbf{L}_k)_j + (\mathbf{L}_k^\alpha)_i \\ &= -(\mathbf{L}_k)_i + \sum_{j=i+1}^{k} (\mathbf{Q}_k^\alpha)_{ij} (\mathbf{L}_k)_j + (\mathbf{L}_k^\alpha)_i \\ &= -\binom{k+1}{i}\tilde{R}_{k-i} + \sum_{j=i+1}^{k} \binom{j}{i}\tilde{R}_{j-i-1}^\alpha \binom{k+1}{j}\tilde{R}_{k-j} + \binom{k+1}{i}\tilde{R}_{k-i}^\alpha \\ &= -\binom{k+1}{i}\left\{\tilde{R}_{k-i} - \tilde{R}_{k-i}^\alpha - \sum_{j=i+1}^{k}\binom{k+1-i}{j-i}\tilde{R}_{j-i-1}^\alpha \tilde{R}_{k-j}\right\} \\ &\overset{(6.18)}{=} 0. \end{aligned}$$

**Lemma 4:**

$$\mathbf{Q}_n \mathbf{L}_n^\alpha = -\mathbf{L}_n. \tag{6.27}$$

**Proof:** Consider an element of the product $\mathbf{Q}_n\mathbf{L}_n^\alpha$. This is

$$\begin{aligned}
(\mathbf{Q}_n)_{ij}(\mathbf{L}_n^\alpha)_j &= -\binom{n+1}{i}\tilde{R}_{n-i}^\alpha + \sum_{j=i+1}^{n}\left[-\binom{j}{i}\tilde{R}_{j-i-1}\right] \\
&\quad \times \binom{n+1}{j}\tilde{R}_{n-j}^\alpha \\
&= -\binom{n+1}{i}\left\{\tilde{R}_{n-i}^\alpha + \sum_{j=i+1}^{n}\binom{j}{i}\binom{n+1}{j}\Big/\binom{n+1}{i}\right. \\
&\quad \left.\times\, \tilde{R}_{n-j}^\alpha\tilde{R}_{j-i-1}\right\} \\
&\overset{(6.18)}{=} -\binom{n+1}{i}\tilde{R}_{n-i} \\
&= -(\mathbf{L}_n)_i.
\end{aligned}$$

Therefore (6.22) is written as

$$\begin{aligned}
\mathbf{F}_n &= (\mathbf{Q}_n^\alpha)^{-1}\,\mathbf{L}_n^\alpha \\
&\overset{(6.25)}{=} \mathbf{Q}_n\mathbf{L}_n^\alpha \\
&\overset{(6.27)}{=} -\mathbf{L}_n. \qquad (6.28)
\end{aligned}$$

Equation (6.20) is equally written as

$$\tilde{R}_n^\alpha = \sum_{i=1}^{n}\tilde{R}_{i-1}^\alpha\frac{\partial f}{\partial y^{(i-1)}},$$

which, when (6.28) and (6.18) are used, results to $(R_n)$ and the proposition is proven.

# References


[1] K. Andriopoulos, P. G. L. Leach and G. P. Flessas, Complete symmetry groups of ordinary differential equations and their integrals: some basic considerations *J. Math. Anal. Applic.* **262** (2001), 256–273.

[2] K. Andriopoulos and P. G. L. Leach, The economy of complete symmetry groups for linear higher-dimensional systems *J. Nonlin. Math. Phys.* **9, s-2** (2002), 10–23.

[3] K. Andriopoulos and P. G. L. Leach, An interpretation of the presence of both positive and negative nongeneric resonances in the singularity analysis *Phys. Lett. A* **359** (2006), 199–203.

[4] K. Andriopoulos and P. G. L. Leach, Autonomous self-similar ordinary differential equations and the Painlevé connection *J. Math. Anal. Applic.* **328** (2007), 625–639.

[5] K. Andriopoulos, Complete symmetry groups and Lie remarkability, *Symmetry and Perturbation Theory – SPT 2007*, G. Gaeta, R. Vitolo and S. Walcher eds. World Scientific, Singapore (2007), 237-238.

[6] K. Andriopoulos, Generalised symmetries and the connection to Differential Sequences *Internat. J. Pure Appl. Math.* **42** (2008), 383–390.

[7] K. Andriopoulos, S. Dimas, P. G. L. Leach and D. Tsoubelis, On the systematic approach to the classification of differential equations by group theoretical methods, *J. Comput. and Appl. Math.* **230** (2009), 224-232.

[8] K. Andriopoulos and P. G. L. Leach, Singularity Analysis for Ordinary Differential Equations, *Appl. Anal. Discrete Math.* (2011), DOI:10.2298/AADM110715016A

[9] V. K. Chandrasekar, M. Senthilvelan and M. Lakshmanan, On the general solution for the modified Emden type equation $\ddot{x}+\alpha x\dot{x}+\beta x^3 = 0$ *J. Phys. A: Math. Theoret.* **40** (2007), 4717–4727.

[10] R. Conte, Singularities of differential equations and integrability in *Introduction to Methods of Complex Analysis and Geometry for Classical Mechanics and Nonlinear Waves* Benest D and Frœschlé C edd (Éditions Frontières, Gif-sur-Yvette, 1994) 49–143.

[11] S. Dimas and D. Tsoubelis, SYM: A new symmetry-finding package for Mathematica *Group Analysis of Differential Equations* Ibragimov NH, Sophocleous C & Damianou PA edd (University of Cyprus, Nicosia, 2005) 64–70.

[12] S. Dimas and D. Tsoubelis, A new Mathematica-based program for solving overdetermined systems of PDEs *8th International Mathematica Symposium* (Avignon, France, 2006).

[13] S. Dimas, K. Andriopoulos, D. Tsoubelis and P. G. L. Leach, Complete specification of some partial differential equations that arise in financial mathematics, *J. Nonlin. Math. Phys.* **16, s-1** (2009), 73-92.

[14] V. Ermakov, Second order differential equations. Conditions of complete integrability *Universita Izvestia Kiev Series III* **9** (1880), 1–25, *Applic. Anal. Disc. Math.* **2** (2008), 123–145, (Translator A. O. Harin, Redactor P. G. L. Leach).

[15] M. Euler, N. Euler and N. Petersson, Linearizable hierarchies of evolution equations in $(1+1)$ dimensions *Stud. Appl. Math.* **111** (2003), 315–337.

[16] M. Euler, N. Euler and P. G. L. Leach, The Riccati and Ermakov-Pinney Hierarchies *J. Nonlin. Math. Phys.* **14** (2006), 290–302.

[17] N. Euler and P. G. L. Leach, Aspects of proper differential sequences of ordinary differential equations *Theoret. Math. Phys.* **159** (2009), 474–487.

[18] M. R. Feix, C. Géronimi, L. Cairó, P. G. L. Leach, R. L. Lemmer and S. É. Bouquet, On the singularity analysis of ordinary differential equations invariant under time translation and rescaling *J. Phys. A: Math. Gen.* **30** (1997), 7437–7461.

[19] E. L. Ince, *Ordinary Differential Equations* (Longmans, Green and Co, London, 1927).

[20] E. Kamke, *Differentialgleichungen Lösungsmethoden und Lösungen* (BG Teubner, Stuttgart, 1983).

[21] J. Krause, On the complete symmetry group of the classical Kepler system *J. Math. Phys.* **35** (1994), 5734–5748.

[22] P. G. L. Leach, A. Karasu (Kalkanlı), M. C. Nucci and K. Andriopoulos, Ermakov's superintegrable toy and nonlocal symmetries *SIGMA:* **1** (2005), Paper 018, 15 pages.

[23] P. G. L. Leach and N. Euler, A Novel Riccati Sequence *J. Nonlin. Math. Phys.* **16 s-1** (2009), 157–164.

[24] F. M. Mahomed and P. G. L. Leach, The linear symmetries of a nonlinear differential equation *Quæst. Math.* **8** (1985), 241–274.

[25] F. M. Mahomed and P. G. L. Leach, Symmetry Lie algebras of $n$th-order ordinary differential equations *J. Math. Anal. Applic.* **151** (1990), 80–107.

[26] P. Painlevé, Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme *Acta math.* **25** (1902), 1–86.

[27] N. Petersson, N. Euler and M. Euler, Recursion operators for a class of integrable third-order evolution equations *Stud. Appl. Math.* **112** (2004), 201–225.

[28] E. Pinney, The nonlinear differential equation $y''(x) + p(x)y + cy^{-3} = 0$ *Proc. Amer. Math. Soc.* **1** (1950), 681.

[29] Jacopo Riccati, *Opere* (Jacopo Guisti, Lucca, 1764) 83-96.

Table 6.1: Singularity analysis for the first four members of the Riccati Sequence.

| Member | Leading-order coefficients | Resonances |
|---|---|---|
| $R_1$ | $\alpha = 1$ | $r = -1$ |
| $R_2$ | $\alpha = 1$<br>$\alpha = 2$ | $r = -1, 1$<br>$r = -1, -2$ |
| $R_3$ | $\alpha = 1$<br>$\alpha = 2$<br>$\alpha = 3$ | $r = -1, 1, 2$<br>$r = -1, 1, -2$<br>$r = -1, -2, -3$ |
| $R_4$ | $\alpha = 1$<br>$\alpha = 2$<br>$\alpha = 3$<br>$\alpha = 4$ | $r = -1, 1, 2, 3$<br>$r = -1, 1, 2, -2$<br>$r = -1, 1, -2, -3$<br>$r = -1, -2, -3, -4$ |

Table 6.2: Singularity analysis for the general member of the Riccati Sequence.

| Member | Leading-order coefficients | Resonances |
|---|---|---|
| $R_n$ | $\alpha = 1$<br>$\alpha = 2$<br>$\vdots$<br>$\alpha = n$ | $r = -1, 1, 2, \ldots, n-1$<br>$r = -1, 1, \ldots, n-2, -2$<br>$\vdots$<br>$r = -1, -2, \ldots, -n$ |