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Certain New Integral Inequalities Involving Erdèlyi-Kober Operators

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Abstract: In the present article, using a family of n positive functions $(n \in \mathbb{N})$, the Erdèlyi-Kober integral operator of fractional type is employed to get generalization of certain classes of integral inequalities. As applications, certain special cases and consequent results of the main inequalities obtained in this paper are also discussed.

Keywords: Integral inequalities, fractional calculus operators, Erdèlyi-Kober operator.

1 Introduction, Motivation and Preliminaries

Fractional calculus, particularly in mathematical analysis, is playing a very important role to perform differentiation and integration with the real number or complex number powers of the differential or integral operators. Because of the significant importance of fractional calculus operators, many research papers have studied and investigated the verity of extensions and applications for these operators. It is fairly well-known that there are a number of different definitions of fractional calculus operators and their applications. Each definition has its own advantages and suitable for applications to different type of scientific or engineering problems. For more details about verity of operators of fractional calculus, reders are refer to see the monographs of Baleanu *et al.* [1], Kiryakova [2], Miller and Ross [3] and Samko *et al.* [4].

In current years, FDEs are one of the most important topics in mathematics and have received consideration due to the options of unfolding nonlinear systems, thus attracting much consideration and growing curiosity due to its prospective physics and engineering applications, see [3,5,6,7]. Integral inequalities are taken up to be important as these are useful in the study of existence and uniqueness of different classes of differential and integral equations [8,9]. Due to this fact, this subject has earned the attention of many researchers and mathematicians during last few decades [10,11]. However, there is a large number of the fractional calculus operators in the literature, but due to their important applications in many fields, the Riemann-Liouville and Hadamard fractional integral operators have been studied extensively [12,13,14,15,16, 17,18,19,20]. Moreover, for the integral inequalities involving generalized fractional operators, one can see the recent papers [21,22,23,24,25,26,27,28,29,30,31,32,33] and references therein.

We obtain a generalization of all the results of [17]. By taking Erdèlyi-Kober fractional integral operators, we investigate certain new classes of integral inequalities for a family of n positive functions, which are defined on the interval $[a,b]$. Interesting new inequalities can also be obtained as particular cases of the main results.

Firstly, we introduced the necessary definition and mathematical notations of fractional calculus operators which are used in our analysis [2].

For real valued continuous function $f(t)$, the fractional integral operators of Erdèlyi-Kober type, that is $I_{\beta}^{\eta,\alpha}$ is defined by

$$I_{\beta}^{\eta,\alpha}\{f(t)\} = \frac{t^{-\beta(\eta+\alpha)}}{\Gamma(\alpha)} \int_0^t \tau^{\beta\eta}\left(t^{\beta}-\tau^{\beta}\right)^{\alpha-1} f(\tau)\, d(\tau^{\beta})$$

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$$= \frac{\beta t^{-\beta(\eta+\alpha)}}{\Gamma(\alpha)} \int_0^t \tau^{\beta(\eta+1)-1} \left(t^\beta - \tau^\beta\right)^{\alpha-1} f(\tau)\, d\tau, \tag{1}$$

where $\alpha > 0$ (order of integration), $\beta > 0$ and $\eta \in \mathbb{R}$. We also give the following properties of the operators, for the convenience of investigating the main results:

$$I_\beta^{\tau,\sigma} I_\beta^{\gamma,\delta} \{f(t)\} = I_\beta^{\gamma,\delta} I_\beta^{\tau,\sigma} \{f(t)\} \quad (\tau \geq 0,\ \sigma \geq 0), \tag{2}$$

and

$$I_\beta^{\gamma+\delta,\sigma} I_\beta^{\gamma,\delta} \{f(t)\} = I_\beta^{\gamma,\sigma+\delta} \{f(t)\} \quad (\sigma \geq 0,\ \delta \geq 0). \tag{3}$$

The aim of this paper is to establish certain new integral inequalities involving fractional integral operators of Erdèlyi-Kober type, for a family of n positive functions, defined on the interval $[a,b]$. Our results generalize, improve and extend the recent results of [17]. Special cases are also presented.

2 Fractional Integral Inequalities

In this segment, we will investigate certain classes of integral inequalities involving the fractional operator (1), for a family of n positive functions. The special cases of these results are also reduces in terms of certain known inequalities in literature. Our main results of this paper are the following theorems:

Theorem 1.*Consider n positive functions $f_1, f_2, \cdots, f_n$, which are continuous and decreasing on the interval $[a,b]$ and $a < t \leq b, \alpha > 0, \delta > 0, \zeta \geq \gamma_p > 0$ where p is fixed integer in $\{1,2,...,n\}$. Then the following inequality holds true*

$$\frac{I_\beta^{\eta,\alpha}\left[\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right]}{I_\beta^{\eta,\alpha}\left[\prod_{i=1}^n f_i^{\gamma_i}(t)\right]} \geq \frac{I_\beta^{\eta,\alpha}\left[(t-a)^\delta \prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right]}{I_\beta^{\eta,\alpha}\left[(t-a)^\delta \prod_{i=1}^n f_i^{\gamma}(t)\right]}. \tag{4}$$

*Proof.*Since $f_1, f_2, \cdots, f_n$ are continuous and decreasing n positive functions on the interval $[a,b]$, hence one can write

$$\left((\rho-a)^\delta - (\tau-a)^\delta\right)\left(f_p^{\zeta-\gamma_p}(\tau) - f_p^{\zeta-\gamma_p}(\rho)\right) \geq 0,$$

which implies that

$$(\rho-a)^\delta f_p^{\zeta-\gamma_p}(\tau) + (\tau-a)^\delta f_p^{\zeta-\gamma_p}(\rho) \geq (\tau-a)^\delta f_p^{\zeta-\gamma_p}(\tau) + (\rho-a)^\delta f_p^{\zeta-\gamma_p}(\rho). \tag{5}$$

Form the conditions, we have $\zeta \geq \gamma_p > 0,\ \delta > 0,\ \tau,\ \rho \in [a,t];\ a < t \leq b$, where $p \in \{1,\ldots,n\}$ is any fixed quantity. Now, let us consider the functional

$$\mathcal{N}_p(t,\tau) = \frac{\beta t^{-\beta(\eta+\alpha)} \tau^{\beta(\eta+1)-1}\left(t^\beta - \tau^\beta\right)^{\alpha-1}}{\Gamma(\alpha)} \prod_{i=1}^n f_i^{\gamma_i}(\tau). \tag{6}$$

We observe that $\alpha, \beta > 0$ before, and hence each factor of the functional (6) is positive in view of the valid conditions mentioned with Theorem 1, which implies that the functional defined above is positive, i.e. $\mathcal{N}_p(t,\tau) \geq 0$ for all $\tau \in (0, t)$ $(t > 0)$.

By multiplying $\mathcal{N}_p(t,\tau)$ (where $\mathcal{N}_p(t,\tau)$ is given by (6)) to both sides of relation (5), making integration with respect to τ between the interval $(0,t)$, and hence using the operator (1), we have

$$\begin{aligned}&(\rho-a)^\delta I_\beta^{\eta,\alpha}\left[\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right] + f_p^{\zeta-\gamma_p}(\rho) I_\beta^{\eta,\alpha}\left[(t-a)^\delta \prod_{i=1}^n f_i^{\gamma_i}(t)\right] \geq \\ &(\rho-a)^\delta f_p^{\zeta-\gamma_p}(\rho) I_\beta^{\eta,\alpha}\left[\prod_{i=1}^n f_i^{\gamma_i}(t)\right] + I_\beta^{\eta,\alpha}\left[(t-a)^\delta \prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right].\end{aligned} \tag{7}$$

Again, on multiplying $\mathcal{N}_p(t,\rho)$ to both the sides of (7) and taking integrating along the variable ρ from $\rho = 0$ to $\rho = t$, and hence on using the operator (1), we arrive at

$$I_\beta^{\eta,\alpha}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\alpha}\left[(t-a)^\delta \prod_{i=1}^{n} f_i^{\gamma_i}(t)\right] \geq$$

$$I_\beta^{\eta,\alpha}\left[(t-a)^\delta \prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\alpha}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right], \tag{8}$$

This completes the proof of inequality (4).

*Remark.*It is remarked that, if the functions $f_1, f_2, \cdots, f_n$ are increasing on $[a,b]$, then the inequality (4) will be reversed.

*Remark.*Taking $\eta = 0, \alpha = 1, \beta = 1, n = 1$ and $t = b$, we get Theorem 3 in [17]. Therefore, this result extend and improve Theorem 3 of [17].

Theorem 2.*Suppose n positive functions $f_1, f_2, \cdots, f_n$ are continuous and decreasing on the interval $[a,b]$ and $a < t \leq b, \alpha > 0, \delta > 0, \zeta \geq \gamma_p > 0, \omega > 0$, Then following inequality holds true*

$$\frac{I_\beta^{\eta,\alpha}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\omega}\left[(t-a)^\delta \prod_{i=1}^{n} f_i^{\gamma_i}(t)\right] + I_\beta^{\eta,\omega}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\alpha}\left[(t-a)^\delta \prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]}{I_\beta^{\eta,\alpha}\left[(t-a)^\delta \prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\omega}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right] + I_\beta^{\eta,\omega}\left[(t-a)^\delta \prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\alpha}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]} \geq 1, \tag{9}$$

where $p \in \{1,2,...,n\}$ is any fixed integer.

*Proof.*Multiplying both sides of (7) by $\frac{\beta t^{-\beta(\eta+\omega)}\rho^{\beta(\eta+1)-1}(t^\beta-\rho^\beta)^{\omega-1}}{\Gamma(\omega)}\prod_{i=1}^{n} f_i^{\gamma_i}(\rho), \omega > 0$, and making integration of the improved inequality with respect to ρ between the interval $(0,t)$ $(a < t \leq b)$, hence on using the Fubini's theorem, we arrive at

$$0 \leq \int_0^t \int_0^t \frac{\beta t^{-\beta(\eta+\omega)}\rho^{\beta(\eta+1)-1}\left(t^\beta-\rho^\beta\right)^{\omega-1}}{\Gamma(\omega)} \prod_{i=1}^{n} f_i^{\gamma_i}(\rho)\mathcal{N}_p(t,\tau)\,d\tau d\rho$$

$$= I_\beta^{\eta,\alpha}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\omega}\left[(t-a)^\delta \prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]$$

$$+ I_\beta^{\eta,\omega}\left[\prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\alpha}\left[(t-a)^\delta \prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]$$

$$- I_\beta^{\eta,\alpha}\left[(t-a)^\delta \prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\omega}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]$$

$$- I_\beta^{\eta,\omega}\left[(t-a)^\delta \prod_{i\neq p}^{n} f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\alpha}\left[\prod_{i=1}^{n} f_i^{\gamma_i}(t)\right]. \tag{10}$$

On further simplification, one can easily arrive at the inequality of Theorem 2.

*Remark.*By letting $\alpha = \omega$ in Theorem 2, we obtain Theorem 1.

*Remark.*Again, by setting $\delta = 0, \alpha = \omega = 1, \beta = 1, n = 1$ and $t = b$ in the Theorem 2, we obtain the well known inequality of Theorem 3 in [17].

To generalize the above theorems, we obtain the another class of integral inequalities involving the fractional operator (1), as under:

Theorem 3. *Suppose $f_1, f_2, \cdots, f_n$ and g be continuous functions, such that $f_1, f_2, \cdots, f_n$ are decreasing and g is increasing on the close interval $[a,b]$, for $a < t \leq b$, $\alpha > 0$, $\beta > 0$, $\zeta \geq \gamma_p > 0$. Then the following inequality holds true*

$$\frac{I_\beta^{\eta,\alpha}\left[\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\alpha}\left[g^\delta(t)\prod_{i=1}^n f_i^{\gamma_i}(t)\right]}{I_\beta^{\eta,\alpha}\left[g^\delta(t)\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\alpha}\left[\prod_{i=1}^n f_i^{\gamma_i}(t)\right]} \geq 1, \tag{11}$$

where $p \in \{1,2,...,n\}$ is any fixed integer.

Proof. Following the valid conditions stated with Theorem 3, we have

$$\left(g^\delta(\rho) - g^\delta(\tau)\right)\left(f_p^{\zeta-\gamma_p}(\tau) - f_p^{\zeta-\gamma_p}(\rho)\right) \geq 0, \tag{12}$$

for all $p = 1, \ldots, n$, $a < t \leq b$, $\alpha > 0$, $\delta > 0$, $\zeta \geq \gamma_p > 0$; $\tau, \rho \in [a\ , b]$.

Now, let's consider the quantity

$$L_p(t,\tau) = \frac{\beta t^{-\beta(\eta+\alpha)}\tau^{\beta(\eta+1)-1}\left(t^\beta - \tau^\beta\right)^{\alpha-1}}{\Gamma(\alpha)}$$

$$\prod_{i=1}^n f_i^{\gamma_i}(\tau)\left(\left(g^\delta(\rho) - g^\delta(\tau)\right)\right)\left(f_p^{\zeta-\gamma_p}(\tau) - f_p^{\zeta-\gamma_p}(\rho)\right). \tag{13}$$

It is clear that

$$L_p(t,\tau) \geq 0, \tag{14}$$

therefore

$$\begin{aligned} 0 \leq \int_0^t L_p(t,\tau)\,d\tau =& g^\delta(\rho) I_\beta^{\eta,\alpha}\left[\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right] + f_p^{\zeta-\gamma_p}(\rho) I_\beta^{\eta,\alpha}\left[g^\delta(t)\prod_{i=1}^n f_i^{\gamma_i}(t)\right] \\ & -I_\beta^{\eta,\alpha}\left[g^\delta(t)\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right] - g^\delta(\rho) f_p^{\zeta-\gamma_p}(\rho) I_\beta^{\eta,\alpha}\left[\prod_{i=1}^n f_i^{\gamma_i}(t)\right]. \end{aligned} \tag{15}$$

Consequently

$$\begin{aligned} & I_\beta^{\eta,\alpha}\left[\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\alpha}\left[g^\delta(t)\prod_{i=1}^n f_i^{\gamma_i}(t)\right] \geq \\ & I_\beta^{\eta,\alpha}\left[g^\delta(t)\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\alpha}\left[\prod_{i=1}^n f_i^{\gamma_i}(t)\right], \end{aligned} \tag{16}$$

which arrives at the result of Theorem 3.

Remark. On putting $\eta = 0, \alpha\ = 1,\ \beta = 1, n = 1$ and $t = b$ in the Theorem 3, we easily obtain the known result of [17].

Now we provide another class of inequalities as follows:

Theorem 4. *Consider $f_1, f_2, \cdots, f_n$ and g be continuous functions, such that $f_1, f_2, \cdots, f_n$ are decreasing and g is increasing on the close interval $[a,b]$. Then the inequality*

$$\frac{I_\beta^{\eta,\alpha}\left[\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\omega}\left[g^\delta(t)\prod_{i=1}^n f_i^{\gamma_i}(t)\right] + I_\beta^{\eta,\omega}\left[\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\alpha}\left[g^\delta(t)\prod_{i=1}^n f_i^{\gamma_i}(t)\right]}{I_\beta^{\eta,\alpha}\left[g(t)\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\omega}\left[\prod_{i=1}^n f_i^{\gamma_i}(t)\right] + I_\beta^{\eta,\omega}\left[g(t)\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\alpha}\left[\prod_{i=1}^n f_i^{\gamma_i}(t)\right]} \geq 1, \tag{17}$$

holds for all $a < t \leq b$, $\omega > 0$, $\alpha > 0$, $\delta > 0$, $\zeta \geq \gamma_p > 0$ and for any fixed $p \in \{1,2,...,n\}$.

*Proof.*Using relation (15), we can write

$$0 \leq \int_0^t \int_0^t \frac{\beta t^{-\beta(\eta+\omega)} \rho^{\beta(\eta+1)-1} \left(t^\beta - \rho^\beta\right)^{\omega-1}}{\Gamma(\omega)} \prod_{i=1}^n f_i^{\gamma_i}(\rho) L_p(t,\tau)\, d\tau d\rho =$$

$$I_\beta^{\eta,\alpha}\left[\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\omega}\left[g^\delta(t)\prod_{i=1}^n f_i^{\gamma_i}(t)\right]$$

$$+I_\beta^{\eta,\omega}\left[\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\alpha}\left[g^\delta(t)\prod_{i=1}^n f_i^{\gamma_i}(t)\right]$$

$$-I_\beta^{\eta,\alpha}\left[g(t)\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\omega}\left[\prod_{i=1}^n f_i^{\gamma_i}(t)\right]$$

$$-I_\beta^{\eta,\omega}\left[g(t)\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\alpha}\left[\prod_{i=1}^n f_i^{\gamma_i}(t)\right]. \tag{18}$$

On some simplification, the above inequality yields to Theorem 4 .

*Remark.*Setting $\alpha = \omega$ for Theorem 4, we get Theorem 3.

*Remark.*Again, on letting $\eta = 0, \alpha = \omega = 1$, $\beta = 1, n = 1$ and $t = b$, Theorem 3 reduces to the Theorem 4 of [17].

Theorem 5.*Suppose the continuous functions $f_1, f_2, \cdots, f_n$ and g be defined on the close interval $[a,b]$. Also assume that for any fixed $p \in \{1,2,...,n\}$, $\left(f_p^\delta(\tau)g^\delta(\rho) - f_p^\delta(\rho)g^\delta(\tau)\right)\left(f_p^{\zeta-\gamma_p}(\tau) - f_p^{\zeta-\gamma_p}(\rho)\right) \geq 0$; $\delta > 0$, $\alpha > 0$, $\zeta \geq \gamma_p > 0$; $\tau, \rho, \in [a,t]$, $t \in (a,b]$ then we have*

$$\frac{I_\beta^{\eta,\alpha}\left[\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta+\delta}(t)\right] I_\beta^{\eta,\alpha}\left[g^\delta(t)\prod_{i=1}^n f_i^{\gamma_i}(t)\right]}{I_\beta^{\eta,\alpha}\left[g^\delta(t)\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\alpha}\left[f_p^\delta \prod_{i=1}^n f_i^{\gamma_i}(t)\right]} \geq 1. \tag{19}$$

*Proof.*It follows from the proof of Theorem 3, if we replace the quantity $\left(g^\delta(\rho) - g^\delta(\tau)\right)$ by $\left(f_p^\delta(\tau)g^\delta(\rho) - f_p^\delta(\rho)g^\delta(\tau)\right)$, one can easily prove the result (19).

Theorem 6.*Suppose $f_1, f_2, \cdots, f_n$ and g be continuous functions on the close interval $[a,b]$. Then for $\left(f_p^\delta(\tau)g^\delta(\rho) - f_p^\delta(\rho)g^\delta(\tau)\right)\left(f_p^{\zeta-\gamma_p}(\tau) - f_p^{\zeta-\gamma_p}(\rho)\right) \geq 0$; $\delta > 0$, $\alpha > 0$, $\zeta \geq \gamma_p > 0$; $\tau, \rho, \in [a,t]$, $t \in (a,b]$, the following inequality holds true:*

$$\frac{I_\beta^{\eta,\alpha}\left[\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta+\delta}(t)\right] I_\beta^{\eta,\omega}\left[g^\delta(t)\prod_{i=1}^n f_i^{\gamma_i}(t)\right] + I_\beta^{\eta,\omega}\left[\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta+\delta}(t)\right] I_\beta^{\eta,\alpha}\left[g^\delta(t)\prod_{i=1}^n f_i^{\gamma_i}(t)\right]}{I_\beta^{\eta,\alpha}\left[g^\delta(t)\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta+\delta}(t)\right] I_\beta^{\eta,\omega}\left[f_p^\delta \prod_{i=1}^n f_i^{\gamma_i}(t)\right] + I_\beta^{\eta,\omega}\left[g^\delta(t)\prod_{i\neq p}^n f_i^{\gamma_i} f_p^{\zeta}(t)\right] I_\beta^{\eta,\alpha}\left[\prod_{i=1}^n f_i^{\gamma_i}(t)\right]} \geq 1, \tag{20}$$

provided $p \in \{1,2,...,n\}$, be any fixed number.

*Proof.*Following the similar procedure of Theorem 4, provided the quantity $\left(g^\delta(\rho) - g^\delta(\tau)\right)$ replaced by $\left(f_p^\delta(\tau)g^\delta(\rho) - f_p^\delta(\rho)g^\delta(\tau)\right)$, we easily prove the Theorem 6.

*Remark.*On substituting $\alpha = \omega$, we observe that the Theorem 6 reduces to Theorem 3.

*Remark.*Again, Theorem 6 for $\eta = 0, \alpha = \omega = 1$, $\beta = 1, n = 1$ and $t = b$, yields to the Theorem 5 of [17].

3 Special Cases

Following Kiryakova [2], the operator $\mathrm{I}_{\beta}^{\eta,\alpha}$ includes a number of generalized integration and differentiation operators as its special cases, used by various researchers. Some important particular cases of the integral operator $\mathrm{I}_{\beta}^{\eta,\alpha}$ are as follows:

1.For $\eta=0,\ \beta=1$, the operator (1) yields to the fractional integral operator of Riemann-Liouville type, by means of the following relationship:

$$R^{\alpha}\{f(t)\}=t^{\alpha}I_{1}^{0,\alpha}\{f(t)\}=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1}f(\tau)\,d\tau. \tag{21}$$

2.If we take $\eta=0$, $\alpha=n\in\mathbb{N}$ and $\beta=1$, then the operator (1) leads the following ordinary n-fold integrations:

$$I^{n}\{f(t)\}=t^{n}I_{1}^{0,n}\{f(t)\}=\frac{1}{(n-1)!}\int_{0}^{t}(t-\tau)^{n-1}f(\tau)\,d\tau. \tag{22}$$

3.Again if $\beta=1$, then the operator (1) leads to the fractional integral operator, which originally considered by Kober [34] and the Erdèlyi [35].

$$I^{\eta,\alpha}\{f(t)\}=I_{1}^{\eta,\alpha}\{f(t)\}=\frac{t^{-\alpha-\eta}}{\Gamma(\alpha)}\int_{0}^{t}\tau^{\eta}(t-\tau)^{\alpha-1}f(\tau)\,d\tau\ \ (\alpha>0,\eta\in\mathbb{R}). \tag{23}$$

4.If $\eta=0$, $\alpha=1$ and $\beta=1$,then the operator (1) reduces to the Hardy-Littlewood (Cesaro) integration operator:

$$L_{1,0}\{f(t)\}=I_{1}^{0,1}\{f(t)\}=\frac{1}{t}\int_{0}^{t}f(\tau)d\tau. \tag{24}$$

5.Further, for $\beta=2$ the operator (1) leads the fractional integral operator of Erdèlyi-Kober type ($I_{\eta,\alpha}$, introduced by Sneddon [36]):

$$I_{\eta,\alpha}=I_{2}^{\eta,\alpha}\{f(t)\}=\frac{2t^{-2(\alpha+\eta)}}{\Gamma\ (\alpha)}\int_{0}^{t}\tau^{2\eta+1}\left(t^{2}-\tau^{2}\right)^{\alpha-1}f(\tau)\,d\tau. \tag{25}$$

Now, by substituting the particular values of the parameters η, α and β, the results presented in this article may generate some more known and possibly new inequalities involving the various types of operators, by taking the relations (21) to (25) into account.

4 Conclusion

Using a family of n positive continous functions, here we have obtained certain new classes of integral inequalities, associated with the Erdèlyi-Kober fractional integral operators. These results provides an important insight about the use of fractional integral operators to generate the well known integral inequalities. Further, in the generalized axially symmetric potential theory and other related physical problems, the operator $I_{\beta}^{\eta,\alpha}$ has number of applications, therefore, the results derived here are expected to find certain applications in this theory and for studying the uniqueness of solutions in FDE's. Additionally, certain new integral inequalities involving the various types of integral operators, can be easily found as special cases of our main results.

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References

[1] D. Baleanu, K. Diethelm, E. Scalas, J. J. Trujillo, *Fractional Calculus Models and Numerical Methods*, World Scientific, Singapore, 2012.
[2] V. Kiryakova, *Generalized Fractional Calculus and Applications* (Pitman Res. Notes Math. Ser.301), Longman Scientific & Technical, Harlow, 1994.
[3] K. S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, A Wiley Interscience Publication. John Wiley & Sons, Inc., New York, 1993.
[4] S. Samko, A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon & Breach. Sci. Publ., New York, 1993.
[5] K. Diethelm, *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Operators of Caputo Type*, Springer, 2004.
[6] S. D. Purohit, Solutions of fractional partial differential equations of quantum mechanics, *Adv. Appl. Math. Mech.* **5**, 639-651(2013).
[7] S. D. Purohit and S. L. Kalla, On fractional partial differential equations related to quantum mechanics, *J. Phys. A Math. Theor.* **44**, (2011).
[8] L. Yang, C. Shen and W. Zhang, Existence of positive solutions for second order differential equations with four point boundary conditions, *Fixed Point Theor.* **12**(2), 467–474(2011).
[9] A. Zerarka, A. Soukeur and H. Saidi, On some Volterra and Fredholm problems via the unified integrodifferential quadrature method, *ISRN Comput. Math.* **2012**, Art. ID 139514, (2012).
[10] D. S. Mitrinovic, J. E. Pecaric and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.
[11] B. G. Pachpatte, *Mathematical Inequalities*, North Holland Mathematical Library 67, 2005.
[12] S. Belarbi and Z. Dahmani, On some new fractional integral inequalities, *J. Inequal. Pure Appl.* Math. **10**(3), Art. 86, (2009).
[13] Z. Dahmani, New classes of integral inequalities of fractional order, *Le Matemat.* **LXIX**– Fasc. I, 237–247 (2014).
[14] Z. Dahmani, L. Tabharit and S. Taf, New generalizations of Gruss inequality using Riemann-Liouville fractional integrals, *Bull. Math. Anal. Appl.* **2**(3), 93–99 (2010).
[15] Z. Dahmani, New classes of integral inequalities of fractional order, *Le Matemat.* **59**-Fasc. I, 237–247 (2014).
[16] M. A. Latif and S. Hussain, New inequalities of Ostrowski type for co-ordinated convex functions via fractional integrals, *J. Fract. Calc. Appl.* **2**, 1–15 (2012).
[17] W. Liu, Q. A. Ngo and V. N. Huy, Several interesting integral inequalities, *J. Math. Inequal.* **3**(2), 201–212 (2009).
[18] S. K. Ntouyas, S. D. Purohit and J. Tariboon, Certain Chebyshev type integral inequalities involving Hadamard's fractional operators, *Abstr. Appl. Anal.* **2014**, Art. ID 249091, (2014).
[19] E. Set, M. Tomar and M. Z. Sarıkaya, On generalized Grüss type inequalities for $k-$fractional integrals, *Appl. Math. Comput.* **269**, 29–34 (2015).
[20] M. Tunc, On new inequalities for h-convex functions via Riemann-Liouville fractional integration, *Filomat* **27**, 559–565 (2013).
[21] M. Aldhaifallah, M. Tomar, K. S. Nisar and S. D. Purohit, Some new inequalities for (k,s)-fractional integrals, *J. Nonlin. Sci. Appl.* **9**, 5374–5381(2016).
[22] D. Baleanu and S. D. Purohit, Chebyshev type integral inequalities involving the fractional hypergeometric operators, *Abstr. Appl. Anal.* **2014**, Art. ID 609160, (2014).
[23] D. Baleanu, S. D. Purohit and P. Agarwal, On fractional integral inequalities involving hypergeometric operators, *Chinese J. Math.* **2014**, Art. ID 609476, (2014).
[24] D. Baleanu, S. D. Purohit and J. C. Prajapati, Integral inequalities involving generalized Erdèlyi-Kober fractional integral operators, *Open Math.* **14**, 89–99 (2016).
[25] D. Baleanu, S. D. Purohit and F. Ucar, On Grüss type integral inequality involving the Saigo's fractional integral operators, *J. Comput. Anal. Appl.* **19**(3), 480–489 (2015).
[26] J. Choi and S. D. Purohit, A Grüss type integral inequality associated with gauss hypergeometric function fractional integral operator, *Commun. Korean Math. Soc.* **30**(2), 81–92 (2015).
[27] S. L. Kalla and A. Rao, On Gruss type inequality for hypergeometric fractional integrals, *Le Matemat.* **66**(1), 57–64 (2011).
[28] Y. Louartassi, E. El Mazoudi and N. Elalami, A new generalization of lemma Gronwall-Bellman, *Appl. Math. Sci.* **6**(13), 621–628 (2012).
[29] S. D. Purohit and S. L. Kalla, Certain inequalities related to the Chebyshev's functional involving Erdèlyi-Kober operators, *Scientia, Ser. A Math. Sci.* **25**, 55–63 (2014).
[30] S. D. Purohit and R. K. Raina, Chebyshev type inequalities for the Saigo fractional integrals and their q-analogues, *J. Math. Inequal.* **7**(2), 239–249 (2013).
[31] S. D. Purohit and R. K. Raina, Certain fractional integral inequalities involving the Gauss hypergeometric function, *Rev. Téc. Ing. Univ. Zulia* **37:2**, 167–175 (2014).
[32] S. D. Purohit, F. Uçar and R. K. Yadav, On fractional integral inequalities and their q-analogues, *Rev. Tecno. URU* **6**, 53–66 (2013).
[33] M. Z. Sarikaya and N. Aktan, On the generalization of some integral inequalities and their applications, *Math. Comput. Model.* **54**, 2175–2182 (2011).
[34] H. Kober, On a fractional integral and derivative, *Quart. J. Math. Oxford* **11**, 193–211 (1940).

[35] A. Erdèlyi, On fractional integration and its applications to the theory of Hankel transforms, *Quart. J. Math. Oxford* **11**, 293–303 (1940).
[36] I. N. Sneddon, *Mixed Boundary Value Problems in Potential Theory*, North-Holand Publ. Co., Amsterdam, 1966.