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Bayesian Estimations using MCMC Approach under Exponentiated Rayleigh Distribution Based on Unified Hybrid Censored Scheme

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Abstract: This paper aims to estimate the unknown parameters, survival and hazard functions for exponentiated Rayleigh distribution based on unified hybrid censored data. The maximum likelihood and Bayes methods are used for estimating the two unknown parameters as well as survival and hazard functions. Lindley's approximation and Markov Chain Monte Carlo (MCMC) method applied to find the Bayes estimation. Approximate confidence intervals for the unknown parameters moreover survival and hazard functions are constructed based on the s-normal approximation to the asymptotic distribution of maximum likelihood estimates (MLEs). The approximate Bayes estimators have been obtained under the assumptions of non-informative priors depending on symmetric and asymmetric loss functions via the Gibbs within Metropolis-Hasting samplers procedure. Finally, the proposed methods can be understood through illustrating the results of the real data analysis.

Keywords: Exponentiated Rayleigh distribution, Unified hybrid censoring scheme, Maximum likelihood estimators, Bayesian estimation, Balanced loss function, Lindley's approximation, MCMC method.

1 Introduction

Epstein [1] considered a hybrid censored scheme (HCS), which is a mixture of Type-I and Type-II censoring schemes. In Type-I HCS, the life-testing experiment is terminated at a random time T^*=min$\{X_{r:n},T\}$, where $1 \le r \le n$ and $T \in (0,\infty)$ are fixed. The disadvantage of Type-I HCS is very few failures occurring until the pre-fixed time T. To overcome this disadvantage, Childs et al. [2] introduced a substitute Type-II HCS that would terminate the experiment at the random time T^*=max$\{X_{r:n},T\}$, where $1 \le r \le n$ and $T \in (0,\infty)$ are fixed. Although the Type-II HCS guarantees a specified number of failures, it has the disadvantage that it might take a very long time to observe r failures and complete the life-test. To avoid the disadvantages in these schemes, Chandrasekar et al. [3] proposed two new schemes which are called generalized Type-I and Type-II HCS. In generalized Type-I HCS, fix $k,r \in (1,2,...,n)$ and $T \in (0,\infty)$ such that $k < r < n$. If the k^{th} failure occurs before time T, the experiment is terminated at min$\{X_{r:n},T\}$. If the k^{th} failure occurs after time T, the experiment is terminated at $X_{k:n}$, so, it is clear that this HCS modifies the Type-I HCS by allowing the experiment to continue after time T if very few failures had observed until that time. In generalized Type-II HCS, fix $r \in (1,2,...,n)$ and $T_1,T_2 \in (0,\infty)$ such that $T_2 > T_1$. If the r^{th} failure occurs before time T_1, the experiment is terminated at T_1. If the r^{th} failure occurs between T_1 and T_2, the experiment is terminated at $X_{r:n}$. If the r^{th} failure occurs after T_2, the experiment is terminated at T_2. There are some drawbacks in generalized HCS, such as, in generalized Type-I HCS. Moreover, because the experiment is terminated at the same time or before T, we cannot guarantee observing r failures, while in the generalized Type-II HCS, we cannot observe any failure at all or observe only few number of failures until the pre-fixed time T_2. To avoid the drowbacks in these schemes, Balakrishnan et al. [4] introduced a mixture of generalized Type-I and Type-II HCS which is called the unified hybrid censoring scheme (UHCS), which can be described as follows, fix $r,k \in \{1,....,n\}$ where $k < r < n$ and $T_1,T_2 \in (0,\infty)$ where $T_2 > T_1$. If the k^{th} failure occurs before time T_1, the experiment is terminated at min$\{$max$\{X_{r:n},T_1\},T_2\}$. If the k^{th} failure occurs between T_1 and T_2, the experiment is terminated at min$\{X_{r:n},T_2\}$ and if the k^{th} failure occurs after time T_2, the experiment is terminated at $X_{k:n}$. Under this censoring scheme, we can guarantee

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that the experiment would be completed at most in time T_2 with at least k failure and if not, we can guarantee exactly k failures. Therefor, under the UHCS, we have the following six cases:
Case I: $0 < x_{k:n} < x_{r:n} < T_1 < T_2$, the experiment is terminated at T_1.
Case II: $0 < x_{k:n} < T_1 < x_{r:n} < T_2$, the experiment is terminated at $x_{r:n}$.
Case III:$0 < x_{k:n} < T_1 < T_2 < x_{r:n}$, the experiment is terminated at T_2.
Case IV: $0 < T_1 < x_{k:n} < x_{r:n} < T_2$, the experiment is terminated at $x_{r:n}$.
Case V: $0 < T_1 < x_{k:n} < T_2 < x_{r:n}$, the experiment is terminated at T_2.
Case VI: $0 < T_1 < T_2 < x_{k:n} < x_{r:n}$, the experiment is terminated at $x_{k:n}$.
The exponentiated Rayleigh (ER) distribution has been used for the lifetime modelling in reliability analysis, life testing problems and acceptance sampling plans. The ER distribution is obtained by generalization of the Rayleigh distribution. It is also called the two parameter (scale and shape) Burr type X distribution. The ER distribution was studied by Sartawi and Abu-Salih [5], Jaheen [6,7], Ahmad et al. [8], Raqab [9] and Surles and Padgett [10].
The cumulative distribution function (CDF) given by

$$F(x;\alpha,\beta) = (1-e^{-\beta x^2})^{\alpha}, \qquad x>0, \ \ (\alpha,\ \beta>0). \tag{1}$$

The probability density function (PDF) is

$$f(x;\alpha,\beta) = 2\alpha\beta \ x \ e^{-\beta x^2}(1-e^{-\beta x^2})^{\alpha-1}, \qquad x>0, \ \ (\alpha,\ \beta>0). \tag{2}$$

The survival function $S(t)$ is given by

$$S(t) = 1-(1-e^{-\beta t^2})^{\alpha}, \qquad t>0. \tag{3}$$

The hazard function $H(t)$ is given by

$$H(t) = \frac{2\alpha\beta \ t \ e^{-\beta t^2}(1-e^{-\beta t^2})^{\alpha-1}}{1-(1-e^{-\beta t^2})^{\alpha}}, \qquad t>0. \tag{4}$$

Here α and β are the shape and scale parameters, respectively. From now on the ER distribution with the shape parameter α and scale parameter β will be denoted by $ER(\alpha,\beta)$.
Recently, Surles and Padgett [11] introduced two parameter Burr Type X distribution and correctly named as the ER distribution. The two parameter ER distribution is a special case of the Weibull distribution originally suggested by Mudholkar and Srivastava [12]. See also, Kundu and Raqab [13], Raqab and Madi [14], Abd-Elfattah [15], Raqab and Madi [16] and Mahmoud and Ghazal [17].
The rest of the paper is organized as follows: In Section 2, we discussed the MLEs of the unknown parameters in addition to $S(t)$ and $H(t)$. In Section 3, approximate confidence interval based the MLEs are presented. In Section 4, we apply Lindley's approximation and MCMC technique to computing the Bayes estimation under three different balanced losses functions and obtain the credible intervals for the parameters of the ER distribution. Real data set has been analyzed for illustrative purposes in Section 5. Finally, conclusions are given in Section 6.

2 Maximum Likelihood Estimation

In this section, we obtained the MLEs of $ER(\alpha,\beta)$ distribution when α and β are unknown. Let $(x_1,\ldots,x_n)$ be a random sample of size n from $ER(\alpha,\beta)$ distribution, then the likelihood function for six cases of the UHCS is as follows:

$$L(\underline{x},\theta) = \frac{n!}{(n-R)!}\left[\prod_{i=1}^{R} f(x_i)\right]\left[1-F(C)\right]^{n-R}, \tag{5}$$

$$(R,C) = \begin{cases} (d,T_1), & \text{for case I,} \\ (r,x_{r:n}), & \text{for case II and case IV,} \\ (d_2,T_2), & \text{for case III and for case V,} \\ (k,x_{k:n}), & \text{for case VI,} \end{cases}$$

where R indicates the number of the total failures in experiment up to time C (the stopping time point) and d_1 and d_2 indicate the number of failures that occur before time points T_1 and T_2, respectively, where $d = d_1 = d_2$ for case I.

From (1), (2) and (5) we get

$$L(\underline{x};\alpha,\beta)=K\alpha^R\beta^R\prod_{i=1}^{R}x_ie^{-\beta\sum_{i=1}^{R}x_i^2}\prod_{i=1}^{R}\left[1-e^{-\beta x_i^2}\right]^{\alpha-1}\left[1-(1-e^{-\beta C^2})^{\alpha}\right]^{n-R}, \tag{6}$$

where $K=\frac{2^Rn!}{(n-R)!}$.
The log-likelihood function for the ER parameters, corresponding to Equation (6) is

$$\begin{aligned}\ell(\underline{x};\alpha,\beta)&=\ln(K)+R\ln(\alpha)+R\ln(\beta)+\sum_{i=1}^{R}\ln(x_i)-\beta\sum_{i=1}^{R}x_i^2\\&+(\alpha-1)\sum_{i=1}^{R}\ln(1-e^{-\beta x_i^2})+(n-R)\ln[1-(1-e^{-\beta C^2})^{\alpha}].\end{aligned} \tag{7}$$

Taking derivatives of Equation (7) with respect to α and β and setting each of these derivatives equal to zero, we obtain the likelihood equations for the parameters α and β as follows

$$\frac{\partial\ell}{\partial\alpha}=\frac{R}{\alpha}+\sum_{i=1}^{R}\ln(1-e^{-\beta x_i^2})-\frac{(n-R)(1-e^{-\beta C^2})^{\alpha}\ln(1-e^{-\beta C^2})}{1-(1-e^{-\beta C^2})^{\alpha}}=0, \tag{8}$$

$$\frac{\partial\ell}{\partial\beta}=\frac{R}{\beta}-\sum_{i=1}^{R}x_i^2+(\alpha-1)\sum_{i=1}^{R}\frac{x_i^2e^{-\beta x_i^2}}{1-e^{-\beta x_i^2}}-\frac{\alpha(n-R)(1-e^{-\beta C^2})^{\alpha-1}C^2e^{-\beta C^2}}{1-(1-e^{-\beta C^2})^{\alpha}}=0. \tag{9}$$

The MLEs of α and β can be found by solving the system of Equations (8) and (9), even if the suggested estimators cannot be expressed in closed forms, we can use a suitable numerical technique to obtain the estimators. Moreover, we can obtain the MLEs of $S(t)$ and $H(t)$ after replacing α and β by their MLEs $\hat{\alpha}$ and $\hat{\beta}$ as following

$$\hat{S}_{ML}(t)=1-(1-e^{-\hat{\beta}t^2})^{\hat{\alpha}}, \tag{10}$$

and

$$\hat{H}_{ML}(t)=\frac{2\hat{\alpha}\hat{\beta}\ t\ e^{-\hat{\beta}}t^2(1-e^{-\hat{\beta}t^2})^{\hat{\alpha}-1}}{1-(1-e^{-\hat{\beta}t^2})^{\hat{\alpha}}}. \tag{11}$$

3 Approximate Confidence Interval

The asymptotic variances-covariances of the MLEs for parameters α and β are given by elements of the inverse of the Fisher information matrix are defined as

$$I_{ij}=-E\left(\frac{\partial^2\ell}{\partial\theta_i\partial\theta_j}\right);\qquad where\ i,j=1,2,\qquad and\ \theta_1=\alpha,\ and\ \theta_2=\beta.$$

Unfortunately, the exact mathematical expressions for the above expectations are very difficult to obtain. Therefore, we give the approximate asymptotic varaince-covariance matrix for the MLE, which is obtained by dropping the expectation operator E

$$I^{-1}(\alpha,\beta)=\begin{pmatrix}-\frac{\partial^2\ell}{\partial\alpha^2} & -\frac{\partial^2\ell}{\partial\alpha\partial\beta}\\ -\frac{\partial^2\ell}{\partial\beta\partial\alpha} & -\frac{\partial^2\ell}{\partial\beta^2}\end{pmatrix}^{-1}_{(\hat{\alpha},\hat{\beta})}=\begin{pmatrix}\widehat{var}(\hat{\alpha}) & cov(\hat{\alpha},\hat{\beta})\\ cov(\hat{\beta},\hat{\alpha}) & \widehat{var}(\hat{\beta})\end{pmatrix}.$$

The asymptotic normality of the MLEs can be used to compute the approximate confidence intervals (ACI) for parameters α and β. Therefore, $(1-\gamma)100\%$ confidence intervals (CIs) for parameters α and β become

$$\left(\hat{\alpha}\pm Z_{\gamma/2}\sqrt{\widehat{var}(\hat{\alpha})}\right)\qquad and\qquad\left(\hat{\beta}\pm Z_{\gamma/2}\sqrt{\widehat{var}(\hat{\beta})}\right), \tag{12}$$

where $Z_{\gamma/2}$ is a standard normal value.
Moreover; to construct the ACI of the $S(t)$ and $H(t)$, which they are functions in the parameters α and β we need to find

the variances of them. In order to find the approximate estimates of the variance of $\hat{S}(t)$ and $\hat{H}(t)$ we use the delta method. The delta method is a general approach for computing confidence intervals for functions of MLEs. According to this method, the variance of $\hat{S}(t)$ and $\hat{H}(t)$ can be approximated, respectively by

$$\hat{\sigma}^2_{S(t)} = \left[\nabla\hat{S}(t)\right]^T\left[\hat{V}\right]\left[\nabla\hat{S}(t)\right] \qquad \text{and} \qquad \hat{\sigma}^2_{H(t)} = \left[\nabla\hat{H}(t)\right]^T\left[\hat{V}\right]\left[\nabla\hat{H}(t)\right],$$

where $\nabla\hat{S}(t)$ and $\nabla\hat{H}(t)$ are the gradient of $\hat{S}(t)$ and $\hat{H}(t)$ respectively, with respect to α and β, and $\hat{V} = I^{-1}(\alpha,\beta)$. Then, $(1-\gamma)100\%$ confidence intervals for $S(t)$ and $H(t)$ become

$$\left(\hat{S}(t) \pm Z_{\gamma/2}\sqrt{\hat{\sigma}^2_{S(t)}}\right) \qquad \text{and} \qquad \left(\hat{H}(t) \pm Z_{\gamma/2}\sqrt{\hat{\sigma}^2_{H(t)}}\right).$$

4 Bayes Estimation

In this section, we obtain Bayesian estimates of the unknown parameters α and β as well as $S(t)$ and $H(t)$ of $ER(\alpha,\beta)$ distribution under balanced squared error (BSE) loss, balanced linear exponential (BLINEX) loss and balanced general entropy (BGE) loss functions based on UHCS. It is assumed here that the parameters α and β are independent and follow the gamma prior distributions.

$$\pi_1(\alpha) \propto \alpha^{a_1-1}e^{-b_1\alpha}, \qquad \alpha > 0, \tag{13}$$

$$\pi_2(\beta) \propto \beta^{a_2-1}e^{-b_2\beta}, \qquad \beta > 0. \tag{14}$$

Here all the hyper parameters a_1, a_2, b_1 and b_2 are assumed to be known and non-negative.
The joint prior distribution for α and β is

$$\pi(\alpha,\beta) \propto \alpha^{a_1-1}\beta^{a_2-1}e^{-(b_1\alpha+b_2\beta)}. \tag{15}$$

From (6) and (15) we obtained the joint posterior density function as follows

$$\begin{aligned}\pi^*(\alpha,\beta|\underline{x}) \propto\ & \alpha^{a_1+R-1}\beta^{a_2+R-1}e^{-\beta(b_2+\sum_{i=1}^R x_i^2)}e^{-\alpha[b_1-\sum_{i=1}^R \ln(1-e^{-\beta x_i^2})]} \\ & e^{-\sum_{i=1}^R \ln(1-e^{-\beta x_i^2})}\left[1-(1-e^{-\beta C^2})^\alpha\right]^{n-R}.\end{aligned} \tag{16}$$

It is evident that is not possible to compute (16) analytically because it very difficult to get explicit forms for the marginal posterior distributions for each parameter. Then, we suggested using MCMC method and Lindley's approximation to approximate these (16) under BSE, BLINEX and BGE functions.

4.1 Estimation based on balanced loss functions

Under balanced loss functions of the form, see Jozani et al. [18]

$$L_{\rho,\omega,\theta_0}(\theta,\hat{\theta}) = \omega\rho(\hat{\theta},\hat{\theta}_0) + (1-\omega)\rho(\theta,\hat{\theta}), \tag{17}$$

where $0 \le \omega \le 1$ and ρ is an arbitrary loss function, while θ_0 is a general "target"estimator of θ, obtained for instance using the criterion of maximum likelihood, least-squares or unbiasedness. $L_{\rho,\omega,\theta_0}(\theta,\hat{\theta})$ can be specialized to various choices of loss function, such as for absolute value, square error, LINEX and general entropy loss functions. By choosing $\rho(\theta,\hat{\theta}) = (\hat{\theta}-\theta)^2$, (17) reduced to the BSE loss function, used by Ahmadi et al. [19] in the form

$$L(\theta,\hat{\theta}) = \omega(\hat{\theta}_0-\hat{\theta})^2 + (1-\omega)(\hat{\theta}-\theta)^2, \tag{18}$$

and the corresponding Bayes estimate of the unknown parameter θ is given by

$$\hat{\theta}(\underline{x}) = \omega\hat{\theta}_0 + (1-\omega)E(\theta|\underline{x}). \tag{19}$$

The BLINEX loss function with shape parameter a is obtained with the choice of $\rho(\theta,\hat{\theta}) = e^{a(\hat{\theta}-\theta)} - a(\hat{\theta}-\theta) - 1$, where $a \neq 0$, see Zellner [20]. Therefore, the Bayes estimation of θ under BLINEX function is given by

$$\hat{\theta}(\underline{x}) = \frac{-1}{a}\ln[\omega e^{-a\hat{\theta}_0} + (1-\omega)E(e^{-a\theta}|\underline{x})]. \tag{20}$$

The BGE loss function with shape parameter a is obtained with the choice of $\rho(\theta,\hat{\theta}) = (\frac{\hat{\theta}}{\theta})^a - a\ln(\frac{\hat{\theta}}{\theta}) - 1$. Then, the Bayes estimation of θ under BGE loss function is given by

$$\hat{\theta}(\underline{x}) = \left[\omega(\hat{\theta}_0)^{-a} + (1-\omega)E(\theta^{-a}|\underline{x})\right]^{\frac{-1}{a}}. \tag{21}$$

It is evident that the balanced loss functions are more general, which contain the MLE and both symmetric and asymmetric Bayes estimates as special cases. For examples, the Bayes estimate under BSE loss function in (19) reduces to MLE when $\omega = 1$, and for $\omega = 0$, the BSE loss function reduces to the Bayes estimate relative to squared error loss function (symmetric). Also, the Bayes estimator under BLINEX loss function reduces to MLE when $\omega = 1$, in (20), and for $\omega = 0$, it reduces to the case of LINEX loss function (asymmetric). Also, the Bayes estimator under BGE loss function reduces to MLE when $\omega = 1$, in (21), and for $\omega = 0$, it reduces to the case of GE loss function.

4.2 Lindley's approximation

Lindley [21] suggested a procedure to approximate the ratio of integrals. This approach has been used by several authors such as AL-Hussaini and Jaheen [22,23], to obtain the approximate Bayes estimators for some lifetime distributions. Lindley [21] developed approximate procedures for the evaluation of the ratio of two integrals in the form

$$\frac{\int_{\Theta}\omega(\Theta)e^{\ell(\Theta)}d\Theta}{\int_{\Theta}\nu(\Theta)e^{\ell(\Theta)}d\Theta}, \tag{22}$$

where $\Theta = (\theta_1,....,\theta_m)$, $\ell(\Theta)$ is the log-likelihood function, $\omega(\Theta)$ and $\nu(\Theta)$ are arbitrary functions of Θ. Suppose that $\nu(\Theta)$ is the prior distribution for Θ and $\omega(\Theta) = u(\Theta)\nu(\Theta)$ where $u(\Theta)$ is arbitrary function. In our study $\theta = \left[\alpha,\beta,H(t),S(t)\right]$ then the above ratio yields the posterior expectation of $u(\alpha,\beta)$ is given as

$$\hat{u}_B(\alpha,\beta) = E[u(\alpha,\beta)|\underline{x}] = \frac{\int_{\alpha}\int_{\beta}u(\alpha,\beta)e^{\ell(\underline{x}|\alpha,\beta)+\rho(\alpha,\beta)}d\alpha d\beta}{\int_{\alpha}\int_{\beta}e^{\ell(\underline{x}|\alpha,\beta)+\rho(\alpha,\beta)}d\alpha d\beta}. \tag{23}$$

Expanding $\rho(\alpha,\beta) = \ln[\pi(\alpha,\beta)]$ and $\ell(\alpha,\beta)$ in (23) into a Taylor series expansion about the MLE of (α,β), Lindley's procedure can be approximated asymptotically by

$$\hat{u}_B(\alpha,\beta) = E[u(\alpha,\beta)|\underline{x}] = u(\hat{\alpha},\hat{\beta}) + \frac{1}{2}\sum_{i,j}^{m}\left[u_{ij}(\alpha,\beta) + 2u_i(\alpha,\beta)\rho_j(\alpha,\beta)\right]\sigma_{ij} + \frac{1}{2}\sum_{i,j,s,k}^{m}\hat{\ell}_{ijs}u_k(\alpha,\beta)\sigma_{ij}\sigma_{sk}, \tag{24}$$

where, i,j,s,k=1, 2, ..., m, then

$$\begin{aligned}\hat{u}_B(\alpha,\beta) =& u(\hat{\alpha},\hat{\beta}) + \frac{1}{2}\Big[(\hat{u}_{\alpha\alpha} + 2\hat{u}_{\alpha}\hat{\rho}_{\alpha})\hat{\sigma}_{\alpha\alpha} + (\hat{u}_{\beta\alpha} + 2\hat{u}_{\beta}\hat{\rho}_{\alpha})\hat{\sigma}_{\beta\alpha} + (\hat{u}_{\alpha\beta} + 2\hat{u}_{\alpha}\hat{\rho}_{\beta})\hat{\sigma}_{\alpha\beta} + (\hat{u}_{\beta\beta} \\ &+ 2\hat{u}_{\beta}\hat{\rho}_{\beta})\hat{\sigma}_{\beta\beta}\Big] + \frac{1}{2}\Big[(\hat{u}_{\alpha}\hat{\sigma}_{\alpha\alpha} + \hat{u}_{\beta}\hat{\sigma}_{\alpha\beta})(\hat{\ell}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha} + \hat{\ell}_{\alpha\beta\alpha}\hat{\sigma}_{\alpha\beta} + \hat{\ell}_{\beta\alpha\alpha}\hat{\sigma}_{\beta\alpha} + \hat{\ell}_{\beta\beta\alpha}\hat{\sigma}_{\beta\beta}) \\ &+ (\hat{u}_{\alpha}\hat{\sigma}_{\beta\alpha} + \hat{u}_{\beta}\hat{\sigma}_{\beta\beta})(\hat{\ell}_{\beta\alpha\alpha}\hat{\sigma}_{\alpha\alpha} + \hat{\ell}_{\alpha\beta\beta}\hat{\sigma}_{\alpha\beta} + \hat{\ell}_{\beta\alpha\beta}\hat{\sigma}_{\beta\alpha} + \hat{\ell}_{\beta\beta\beta}\hat{\sigma}_{\beta\beta})\Big].\end{aligned} \tag{25}$$

where

$$\hat{\ell}_{ht} = \frac{\partial^{h+t}\ell}{\partial\theta_1^h\partial\theta_2^t}, \quad \rho = \ln\pi(\theta_1,\theta_2), \quad \rho_i = \frac{\partial\rho}{\partial\theta_i}, \quad u_{\theta_i\theta_j} = \frac{\partial^2 u}{\partial\theta_i\partial\theta_j}, \quad u_{\theta_i} = \frac{\partial u}{\partial\theta_i}.$$

For $h,\ t = 0,1,2,3,\ h+t = 3,\ i,j = 1,2,\ \theta_1 = \alpha$ and $\theta_2 = \beta$, where $\ell(.,.)$ is the log-likelihood function of the observed data, $\pi(\theta_1,\theta_2) = \pi(\alpha,\beta)$ is the joint prior density function of (α,β), and σ_{ij} is the (i, j)th element of the inverse of the Fisher information matrix. Moreover, $\hat{\alpha}$ and $\hat{\beta}$ are the MLEs of α and β, respectively. Now, we can deduce the values of the Bayes estimates of various parameters under three cases when $\hat{\theta} = \left[\hat{\alpha},\hat{\beta},\hat{S}(t),\hat{H}(t)\right]$.
(1) Case of the BSE loss function

If $u(\hat{\alpha},\hat{\beta})=\hat{\theta}$, then,

$$\begin{aligned}\hat{\theta}_{LBS}=&\omega\hat{\theta}_{MLE}+(1-\omega)\Bigg[\hat{\theta}_{MLE}+\frac{1}{2}\Big[(\hat{u}_{\alpha\alpha}+2\hat{u}_{\alpha}\hat{\rho}_{\alpha})\hat{\sigma}_{\alpha\alpha}+(\hat{u}_{\beta\alpha}+2\hat{u}_{\beta}\hat{\rho}_{\alpha})\hat{\sigma}_{\beta\alpha}+(\hat{u}_{\alpha\beta}+2\hat{u}_{\alpha}\hat{\rho}_{\beta})\hat{\sigma}_{\alpha\beta}\\&+(\hat{u}_{\beta\beta}+2\hat{u}_{\beta}\hat{\rho}_{\beta})\hat{\sigma}_{\beta\beta}\Big]+\frac{1}{2}\Big[(\hat{u}_{\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{u}_{\beta}\hat{\sigma}_{\alpha\beta})(\hat{\ell}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{\ell}_{\alpha\beta\alpha}\hat{\sigma}_{\alpha\beta}+\hat{\ell}_{\beta\alpha\alpha}\hat{\sigma}_{\beta\alpha}+\hat{\ell}_{\beta\beta\alpha}\hat{\sigma}_{\beta\beta})\\&+(\hat{u}_{\alpha}\hat{\sigma}_{\beta\alpha}+\hat{u}_{\beta}\hat{\sigma}_{\beta\beta})(\hat{\ell}_{\beta\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{\ell}_{\alpha\beta\beta}\hat{\sigma}_{\alpha\beta}+\hat{\ell}_{\beta\alpha\beta}\hat{\sigma}_{\beta\alpha}+\hat{\ell}_{\beta\beta\beta}\hat{\sigma}_{\beta\beta})\Bigg].\end{aligned}\tag{26}$$

(2) Case of the BLINEX loss function
If $u(\hat{\alpha},\hat{\beta})=e^{-a\hat{\theta}}$, then,

$$\begin{aligned}\hat{\theta}_{LBL}=&\frac{-1}{a}\ln\Bigg[\omega\ e^{-a\hat{\theta}_{MLE}}+(1-\omega)\Big[e^{-a\hat{\theta}_{MLE}}+\frac{1}{2}\Big[(\hat{u}_{\alpha\alpha}+2\hat{u}_{\alpha}\hat{\rho}_{\alpha})\hat{\sigma}_{\alpha\alpha}+(\hat{u}_{\beta\alpha}+2\hat{u}_{\beta}\hat{\rho}_{\alpha})\hat{\sigma}_{\beta\alpha}\\&+(\hat{u}_{\alpha\beta}+2\hat{u}_{\alpha}\hat{\rho}_{\beta})\hat{\sigma}_{\alpha\beta}+(\hat{u}_{\beta\beta}+2\hat{u}_{\beta}\hat{\rho}_{\beta})\hat{\sigma}_{\beta\beta}\Big]+\frac{1}{2}\Big[(\hat{u}_{\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{u}_{\beta}\hat{\sigma}_{\alpha\beta})(\hat{\ell}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{\ell}_{\alpha\beta\alpha}\hat{\sigma}_{\alpha\beta}\\&+\hat{\ell}_{\beta\alpha\alpha}\hat{\sigma}_{\beta\alpha}+\hat{\ell}_{\beta\beta\alpha}\hat{\sigma}_{\beta\beta})+(\hat{u}_{\alpha}\hat{\sigma}_{\beta\alpha}+\hat{u}_{\beta}\hat{\sigma}_{\beta\beta})(\hat{\ell}_{\beta\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{\ell}_{\alpha\beta\beta}\hat{\sigma}_{\alpha\beta}+\hat{\ell}_{\beta\alpha\beta}\hat{\sigma}_{\beta\alpha}+\hat{\ell}_{\beta\beta\beta}\hat{\sigma}_{\beta\beta})\Big]\Big]\Bigg].\end{aligned}\tag{27}$$

(3) Case of the BGE loss function
If $u(\hat{\alpha},\hat{\beta})=\left[\hat{\theta}\right]^{-a}$, then,

$$\begin{aligned}\hat{\theta}_{LBG}=&\Bigg[\omega\ [\hat{\theta}_{MLE}]^{-a}+(1-\omega)\Big[[\hat{\theta}_{MLE}]^{-a}+\frac{1}{2}\Big[(\hat{u}_{\alpha\alpha}+2\hat{u}_{\alpha}\hat{\rho}_{\alpha})\hat{\sigma}_{\alpha\alpha}+(\hat{u}_{\beta\alpha}+2\hat{u}_{\beta}\hat{\rho}_{\alpha})\hat{\sigma}_{\beta\alpha}\\&+(\hat{u}_{\alpha\beta}+2\hat{u}_{\alpha}\hat{\rho}_{\beta})\hat{\sigma}_{\alpha\beta}+(\hat{u}_{\beta\beta}+2\hat{u}_{\beta}\hat{\rho}_{\beta})\hat{\sigma}_{\beta\beta}\Big]+\frac{1}{2}\Big[(\hat{u}_{\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{u}_{\beta}\hat{\sigma}_{\alpha\beta})(\hat{\ell}_{\alpha\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{\ell}_{\alpha\beta\alpha}\hat{\sigma}_{\alpha\beta}\\&+\hat{\ell}_{\beta\alpha\alpha}\hat{\sigma}_{\beta\alpha}+\hat{\ell}_{\beta\beta\alpha}\hat{\sigma}_{\beta\beta})+(\hat{u}_{\alpha}\hat{\sigma}_{\beta\alpha}+\hat{u}_{\beta}\hat{\sigma}_{\beta\beta})(\hat{\ell}_{\beta\alpha\alpha}\hat{\sigma}_{\alpha\alpha}+\hat{\ell}_{\alpha\beta\beta}\hat{\sigma}_{\alpha\beta}+\hat{\ell}_{\beta\alpha\beta}\hat{\sigma}_{\beta\alpha}+\hat{\ell}_{\beta\beta\beta}\hat{\sigma}_{\beta\beta})\Big]\Big]\Bigg]^{\frac{-1}{a}}.\end{aligned}\tag{28}$$

4.3 MCMC Method

We suggested using MCMC to generate samples from (16) and then compute the Bayes estimates of the parameter α and β as well as $S(t)$ and $H(t)$ under UHCS and also to construct the corresponding credible intervals based on the generated posterior sample. For more details about the MCMC methods, see, for example, Chen and Shao [24]. We supposed the Metropolis-within-Gibbs samplers. From (16) we get the posterior density function of α given β is

$$\pi_1^*(\alpha|\beta,\underline{x})\propto\alpha^{a_1+R-1}e^{-\alpha\left[b_1-\sum_{i=1}^{R}\ln(1-e^{-\beta x_i^2})\right]}.\tag{29}$$

Also the posterior density function of β given α can be written as

$$\pi_2^*(\beta|\alpha,\underline{x})=\beta^{a_2+R-1}e^{-\beta(b_2+\sum_{i=1}^{R}x_i^2)}e^{(\alpha-1)\sum_{i=1}^{R}\ln(1-e^{-\beta x_i^2})}\left[1-(1-e^{-\beta C^2})^{\alpha}\right]^{n-R}.\tag{30}$$

Therefore, the posterior density function of α given β is gamma with the shape parameter (a_1+R) and scale parameter $\left(b_1-\sum_{i=1}^{R}\ln(1-e^{-\beta x_i^2})\right)$ and, therefore, samples of α can be easily generated using any gamma generating routine. Furthermore, the conditional posterior distribution of β given α in (30) cannot be reduced analytically to well known distributions and therefore it is impossible to sample directly by standard methods. So, to generate random numbers from this distribution, we use the Metropolis-Hastings method with normal proposal distribution, see Metropolis et al. [25]). Now, we suppose the next MCMC algorithm to draw samples from the posterior density (16) and in turn to obtain the Bayes estimates α and β ?and any function of them such as $S(t)$ and $H(t)$ and the corresponding credible intervals.
Algorithm of MCMC method:

Step 1. Take some initial guess of α and β, say $\alpha^{(0)}$ and $\beta^{(0)}$ respectively, M = burn-in.
Step 2. Set $j = 1$.
Step 3. Generate $\alpha^{(j)}$ from Gamma$(a_1 + R, b_1 - \sum_{i=1}^{R} \ln(1 - e^{-\beta^{(j-1)} x_i^2}))$.
Step 4. Using Metropolis-Hastings, generate $\beta^{(j)}$ from $\pi_2^*(\beta|\alpha, \underline{x})$ with the $N(\beta^{(j-1)}, \sigma^2)$ proposal distribution where σ^2 is the variance of β obtained using variance-covariance matrix.
(i) Calculate the acceptance probability

$$r = min\left[1, \frac{\pi_2^*(\beta^*|\alpha^j, \underline{x})}{\pi_2^*(\beta^{j-1}|\alpha^j, \underline{x})}\right]. \tag{31}$$

(ii) Generate u from a Uniform $(0,1)$ distribution.
(iii) If $u \le r$, accept the proposal and set $\beta^i = \beta^*$, else set $\beta^i = \beta^{i-1}$.
Step 5. Compute $S(t)$ and $H(t)$ as

$$S(t)^{(j)} = 1 - (1 - e^{-\beta^{(j)} t^2})^{\alpha^{(j)}}, \tag{32}$$

$$H^{(j)}(t) = \frac{2\alpha^{(j)} \beta^{(j)} t e^{-\beta^{(j)} t^2} (1 - e^{-\beta^{(j)} t^2})^{\alpha^{(j)} - 1}}{1 - (1 - e^{-\beta^{(j)} t^2})^{\alpha^{(j)}}}. \tag{33}$$

Step 6. Set $j = j + 1$.
Step 7. Repeat steps $3 - 6$ N times and obtain $\alpha^{(j)}$, $\beta^{(j)}$, $S^{(j)}(t)$ and $H^{(j)}(t)$, $j = M + 1, ..., N$.
Step 8. To compute the credible intervals of $\alpha, \beta, S(t)$ and $H(t)$, order $\alpha^{(j)}, \beta^{(j)}, S^{(j)}(t)$ and $H^{(j)}(t)$, $j = M + 1, ..., N$, as $\left(\alpha^{(1)} < \ldots, \alpha^{(N-M)}\right)$, $\left(\beta^{(1)} < \ldots, < \beta^{(N-M)}\right)$, $\left(S^{(1)}(t) < \ldots, S^{(N-M)}(t)\right)$ and $\left(H^{(1)}(t) < \ldots, H^{(N-M)}(t)\right)$. Then the $100(1-\gamma)\%$ credible intervals of α, β, $S(t)$ and $H(t)$ is $\left(\theta_{(N-M)_{\gamma/2}}, \theta_{(N-M)_{(1-\gamma/2)}}\right)$.

Then, the Bayes estimates of $\theta = [\alpha,\ \beta,\ S(t),\ H(t)]$ where $\hat{\theta}_{MLE} = [\hat{\alpha}_{MLE},\ \hat{\beta}_{MLE},\ \hat{S}(t)_{MLE},\ \hat{H}(t)_{MLE}]$ under BSE loss function, from (19) are given by

$$\hat{\theta}_{MBS} = \omega \hat{\theta}_{MLE} + \frac{(1-\omega)}{N-M} \sum_{i=M+1}^{N} \theta^{(i)}. \tag{34}$$

The Bayes estimates of $\theta = [\alpha,\ \beta,\ S(t),\ H(t)]$ where $\hat{\theta}_{MLE} = [\hat{\alpha}_{MLE},\ \hat{\beta}_{MLE},\ \hat{S}(t)_{MLE},\ \hat{H}(t)_{MLE}]$ under BLINEX loss functio, from (20) are given by

$$\hat{\theta}_{MBL} = -\frac{1}{a} \ln\left[\omega e^{-a\hat{\theta}_{MLE}} + \frac{(1-\omega)}{N-M} \sum_{i=M+1}^{N} e^{-a\theta^{(i)}}\right]. \tag{35}$$

The Bayes estimates of $\theta = [\alpha,\ \beta,\ S(t),\ H(t)]$ where $\hat{\theta}_{MLE} = [\hat{\alpha}_{MLE},\ \hat{\beta}_{MLE},\ \hat{S}(t)_{MLE},\ \hat{H}(t)_{MLE}]$ under BGE loss function, from (21) are given by

$$\hat{\theta}_{MBG} = \left[\omega(\hat{\theta}_{MLE})^{-a} + \frac{(1-\omega)}{N-M} \sum_{i=M+1}^{N} \left[\theta^{(i)}\right]^{-a}\right]^{-\frac{1}{a}}. \tag{36}$$

5 Real Life Data

Wind power is renewable and environmentally friendly. It is an alternative clean energy source compared with the fossil fuels that pollute the lower layer of the atmosphere. Wind power is a form of solar energy, driven by the unequal heating of the earth's surfase. The most important parameter of the wind power is wind speed. Statistical methods are useful for estimating wind speed because it is a random phenomenon. For this reason, wind speed probabilities can be modelled by using probability distributions. We have taken the daily average wind speeds from 1 / 6 /2015 to 8 / 9 / 2015 for cairo city as follows

5.3	4.4	5.9	4.6	4.5	4	8.7	7.8	6.4	6.3
5	6.2	6	6.5	5.1	5.9	4.5	1.8	6.2	4.6
5.1	5.8	6.3	5.7	5.9	5	9.5	6	5.6	5.4
6.9	7	6.1	3.7	5.7	5.4	4.4	3.7	4.2	5.8
3.9	4.8	4.9	3.6	3.1	3.2	3.2	3.3	3.8	2.4
3.2	4.9	4.6	7.2	6.4	5.7	4.9	5.5	4.1	3.1
3.6	6	6	5.6	3	4.5	3.8	7.4	6.8	6.5
6.5	6.3	6.2	5.6	6.2	8.1	5.6	7.6	7.2	6.9
6.8	8	7.3	5.8	5.9	6.5	4	5.7	5.9	5.3
3.9	4.5	3.5	4.6	6	4.5	5.7	4.3	5.7	6.3.

This data was produced by the national climatic data center (NCDC) in Asheville in the United States of America. Now, one of the most important subjects is type of distribution of any set of data will be known during statistical tests which are called the goodness of fit. We depended on Kolmogorov-Smirnov (K-S) test to fit whether the data distribution as $ER(\alpha,\beta)$ distribution or not. The calculated value of the K-S test is 0.116167 for the $ER(\alpha,\beta)$ distribution and this value is smaller than their corresponding values expected at 5% significance level, which is 0.13404 at $n = 100$. We have just plotted the empirical $S(t)$ and the fitted $S(t)$ in Fig. (1) . Observe that the $ER(\alpha,\beta)$ distribution can be a good model fitting this data. So, it can be seen that the $ER(\alpha,\beta)$ distribution fits the data very well. P-value = 0.124203, therefore, the high p-value indicates that $ER(\alpha,\beta)$ distribution can be used to analyze this data set.

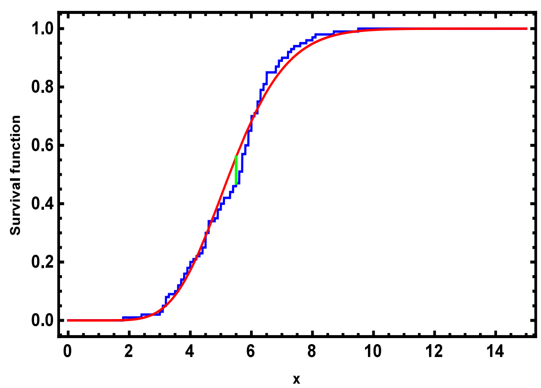

Fig. 1: Empirical and fitted survival functions.

Now, we consider the case when the data are censored. We generate six artificially UHCS data sets from the uncensored data set as:
Case I: $T_1 = 6.45, T_2 = 6.7, k = 70, r = 75$. In this case: $R = 80, C = T_1 = 6.45$.
Case II: $T_1 = 6.45, T_2 = 6.7, k = 70, r = 82$. In this case: $R = 82, C = x_{r:n} = 6.55$.
Case III: $T_1 = 6.45, T_2 = 6.7, k = 70, r = 87$. In this case: $R = 85, C = T_2 = 6.7$.
Case IV: $T_1 = 6.05, T_2 = 7.1, k = 75, r = 88$. In this case: $R = 88, C = x_{r:n} = 6.95$.
Case V: $T_1 = 6.05, T_2 = 7.1, k = 80, r = 91$. In this case: $R = 90, C = T_2 = 7.1$.
Case VI: $T_1 = 6.05, T_2 = 6.25, k = 92, r = 94$. In this case: $R = 92, C = x_{k:n} = 7.25$.
In all the six cases, we estimate the unknown parameters α and β as well as $S(t)$ and $H(t)$ using the MLEs and the Bayesian estimates. The MLEs of α and β as well as $S(t)$ and $H(t)$ based on the complete sample $n = 100$ are obtained to be $\left(\hat{\alpha}_{ML},\ \hat{\beta}_{ML},\ \hat{S}(t=3)_{ML},\ \hat{H}(t=3)_{ML}\right) = (4.3778,\ 0.06938,\ 0.9652,\ 0.0758)$. The computations for different methods of estimations are as follows:
MLEs and ACIs: From (8), (9), (10) and (11), we obtain the MLEs of α, β, $S(t=3)$ and $H(t=3)$, the results are given in Table (3). Also, we computed the 95% CIs the results are given in Table(1).
Bayes estimates: We used MCMC method based on 11000 MCMC sample and discard the first 1000 values as 'burn-in'. We used the non-informative gamma priors for α and β when the hyperparameters are 0, $a_1 = a_2 = b_1 = b_2 = 0$. and we considered the BSE loss, BLINEX loss and BGE loss functions with different values of the shape parametere a of BLINEX loss and BGE loss functions and various values of ω for the parameters α, β, $S(t=2)$ and $H(t=2)$ are given in Table(3). Also, we computed the 95% ACI based on the MCMC samples. The result are given in Table(1). The characteristics of the unknown parameters for MCMC method under non-informative are given in Table(2). Fig. (2) plot the MCMC output of α, β, $S(t)$ and $H(t)$. The histogram of α, β, $S(t)$ and $H(t)$ are displayed in Fig. (3).

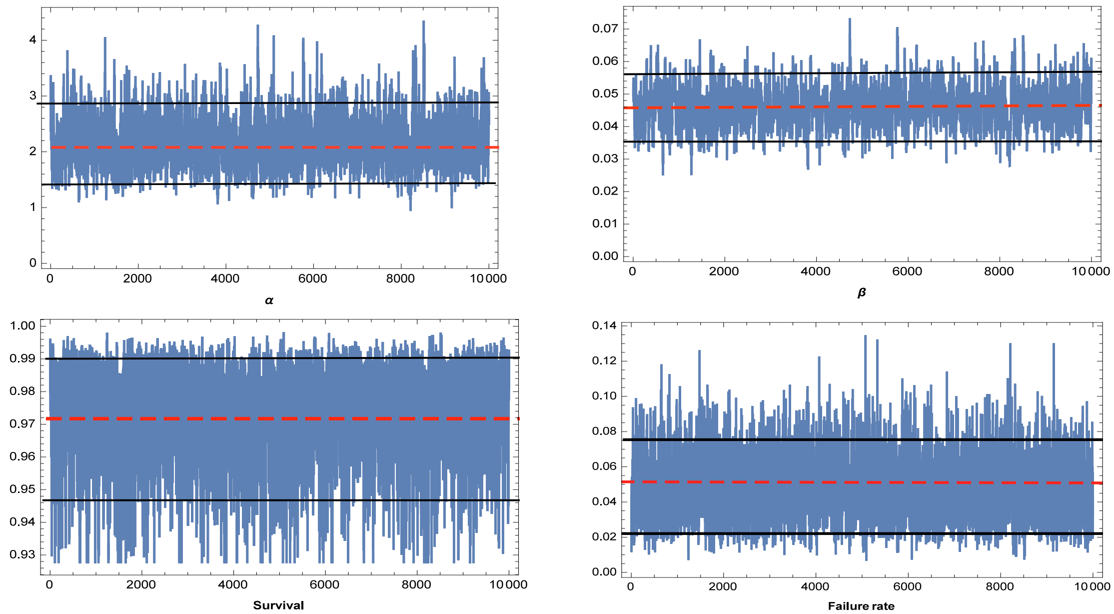

Fig. 2: Simulation number of α, β, $S(t)$ and $H(t)$ generated by MCMC method for case I. Dashed line (- - -) represents the posterior mean and solid lines (—) represent lower and upper bounds of 95% probability intervals.

Also, for Lindley's approximation we considered the BSE loss, BLINEX loss and BGE loss functions with different values of the shape parametere a of BLINEX loss and BGE loss functions and various values of ω for the parameters α, β, $S(t=3)$ and $H(t=3)$ are given in Table(4).
It is well known for the BLINEX loss function when $\omega = 0$ which $a > 0$ indicate that the overestimation is more serious than the underestimation and vice versa for $a < 0$. Also BLINEX loss function becomes symmetric for a close to zero and hence approximately behaves as the BSE loss function itself when $\omega = 0$, and from Table (3) and (4), when $a = 0.2$ we note that the resulting estimates are approximately similar to the corresponding BSE bayes estimates.

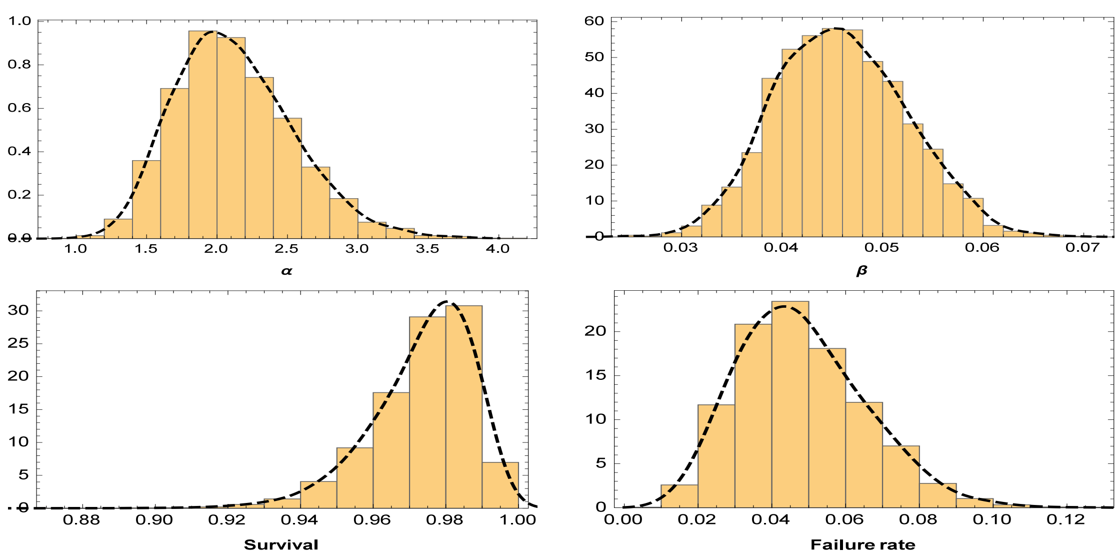

Fig. 3: Histogram of α, β, $S(t)$ and $H(t)$ generated by MCMC method for case I.

Table 1: The 95% confidence interval of MLE and 95% credible intervals of MCMC for α, β, S and H for six cases of UHCS with non-informative prior

		MLE			MCMC		
Cases	Parameters	Lower	Upper	Length	Lower	Upper	Length
case I	α	2.5964	5.5652	2.9688	1.5325	2.9135	1.3810
	β	0.0518	0.0800	0.0282	0.0360	0.0573	0.0213
	S	0.9346	0.9902	0.0556	0.9492	0.9913	0.0421
	H	0.0394	0.1162	0.0768	0.0223	0.0771	0.0548
case II	α	2.6349	5.6105	2.9756	1.6570	3.1717	1.5147
	β	0.0525	0.0804	0.0280	0.0385	0.0602	0.0217
	S	0.9353	0.9904	0.0551	0.9564	0.9934	0.0370
	H	0.0391	0.1160	0.0769	0.0186	0.0711	0.0526
case III	α	2.7079	5.7146	3.0067	1.8752	3.4540	1.5788
	β	0.0537	0.0813	0.0277	0.0423	0.0631	0.0208
	S	0.9365	0.9907	0.0541	0.9667	0.9952	0.0285
	H	0.0386	0.1156	0.0770	0.0146	0.0599	0.0453
case IV	α	2.7297	5.7049	2.9753	2.0921	3.9191	1.8270
	β	0.0540	0.0812	0.0271	0.0453	0.0675	0.0222
	S	0.9367	0.9906	0.0539	0.9739	0.9970	0.0230
	H	0.0386	0.1155	0.0770	0.0103	0.0512	0.0408
case V	α	2.7683	5.7528	2.9845	2.2803	4.1641	1.8839
	β	0.0546	0.0815	0.0269	0.0478	0.0696	0.0217
	S	0.9374	0.9907	0.0534	0.9789	0.9976	0.0187
	H	0.0383	0.1152	0.0770	0.0085	0.0448	0.0364
case VI	α	2.8086	5.8039	2.9953	2.4387	4.4482	2.0095
	β	0.0552	0.0820	0.0267	0.0498	0.0716	0.0218
	S	0.9381	0.9909	0.0528	0.9827	0.9983	0.0156
	H	0.0379	0.1149	0.0769	0.0066	0.0389	0.0323

Table 2: The characteristics for α, β, S and H for six cases of UHCS with non informative prior

Cases	Parameters	Mean	Median	Mode	Standard deviation	Root MS	Skewness
Case I	α	2.1711	2.1383	2.0980	0.4212	2.2115	0.4288
	β	0.0466	0.0465	0.0469	0.0064	0.0470	0.0431
	S	0.9749	0.9775	0.9852	0.0136	0.9750	-1.3726
	H	0.0465	0.0446	0.0400	0.0170	0.0495	0.6909
Case II	α	2.3382	2.2957	2.2920	0.4651	2.3840	0.5933
	β	0.0490	0.0487	0.0490	0.0067	0.0495	0.2074
	S	0.9788	0.9812	0.9848	0.0119	0.9789	-1.3982
	H	0.0416	0.0398	0.0351	0.0162	0.0446	0.6870
Case III	α	2.6231	2.5944	2.4960	0.4837	2.6673	0.3952
	β	0.0527	0.0528	0.0531	0.0063	0.0531	-0.0470
	S	0.9844	0.9866	0.9874	0.0092	0.9845	-1.4942
	H	0.0339	0.0317	0.0274	0.0141	0.0367	0.8132
Case IV	α	2.9243	2.8789	2.7050	0.5570	2.9769	0.4978
	β	0.0561	0.0560	0.05514	0.0067	0.0565	0.1346
	S	0.9885	0.9902	0.9926	0.0074	0.9885	-1.5691
	H	0.0274	0.0254	0.0229	0.0126	0.0301	0.8898
Case V	α	3.1238	3.0603	2.8800	0.5803	3.1773	0.6096
	β	0.0583	0.0579	0.05689	0.0065	0.0586	0.2485
	S	0.9907	0.9920	0.9925	0.0060	0.9907	-1.4402
	H	0.0236	0.0219	0.0176	0.0112	0.0261	0.8635
Case VI	α	3.3620	3.3076	3.1000	0.6192	3.4186	0.5001
	β	0.0606	0.0605	0.0591	0.0066	0.0610	0.0402
	S	0.9926	0.9939	0.9975	0.0052	0.9926	-1.8873
	H	0.0198	0.0180	0.0127	0.0102	0.0223	1.1206

Table 3: Estimation of α, β, S and H for MCMC method for six cases of UHCS with non informative prior

Cases	Parameters	MLE	ω	BSEL	BLINEX			BGE		
					a = -7	a = 0.2	a = 7	a = -7	a = 0.2	a = 7
case I	α	4.0808	0	2.1711	2.9594	2.1535	1.7311	2.4125	2.1225	1.8495
			0.4	2.9350	3.9500	2.8381	1.8041	3.5992	2.7289	1.9887
			0.8	3.6989	4.0490	3.6314	1.961	3.9563	3.5557	2.3224
	β	0.0659	0	0.0466	0.0467	0.0466	0.0464	0.0490	0.0461	0.0428
			0.4	0.0543	0.0547	0.0543	0.0539	0.0593	0.0530	0.0458
			0.8	0.0621	0.0623	0.062	0.0618	0.0641	0.0612	0.0525
	S	0.9624	0	0.9749	0.9755	0.9749	0.9742	0.9754	0.9748	0.9741
			0.4	0.9839	0.9847	0.9839	0.9831	0.9846	0.9838	0.9829
			0.8	0.9929	0.9933	0.9929	0.9925	0.9933	0.9929	0.9924
	H	0.0778	0	0.0465	0.0475	0.0465	0.0455	0.0637	0.0428	0.0228
			0.4	0.0316	0.0334	0.0315	0.0299	0.0592	0.0218	0.0104
			0.8	0.0166	0.0177	0.0166	0.0157	0.0506	0.0120	0.0095
case II	α	4.1227	0	2.3382	3.6098	2.3170	1.8423	2.6254	2.2843	1.9917
			0.4	3.0520	3.9976	2.9632	1.9153	3.6489	2.8689	2.1412
			0.8	3.7658	4.0918	3.7056	2.0723	3.9994	3.6426	2.4979
	β	0.0664	0	0.0490	0.0492	0.0490	0.0488	0.0516	0.0485	0.0452
			0.4	0.056	0.0563	0.0560	0.0556	0.0602	0.0548	0.0484
			0.8	0.0630	0.0631	0.0629	0.0627	0.0647	0.0623	0.0550
	S	0.9628	0	0.9788	0.9793	0.9788	0.9783	0.9792	0.9787	0.9782
			0.4	0.9863	0.9869	0.9863	0.9857	0.9868	0.9862	0.9856
			0.8	0.9938	0.9941	0.9938	0.9935	0.9940	0.9937	0.9934
	H	0.0776	0	0.0416	0.0426	0.0416	0.0407	0.0587	0.0378	0.0164
			0.4	0.0285	0.0300	0.0285	0.0272	0.0546	0.0202	0.0102
			0.8	0.0155	0.0163	0.0155	0.0148	0.0466	0.0116	0.0092
case III	α	4.2113	0	2.6231	3.6881	2.6000	2.0387	2.8878	2.5698	2.2603
			0.4	3.2583	4.0858	3.1837	2.1117	3.7486	3.1130	2.4284
			0.8	3.8936	4.1803	3.8448	2.2686	4.0895	3.8000	2.8243
	β	0.0675	0	0.0527	0.0529	0.0527	0.0526	0.0548	0.0523	0.0493
			0.4	0.0586	0.0589	0.0586	0.0584	0.0618	0.0578	0.0525
			0.8	0.0646	0.0647	0.0645	0.0644	0.0659	0.0641	0.0589
	S	0.9636	0	0.9844	0.9847	0.9844	0.9841	0.9847	0.9844	0.9841
			0.4	0.9897	0.9901	0.9897	0.9894	0.9900	0.9897	0.9893
			0.8	0.9950	0.9952	0.9950	0.9949	0.9952	0.9950	0.9948
	H	0.0771	0	0.0339	0.0346	0.0338	0.0332	0.0496	0.0304	0.0127
			0.4	0.0237	0.0247	0.0237	0.0228	0.0461	0.0175	0.0096
			0.8	0.0136	0.0141	0.0136	0.0131	0.0394	0.0107	0.0088

Continue Table (3)

Cases	Parameters	MLE	ω	BSEL	BLINEX			BGE		
					a = -7	a = 0.2	a = 7	a = -7	a = 0.2	a = 7
case IV	α	4.2173	0	2.9243	4.3559	2.8938	2.2364	3.2454	2.8618	2.5194
			0.4	3.4415	4.3151	3.3820	2.3094	3.8152	3.3299	2.7032
			0.8	3.9587	4.2578	3.9231	2.4664	4.1079	3.8931	3.1243
	β	0.0676	0	0.0561	0.0563	0.0561	0.0560	0.0585	0.0557	0.0527
			0.4	0.0607	0.0609	0.0607	0.0605	0.0631	0.0601	0.0558
			0.8	0.0653	0.0654	0.0653	0.0652	0.0663	0.065	0.0615
	S	0.9637	0	0.9885	0.9887	0.9885	0.9883	0.9886	0.9884	0.9883
			0.4	0.9922	0.9923	0.9922	0.9920	0.9923	0.9921	0.9919
			0.8	0.9959	0.9959	0.9959	0.9958	0.9959	0.9958	0.9958
	H	0.0770	0	0.0274	0.0279	0.0273	0.0268	0.0427	0.0239	0.0086
			0.4	0.0198	0.0205	0.0198	0.0192	0.0397	0.0154	0.0086
			0.8	0.0123	0.0126	0.0122	0.0120	0.0339	0.0103	0.0085
case V	α	4.2605	0	3.1238	4.8648	3.0909	2.4502	3.4608	3.0613	2.7426
			0.4	3.5785	4.7932	3.5265	2.5231	3.9015	3.4850	2.9376
			0.8	4.0332	4.643	4.0037	2.6801	4.1604	3.9809	3.3696
	β	0.0681	0	0.0583	0.0584	0.0583	0.0581	0.0605	0.0578	0.0554
			0.4	0.0622	0.0624	0.0622	0.0620	0.0642	0.0617	0.0584
			0.8	0.0661	0.0662	0.0661	0.066	0.0669	0.0659	0.0634
	S	0.9641	0	0.9907	0.9908	0.9907	0.9905	0.9908	0.9906	0.9905
			0.4	0.9935	0.9936	0.9935	0.9934	0.9936	0.9935	0.9934
			0.8	0.9964	0.9964	0.9964	0.9963	0.9964	0.9963	0.9963
	H	0.0768	0	0.0236	0.0240	0.0236	0.0232	0.0372	0.0204	0.0061
			0.4	0.0175	0.0179	0.0174	0.0170	0.0346	0.0139	0.0065
			0.8	0.0113	0.0116	0.0113	0.0111	0.0296	0.0098	0.0073
case VI	α	4.3062	0	3.3620	5.3405	3.3244	2.5190	3.7115	3.2948	2.9208
			0.4	3.7397	5.2676	3.6944	2.5919	4.0145	3.6609	3.1226
			0.8	4.1174	5.111	4.0938	2.7489	4.2219	4.0770	3.5548
	β	0.0686	0	0.0606	0.0607	0.0606	0.0604	0.0627	0.0602	0.0575
			0.4	0.0638	0.0639	0.0638	0.0636	0.0654	0.0634	0.0603
			0.8	0.0670	0.0671	0.0670	0.0669	0.0676	0.0668	0.0648
	S	0.9645	0	0.9926	0.9927	0.9926	0.9925	0.9927	0.9926	0.9925
			0.4	0.9947	0.9948	0.9947	0.9946	0.9948	0.9947	0.9946
			0.8	0.9968	0.9968	0.9968	0.9968	0.9968	0.9968	0.9968
	H	0.0764	0	0.0198	0.0202	0.0198	0.0195	0.0347	0.0168	0.0035
			0.4	0.0151	0.0155	0.0151	0.0148	0.0322	0.0123	0.0038
			0.8	0.0104	0.0106	0.0104	0.0102	0.0275	0.0092	0.0044

Table 4: Estimation of α, β, S and H for Lindley's approximation for six cases of UHCS with non informative prior

Cases	Parameters	ω	BSEL	BLINEX			BGE		
				a = -7	a = 0.2	a = 7	a = -7	a = 0.2	a = 7
case I	α	0	4.0760	4.4679	4.019	3.6931	4.4077	3.9928	3.7034
		0.4	4.0779	4.4011	4.0190	3.7599	4.2943	4.0277	3.8214
		0.8	4.0799	4.2717	4.0684	3.8895	4.1595	4.063	3.9785
	β	0	0.0655	0.0657	0.0655	0.0653	0.0677	0.0650	0.0630
		0.4	0.0657	0.0658	0.0657	0.0656	0.0670	0.0654	0.064
		0.8	0.0658	0.0659	0.0658	0.0658	0.0663	0.0657	0.0652
	S	0	0.9593	0.9600	0.9593	0.9586	0.9599	0.9592	0.9585
		0.4	0.9605	0.9610	0.9605	0.9601	0.9609	0.9605	0.9601
		0.8	0.9618	0.9619	0.9618	0.9617	0.9619	0.9618	0.9616
	H	0	0.0802	0.0815	0.0802	0.0789	0.0889	0.0773	0.0680
		0.4	0.0793	0.0800	0.0792	0.0784	0.0855	0.0775	0.0708
		0.8	0.0783	0.0786	0.0783	0.0780	0.0809	0.0777	0.0749
case II	α	0	4.1185	4.5104	4.0613	3.7344	4.4492	4.0358	3.7458
		0.4	4.1202	4.4436	4.0613	3.8012	4.3357	4.0703	3.8641
		0.8	4.1218	4.3140	4.1103	3.9309	4.2011	4.1051	4.0209
	β	0	0.066	0.0662	0.0660	0.0659	0.0682	0.0656	0.0635
		0.4	0.0662	0.0663	0.0662	0.0661	0.0675	0.0659	0.0646
		0.8	0.0664	0.0664	0.0664	0.0663	0.0668	0.0663	0.0658
	S	0	0.9597	0.9604	0.9597	0.9591	0.9603	0.9596	0.9589
		0.4	0.9609	0.9614	0.9609	0.9605	0.9613	0.9609	0.9605
		0.8	0.9622	0.9623	0.9622	0.962	0.9623	0.9622	0.9620
	H	0	0.0800	0.0813	0.0800	0.0787	0.0887	0.0770	0.0678
		0.4	0.0790	0.0798	0.0790	0.0782	0.0853	0.0772	0.0706
		0.8	0.0781	0.0783	0.0781	0.0778	0.0807	0.0775	0.0746
case III	α	0	4.2081	4.6018	4.1496	3.8203	4.5396	4.1253	3.8326
		0.4	4.2094	4.5349	4.1496	3.8873	4.4253	4.1595	3.9519
		0.8	4.2106	4.4049	4.1989	4.0173	4.2899	4.1939	4.1095
	β	0	0.0671	0.0673	0.0671	0.0670	0.0692	0.0667	0.0647
		0.4	0.0673	0.0674	0.0673	0.0672	0.0686	0.0670	0.0657
		0.8	0.0674	0.0675	0.0674	0.0674	0.0679	0.0673	0.0669
	S	0	0.9606	0.9612	0.9605	0.9599	0.9611	0.9604	0.9598
		0.4	0.9618	0.9622	0.9618	0.9614	0.9621	0.9617	0.9613
		0.8	0.9630	0.9631	0.9630	0.9629	0.9631	0.9630	0.9628
	H	0	0.0795	0.0808	0.0795	0.0782	0.0883	0.0765	0.0673
		0.4	0.0786	0.0793	0.0785	0.0777	0.0848	0.0768	0.0700
		0.8	0.0776	0.0778	0.0776	0.0773	0.0802	0.0770	0.0741

Continue Table (4)

Cases	Parameters	ω	BSEL	BLINEX a = -7	BLINEX a = 0.2	BLINEX a = 7	BGE a = -7	BGE a = 0.2	BGE a = 7
case IV	α	0	4.2150	4.6051	4.1578	3.8292	4.5403	4.1341	3.8447
		0.4	4.2159	4.5383	4.1578	3.8960	4.4276	4.1671	3.9627
		0.8	4.2168	4.4087	4.2053	4.0256	4.2944	4.2005	4.1177
	β	0	0.0672	0.0674	0.0672	0.0671	0.0692	0.0668	0.0649
		0.4	0.0674	0.0675	0.0674	0.0673	0.0686	0.0671	0.0659
		0.8	0.0675	0.0675	0.0675	0.0675	0.0679	0.0674	0.0670
	S	0	0.9607	0.9613	0.9606	0.9600	0.9612	0.9605	0.9599
		0.4	0.9619	0.9622	0.9618	0.9615	0.9622	0.9618	0.9614
		0.8	0.9631	0.9632	0.9630	0.9629	0.9632	0.9630	0.9629
	H	0	0.0795	0.0808	0.0794	0.0781	0.0882	0.0765	0.0672
		0.4	0.0785	0.0793	0.0785	0.0777	0.0848	0.0767	0.0700
		0.8	0.0775	0.0778	0.0775	0.0773	0.0801	0.0769	0.0741
case V	α	0	4.2589	4.6492	4.2013	3.8717	4.5836	4.1783	3.8882
		0.4	4.2596	4.5824	4.2013	3.9385	4.4707	4.2109	4.0064
		0.8	4.2602	4.4527	4.2486	4.0682	4.3375	4.2439	4.1613
	β	0	0.0677	0.0679	0.0677	0.0676	0.0697	0.0673	0.0654
		0.4	0.0679	0.0680	0.0679	0.0678	0.0691	0.0676	0.0664
		0.8	0.0680	0.0680	0.0680	0.0680	0.0684	0.0679	0.0675
	S	0	0.9611	0.9617	0.9611	0.9605	0.9617	0.9610	0.9604
		0.4	0.9623	0.9627	0.9623	0.9619	0.9626	0.9622	0.9618
		0.8	0.9635	0.9636	0.9635	0.9633	0.9636	0.9634	0.9633
	H	0	0.0792	0.0805	0.0792	0.0778	0.0880	0.0762	0.0669
		0.4	0.0782	0.0790	0.0782	0.0774	0.0845	0.0764	0.0697
		0.8	0.0772	0.0775	0.0772	0.0770	0.0799	0.0766	0.0738
case VI	α	0	4.3053	4.6959	4.2472	3.9164	4.6294	4.2248	3.9339
		0.4	4.3056	4.6290	4.2472	3.9833	4.5163	4.2572	4.0524
		0.8	4.3060	4.4992	4.2944	4.1132	4.3831	4.2898	4.2073
	β	0	0.0683	0.0684	0.0683	0.0681	0.0702	0.0679	0.0660
		0.4	0.0684	0.0685	0.0684	0.0683	0.0696	0.0682	0.0669
		0.8	0.0685	0.0686	0.0685	0.0685	0.0689	0.0684	0.0680
	S	0	0.9616	0.9622	0.9615	0.9610	0.9621	0.9615	0.9609
		0.4	0.9627	0.9631	0.9627	0.9624	0.9631	0.9627	0.9623
		0.8	0.9639	0.9640	0.9639	0.9638	0.9640	0.9639	0.9638
	H	0	0.0789	0.0802	0.0788	0.0775	0.0876	0.0758	0.0666
		0.4	0.0779	0.0787	0.0779	0.0771	0.0842	0.0761	0.0693
		0.8	0.0769	0.0772	0.0769	0.0766	0.0795	0.0763	0.0734

6 Conclusion

In this paper, we considered the Bayes estimation of the unknown parameters of the exponentiated Rayleigh distribution when the data is collected under the unified hybrid censored data. The MLEs and the CIs based on the observed Fisher information matrix have been discussed. Another estimation procedure is also considered which is based on the Bayes estimates. The Bayes estimates have been obtained under balanced loss functions. In our study, the Bayes estimates cannot be obtained in explicit form. So, we have suggested to use Lindley's approximation and MCMC technique to computing the Bayes estimation under three different balanced losses functions. We have applied the developed techniques on a real data set.

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