Applied Mathematics & Information Sciences **5**(3) (2011), 484-499
– An International Journal

A Geometric Approach to Solve Fuzzy Linear Systems of Differential Equations

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Received November 24, 2010; Accepted April 06, 2011

In this paper, systems of linear differential equations with crisp real coefficients and with initial condition described by a vector of fuzzy numbers are studied. A new method based on geometric representations of linear transformations is proposed to find a solution. The most important difference between this method and methods offered in other papers is that the solution is considered to be a fuzzy set of real vector-functions rather than a vector of fuzzy functions. Each member of the solution set satisfies the given system with a certain possibility. It is shown that at any time the solution constitutes a fuzzy region in the coordinate space, alpha-cuts of which are nested parallelepipeds. The proposed method is illustrated on examples.

Keywords: Fuzzy linear system of differential equations, fuzzy number, linear transformation.

2000 Mathematics Subject Classification: 03E72, 34A07, 65L05, 68W99.

1 Introduction

Behaviors of many dynamic systems with uncertainty can be modelled effectively by fuzzy systems of differential equations (FSDE). In particular, linear FSDEs appear in some important applications. The approaches, proposed to solve FSDEs, use different definitions for fuzzy derivative and fuzzy solution.

The concept of a fuzzy derivative was defined by Chang and Zadeh [11]. It was followed up by Dubois and Prade [14], who used the extension principle. The term "fuzzy differential equation" was introduced in 1987 by Kandel and Byatt [22, 23].

There have been many suggestions for the definition of fuzzy derivative to study fuzzy differential equations. One of the earliest suggestions was to generalize the Hukuhara derivative [17] of a set-valued function. This generalization was made by Puri and Ralescu [26] and studied by Kaleva [19,20]. It soon appeared that the solution of a fuzzy differential equation defined by means of Hukuhara derivative has a deficiency: it becomes fuzzier as time goes [13, 21]. Hence, behavior of the fuzzy solution is quite different from that of the crisp solution. Seikkala [28] introduced the notion of fuzzy derivative as an extension of the Hukuhara derivative, which was the same as what Dubois and Prade [14] proposed. To circumvent problems arising in connection with Hukuhara derivative, Hüllermeier [18] considered fuzzy differential equation as a family of differential inclusions. The main downside of using differential inclusions is that we do not have an adequate definition for derivative of a fuzzy-number-valued function.

The concept of strongly generalized differentiability was introduced in [3] and studied in [4,5,9]. In [24] a generalized concept of higher-order differentiability for fuzzy functions is presented to solve Nth-order fuzzy differential equations.

Buckley and Feuring [6, 7] and Buckley et al. [8] gave a very general formulation of fuzzy initial value problem. They firstly find the crisp solution, fuzzify it and then check to see if it satisfies the FSDE.

Rodriguez-Lopez [27] considered several comparison results for the solutions of FSDE obtained through different methods using the Hukuhara derivative. Allahviranloo et al. [1] applied differential transformation method by using generalized H-differentiability. Mizukoshi et al. [25] showed that the solutions of the Cauchy problem obtained by the Zadeh extension principle and by using a family of differential inclusions are same.

Xu et al. [29] used complex number representation of α-level sets of a fuzzy system and proved theorems that provide the solutions in this representation. Chalco-Cano and Román-Flores [10] studied the class of fuzzy differential equations where the dynamics is given by a continuous fuzzy mapping which is obtained via Zadeh's extension principle.

In this paper, we apply a geometric approach to fuzzy linear system of differential equations (FLSDE) with crisp real coefficients and with initial condition described by a vector of fuzzy numbers. We interpret a vector of fuzzy numbers as a rectangular prism in n dimensional space, and show that at any time the solution corresponds to an n dimensional parallelepiped. Unlike earlier researches, we are not looking for solutions of FLSDE in the form of vector of fuzzy functions. Instead, our solutions constitute a fuzzy set of real vector-functions. Each member in the solution set satisfies the system with a certain possibility. The most important difference of the approach proposed in the present paper from the others will be explained in details later where the concept of the solution will be discussed.

In articles [15, 16], by using the same geometric approach an algorithm to solve linear systems of algebraic equations with crisp coefficients and with fuzzy numbers on the right-hand side is proposed. Here we adopt the approach to the fuzzy linear system of differential equations.

This paper is comprised of 6 sections including the Introduction. Preliminaries are given in Section 2. In Section 3, we define FLSDE. In Section 4, we apply the geometric approach to find the solution of FLSDE and present the main results. In Section 5, we solve samples of FLSDE. In Section 6, we summarize the results.

2 Preliminaries

We define a fuzzy number $\tilde{u}$ in parametric form according to [12].

Definition 1. A fuzzy number $\tilde{u}$ in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r),\ \overline{u}(r)$, $0 \leq r \leq 1$, which satisfy the following requirements:

1. $\underline{u}(r)$ is a bounded monotonic nondecreasing left continuous function over $[0,1]$
2. $\overline{u}(r)$ is a bounded monotonic nonincreasing left continuous function over $[0,1]$
3. $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$

A popular fuzzy number is the triangular number $\widetilde{u} = (a, c, b)$ with the membership function

$$\mu(x) = \begin{cases} \frac{x-a}{c-a}, & a \leq x \leq c \\ \frac{x-b}{c-b}, & c \leq x \leq b \end{cases}$$

where $c \neq a$ and $c \neq b$. For triangular numbers we have $\underline{u}(r) = a + (c-a)r$ and $\overline{u}(r) = b + (c-b)r$.

We will denote $\underline{\underline{u}} = a$ and $\overline{\overline{u}} = b$ to indicate the left and the right limits of $\widetilde{u}$, respectively.

We can represent a crisp number a by taking $\underline{u}(r) = \overline{u}(r) = a$, $0 \leq r \leq 1$.

For two arbitrary fuzzy numbers $\widetilde{u}$ and $\widetilde{v}$ the equality $\widetilde{u} = \widetilde{v}$ means that $\underline{u}(r) = \underline{v}(r)$ and $\overline{u}(r) = \overline{v}(r)$ for all $r \in [0,1]$.

For two arbitrary fuzzy numbers $\widetilde{u}$ and $\widetilde{v}$ the following arithmetic operations are defined:

$a)$ Addition: $\widetilde{u} + \widetilde{v} = (\underline{u}(r) + \underline{v}(r),\ \overline{u}(r) + \overline{v}(r))$

$b)$ Multiplication by a real number k:

$$k\widetilde{u} = \begin{cases} (k\underline{u}(r),\ k\overline{u}(r)), & k \geq 0 \\ (k\overline{u}(r),\ k\underline{u}(r)), & k < 0 \end{cases}$$

$c)$ Subtraction: $\widetilde{u} - \widetilde{v} = \widetilde{u} + (-1)\widetilde{v}$

A fuzzy subset $\widetilde{X}$ of R^n is characterized by its membership function $\mu : R^n \to [0,1]$. It is denoted by E^n the space of fuzzy sets whose membership functions satisfy next four conditions:

i) μ is normal, that is there exists an $\mathbf{x}_0$ such that $\mu(\mathbf{x}_0) = 1$;

ii) μ is fuzzy convex, that is for all $\mathbf{x},\ \mathbf{y} \in R^n$ and $0 \leq \lambda \leq 1$: $\mu(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \geq \min\{\mu(\mathbf{x}),\ \mu(\mathbf{y})\}$;

iii) μ is upper semicontinuous;

iv) the closure of $\{\mathbf{x} \in R^n \mid \mu(\mathbf{x}) > 0\}$ is compact.

For $0 < \alpha \leq 1$ the α-level set of fuzzy set $\widetilde{X}$ is defined by $X_\alpha = \{\mathbf{x} \in R^n \mid \mu(\mathbf{x}) \geq \alpha\}$.

It can be seen that any $\widetilde{u} \in E^1$ is a fuzzy number in the sense of Definition 1.

Note that $E^1 \times E^1 \times \ldots \times E^1$ denotes the set of all n-dimension vectors whose components are fuzzy numbers. Note also that $E^n \neq E^1 \times E^1 \times \ldots \times E^1$, namely E^n is wider. For instance, let $\widetilde{X}$ be a fuzzy subset of R^2 with membership function $\mu(x, y) = \max\left\{1 - (x^2 + y^2);\ 0\right\}$. Then, we have $\widetilde{X} \in E^2$, but $\widetilde{X} \notin E^1 \times E^1$.

3 Fuzzy linear systems of differential equations

Definition 2. Let $a_{ij},\ 1 \leq i,\ j \leq n$, be crisp numbers, $f_i(t),\ 1 \leq i \leq n$, be given crisp functions and $\widetilde{u}_i = (\underline{u_i}(r), \overline{u_i}(r)),\ 0 \leq r \leq 1,\ 1 \leq i \leq n$, be fuzzy numbers. The system

$$\begin{cases} x_1'(t) = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + f_1(t) \\ x_2'(t) = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + f_2(t) \\ \vdots \\ x_n'(t) = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n + f_n(t) \end{cases} \tag{3.1}$$

with the fuzzy initial condition

$$\begin{cases} x_1(t_0) = \widetilde{u}_1 \\ x_2(t_0) = \widetilde{u}_2 \\ \vdots \\ x_n(t_0) = \widetilde{u}_n \end{cases} \tag{3.2}$$

is called a fuzzy linear system of differential equations (FLSDE).

One can rewrite the problem (3.1)-(3.2) as follows, using matrix notation.

$$\begin{cases} X' = AX + F(t) \\ X(t_0) = \widetilde{B} \end{cases} \tag{3.3}$$

where $A = [a_{ij}]$ is an $n \times n$ crisp matrix, $F(t) = (f_1(t), f_2(t), \ldots, f_n(t))^T$ is a crisp vector-function and $\widetilde{B} = (\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_n)^T$ is a vector of fuzzy numbers.

If differential equations are considered to describe motion of a body, then fuzzy initial conditions indicate some uncertainty regarding the location of the body at time t_0.

We assume the solution $\widetilde{X}$ of the system (3.1)-(3.2) be a fuzzy set of real vector-functions such as $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))$. Each vector-function $\mathbf{x}(t)$ must satisfy the system (3.1) and must have an initial value $\mathbf{x}(t_0) = (x_1(t_0), x_2(t_0), \ldots, x_n(t_0))$ from the set $\widetilde{B}$. We define the possibility (membership) of the vector-function $\mathbf{x}(t)$ to be equal to the possibility of its initial value $\mathbf{x}(t_0)$ in $\widetilde{B}$.

The solution $\widetilde{X}$, defined above, can be interpreted as a fuzzy bunch of vector-functions.

One can also interpret that we consider a FLSDE as a set of crisp Cauchy problems whose initial values belong to the fuzzy set $\widetilde{B}$.

Mathematically, the fuzzy solution set $\widetilde{X}$ can be defined as follows:

$$\widetilde{X} = \left\{ \mathbf{x}(t)=(x_1(t), x_2(t), \ldots, x_n(t)) \mid \mathbf{x}' = A\mathbf{x} + F, \;\; \mathbf{x}(t_0) = \mathbf{x}_0 \in \widetilde{B} \right\}$$

with membership function $\mu_{\widetilde{X}}(\mathbf{x}(t)) = \mu_{\widetilde{B}}(\mathbf{x}(t_0))$.

The most of the studies on the system (3.1)-(3.2) are inspired by [8]. Let us highlight the main difference between our concept of solution and the one, proposed in [8]. In [8] the solution is assumed to be a vector, components of which are fuzzy functions. Under this assumption, the value $\widetilde{X}(t)$ of the solution at a time t is a vector of fuzzy numbers (i.e. $\widetilde{X}(t) \in E^1 \times E^1 \times \ldots \times E^1$) and, consequently, forms a rectangular prism in the coordinate space. Therefore, as time goes if the solution set constitutes a body which is different from prism, the assumption will fail. Differing from [8], in our approach, the condition $\widetilde{X}(t) \in E^n$ holds. This circumstance allows the solution to change its shape with time and, as a result, our approach works even in the cases when [8] fails, as it will be seen later.

4 The solution method

In this section we develop a method to find the solution of the problem (3.1)-(3.2) (or, (3.3) in matrix form) as a fuzzy set.

Without loss of generality, we put $t_0 = 0$.

Let us write the initial value vector as $\widetilde{B} = \mathbf{b}_{cr} + \widetilde{\mathbf{b}}$, where $\mathbf{b}_{cr}$ is a vector with possibility 1 and denotes the crisp part (the vertex) of $\widetilde{B}$, while $\widetilde{\mathbf{b}}$ denotes the uncertain part with vertex at the origin. It is easy to see that, the solution of the given system is of the form $\widetilde{X}(t) = \mathbf{x}_{cr}(t) + \widetilde{\mathbf{x}}(t)$ (crisp solution + uncertainty). Here $\mathbf{x}_{cr}(t)$ is a solution of the non-homogeneous crisp problem

$$\begin{cases} X' = AX + F(t) \\ X(0) = \mathbf{b}_{cr} \end{cases}$$

while $\widetilde{\mathbf{x}}(t)$ is a solution of the homogeneous system with fuzzy initial condition

$$\begin{cases} X' = AX \\ X(0) = \widetilde{\mathbf{b}} \end{cases}$$

In regard to motion of a body, one could interpret $\mathbf{x}_{cr}(t)$ as the main trajectory. $\mathbf{x}_{cr}(t)$ can be computed by means of analytical or numerical methods. Hence, (3.3) is reduced to solving a homogeneous system with fuzzy initial condition.

The basis of our solution method is summarized below:

We shall make use of the following facts about linear transformations [2]:

1. A linear transformation maps the origin (zero vector) to the origin (zero vector).

2. A linear transformation maps a pair of parallel straight lines to a pair of parallel straight lines (thus a pair of parallel faces to a pair of parallel faces). Consequently, a linear transformation maps a parallelepiped to a parallelepiped.

In addition, we shall reference a property of fuzzy number vectors.

3. The fuzzy vector $\widetilde{\mathbf{b}}$ forms a fuzzy region in R^n, vertex of which is located at the origin and boundary of which is a rectangular prism. Furthermore, the α-cuts of the region are rectangular prisms nested within one another.

The next three properties are in connection with the crisp initial value problem

$$\begin{cases} \mathbf{x}' = A\mathbf{x} \\ \mathbf{x}(0) = \mathbf{v} \end{cases} \tag{4.1}$$

4. By the existence and uniqueness theorem, two solutions with different initial points have distinct values for any t.

5. The solution of (4.1) is of the form $\mathbf{x}(t) = e^{At}\mathbf{v}$.

6. For a fixed $t = t_*$: $\mathbf{x}(t_*) = e^{At_*}\mathbf{x}(0) = M\mathbf{x}(0)$, where M is a fixed invertible matrix. Hence, the value of the solution function at $t = t_*$ is determined by a linear transformation described by M.

The facts 1-6 bring us to the following conclusion: The set of initial points form a rectangular prism (or more generally speaking, a parallelepiped) and, therefore, the solution at any time forms a parallelepiped.

In particular, for $n = 2$, rectangular prism and parallelepiped turn into rectangle and parallelogram, respectively. According to the discussion above, solution curves make up nested surfaces extending along t direction (like a coaxial cable). Cross-sections of these surfaces at any $t = t_*$ are nested parallelograms.

We can liken the behavior of the solution in xy plane to that of a cloud of dust. In the cloud, there exists a point ("the center") where the density of dust is the highest. As moving away from the center, the density decreases along the parallelograms. In other words, the level curves of the density are nested parallelograms. The motion of the center is governed by the crisp solution. The cloud of dust moves along with the center, but the parameters of the parallelograms (orientation, sides and their ratio) may change in time.

The foregoing discussion qualitatively indicates how the solution would behave in n dimensional coordinate space, in general. Now, we shall find a formula for the solution.

Firstly, we consider the case where the initial values in (3.2) are triangular fuzzy numbers $\widetilde{u}_i = (l_i, m_i, r_i)$. We have $(b_{cr})_i = m_i$ and $\widetilde{b}_i = (\underline{\underline{b_i}}, 0, \overline{\overline{b_i}}) = (l_i - m_i, 0, r_i - m_i)$. Let us denote $\widetilde{\mathbf{b}} = (\underline{\underline{\mathbf{b}}}, \mathbf{0}, \overline{\overline{\mathbf{b}}})$, where $\underline{\underline{\mathbf{b}}} = (\underline{\underline{b_1}}, \underline{\underline{b_2}}, \ldots, \underline{\underline{b_n}})$ and $\overline{\overline{\mathbf{b}}} = (\overline{\overline{b_1}}, \overline{\overline{b_2}}, \ldots, \overline{\overline{b_n}})$.

One can express the vectors $\underline{\underline{\mathbf{b}}}$ and $\overline{\overline{\mathbf{b}}}$ through standard basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, ..., $\mathbf{e}_n$:

$$\underline{\underline{\mathbf{b}}} = \underline{\underline{b_1}}\mathbf{e}_1 + \underline{\underline{b_2}}\mathbf{e}_2 + \ldots + \underline{\underline{b_n}}\mathbf{e}_n; \quad \overline{\overline{\mathbf{b}}} = \overline{\overline{b_1}}\mathbf{e}_1 + \overline{\overline{b_2}}\mathbf{e}_2 + \ldots + \overline{\overline{b_n}}\mathbf{e}_n.$$

Let $\mathbf{v}_i = \underline{\underline{b_i}}\mathbf{e}_i$ and $\mathbf{u}_i = \overline{\overline{b_i}}\mathbf{e}_i$. Note that $\mathbf{v}_i$ and $\mathbf{u}_i$ are vectors with all but i-th coordinates zero. The i-th coordinate of $\mathbf{v}_i$ is negative, while the i-th coordinate of $\mathbf{u}_i$ is positive. Any crisp vector from R^n can be expressed uniquely as a linear combination, with non-negative coefficients, of the vectors $\mathbf{v}_i$ and $\mathbf{u}_i$ (under the condition that only one of vectors, either v_i or u_i is used separately for each i).

The fuzzy initial vector $\widetilde{\mathbf{b}}$ forms a rectangular prism in the coordinate space:

$$\widetilde{\mathbf{b}} = \{\mathbf{z} = \alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2 + \ldots + \alpha_n \mathbf{w}_n \mid \alpha_i \in [0,1]\,;\ \mathbf{w}_i = \mathbf{v}_i \text{ or } \mathbf{w}_i = \mathbf{u}_i\}$$

with membership function $\mu_{\widetilde{\mathbf{b}}}(\mathbf{z}) = 1 - \max\limits_{1 \le i \le n} \alpha_i$.

Let $\mathbf{q}_i(t) = e^{At}\mathbf{v}_i$ and $\mathbf{p}_i(t) = e^{At}\mathbf{u}_i$. Then the solution of FLSDE can be expressed as follows:

$$\begin{aligned} \widetilde{X} \quad = \quad & \{\mathbf{x}(t) = \mathbf{x}_{cr}(t) + \alpha_1 \mathbf{r}_1(t) + \alpha_2 \mathbf{r}_2(t) + \ldots + \alpha_n \mathbf{r}_n(t) \\ & \mid \alpha_i \in [0,1]\,;\ \mathbf{r}_i = \mathbf{q}_i \text{ or } \mathbf{r}_i = \mathbf{p}_i\} \end{aligned} \tag{4.2}$$

$$\mu_{\widetilde{X}}(\mathbf{x}(t)) = 1 - \max_{1 \le i \le n} \alpha_i \tag{4.3}$$

If initial values (3.2) are triangular fuzzy numbers, then one could determine an α-cut of the solution, X_α, through geometric similarity (with coefficient $1-\alpha$) without additional computation.

Below we give another representation for the solution and then we generalize the results for the case where the initial values of the problem (3.3) are parametric fuzzy numbers.

The rectangular prism, corresponding to the initial values in the form of triangular fuzzy numbers, and its α-cuts can also be represented as follows:

$$\widetilde{\mathbf{b}} = \left\{k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + \ldots + k_n\mathbf{e}_n \mid k_i \in \left[\underline{\underline{b_i}},\ \overline{\overline{b_i}}\right]\right\}$$

$$\mathbf{b}_\alpha = \left\{k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + \ldots + k_n\mathbf{e}_n \mid k_i \in \left[(1-\alpha)\underline{\underline{b_i}},\ (1-\alpha)\overline{\overline{b_i}}\right]\right\}$$

Let $\mathbf{g}_i(t) = e^{At}\mathbf{e}_i$. Then we can obtain the following formulas for α-cuts of the solution and the solution itself:

$$\begin{aligned} X_\alpha \quad = \quad & \{\mathbf{x}(t) = \mathbf{x}_{cr}(t) + k_1\mathbf{g}_1(t) + k_2\mathbf{g}_2(t) + \ldots + k_n\mathbf{g}_n(t) \\ & \Big|\ k_i \in \left[(1-\alpha)\underline{\underline{b_i}},\ (1-\alpha)\overline{\overline{b_i}}\right]\Big\} \end{aligned} \tag{4.4}$$

$$\widetilde{X} = X_0 \text{ with } \mu_{\widetilde{X}}(\mathbf{x}(t)) = 1 - \max_{1 \le i \le n} \gamma_i, \text{ where } \gamma_i = \begin{cases} k_i/\overline{\overline{b_i}}, & k_i \ge 0 \\ k_i/\underline{\underline{b_i}}, & k_i < 0 \end{cases} \tag{4.5}$$

For the case when initial values consist of parametric fuzzy numbers $\widetilde{u_i} = (\underline{u_i}(r), \overline{u_i}(r))$ the solution can be represented as follows:

$$\begin{aligned} X_\alpha \quad = \quad & \{\mathbf{x}(t) = \mathbf{x}_{cr}(t) + k_1\mathbf{g}_1(t) + k_2\mathbf{g}_2(t) + \ldots + k_n\mathbf{g}_n(t) \\ & \big|\ k_i \in \left[\underline{u_i}(\alpha) - (b_{cr})_i,\ \overline{u_i}(\alpha) - (b_{cr})_i\right]\big\} \end{aligned} \tag{4.6}$$

$$\widetilde{X} = X_0 \tag{4.7}$$

with

$$\mu_{\widetilde{X}}(\mathbf{x}(t)) = \min_{1 \leq i \leq n} \beta_i \tag{4.8}$$

where $\beta_i = \begin{cases} \overline{u_i}^{-1}(s_i), & s_i > \overline{u_i}(1) \\ 1, & \underline{u_i}(1) \leq s_i \leq \overline{u_i}(1) \\ \underline{u_i}^{-1}(s_i), & s_i < \underline{u_i}(1) \end{cases}$. Here $s_i = (b_{cr})_i + k_i$.

Hence, to determine the solution set we need to calculate $e^{At} = [\mathbf{g}_1(t)\ \mathbf{g}_2(t)\ \ldots\ \mathbf{g}_n(t)]$. Note that $\mathbf{g}_i(t) = e^{At}\mathbf{e}_i$ is the solution of the crisp homogeneous system with initial value vector $\mathbf{e}_i$. In other words, we need to determine n linear independent solutions of the homogeneous system by taking each one of the basis vectors $\mathbf{e}_i,\ 1 \leq i \leq n$, as initial value vector.

To summarize, the solution of the problem (3.1)-(3.2) is the fuzzy set of real vector-functions (or, fuzzy bunch of vector-functions), which can be represented by the formulas (4.6)-(4.8) in general case. If triangular fuzzy numbers describe initial conditions, the formulas (4.4)-(4.5) can be applied. To determine the fuzzy solution set we only need to work out the solution of crisp initial value problem for non-homogeneous system and to calculate e^{At}, or to find n solutions $\mathbf{g}_i(t),\ 1 \leq i \leq n$, of the crisp homogeneous system.

We note that if the initial values are in parametric form, then $\mathbf{b}_{cr}$, in general, is not unique. In this case, we can choose the components of $\mathbf{b}_{cr}$ arbitrarily to the extent that $\underline{u_i}(1) \leq (b_{cr})_i \leq \overline{u_i}(1)$. For instance, we can put $(b_{cr})_i = (\underline{u_i}(1) + \overline{u_i}(1))/2$.

5 Examples

Example 1. Solve the system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 5t^2 - 15t - 25 \\ 10t^2 - 10t - 40 \end{bmatrix} \tag{5.1}$$

with the initial values $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} (14.5,\ 15,\ 16) \\ (4,\ \ 6,\ \ 9) \end{bmatrix}$. Find the α-cut of the solution for $\alpha = 0.5$.

Solution: Initial values, given by triangular fuzzy numbers, form a fuzzy region in coordinate space. The boundary of this region (the rectangle $ABCD$) and α-cut for $\alpha = 0.5$ is shown in Fig. 5.1. The vertex of the region, $\mathbf{b}_{cr} = (15, 6)^T$, is marked with a dot.

The problem could be solved in two steps:

1. We determine the crisp solution corresponding to the non-homogenous system with the crisp initial values.

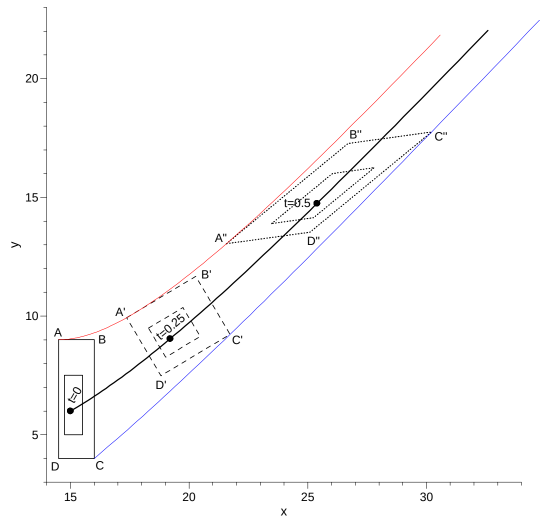

Figure 5.1: $ABCD$ is the boundary of the fuzzy region corresponding to the initial values. Dashed parallelograms are the boundaries of the fuzzy region, corresponding to the solution, and its $\alpha = 0.5$-cut at time $t = 0.25$. Dotted parallelograms are the same, but for time $t = 0.5$. The thick line is the crisp solution.

The solution of (5.1) with the initial value $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 15 \\ 6 \end{bmatrix}$ is

$$\begin{cases} x_{cr}(t) = 5(t+2) + \frac{1}{3}e^{-t} + \frac{14}{3}e^{2t} \\ y_{cr}(t) = \quad 5t^2 \quad + \frac{4}{3}e^{-t} + \frac{14}{3}e^{2t} \end{cases}$$

The solution is graphed with the thick line in Fig. 5.1.

2. We look for the solution corresponding to the homogeneous system with fuzzy initial value, the vertex of which is at the origin.

This means that we are looking for the fuzzy solution of

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{5.2}$$

with initial value

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} (-0.5,\ 0,\ 1) \\ (-2,\ \ 0,\ 3) \end{bmatrix} \tag{5.3}$$

(5.3) determines a rectangle $A_0B_0C_0D_0$ which can be obtained by translation of $ABCD$ by the vector $-\mathbf{b}_{cr}$. The vertices are:

$$A_0(-0.5, 3),\ B_0(1, 3),\ C_0(1, -2),\ D_0(-0.5, -2)$$

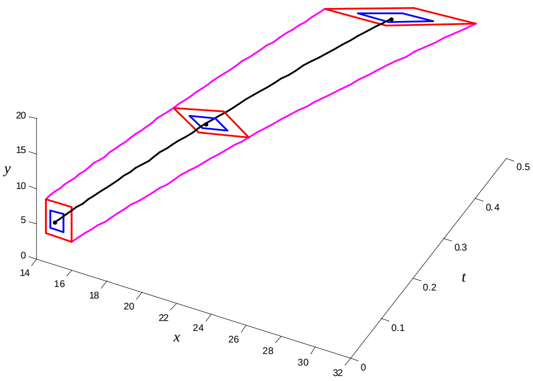

Figure 5.2: The same as in Fig. 5.1, but in the xyt-coordinate space.

The general solution of (5.2) is:

$$\begin{cases} x(t) = \ c_1 e^{-t} + c_2 e^{2t} \\ y(t) = 4c_1 e^{-t} + c_2 e^{2t} \end{cases}$$

For some initial point $P(a, b)$, the constants have the following values:

$c_1 = (-a + b)/3$ and $c_2 = (4a - b)/3$

Based on the solution of (5.2), one could work out A_0', B_0', C_0', the locations of A_0, B_0, C_0 at time t, and hence obtain a parallelogram with three of its vertices at A_0', B_0', C_0'. If this parallelogram is translated by $\mathbf{x}_{cr}(t)$, we wind up with the boundary of region $(A'B'C'D')$ determined by the fuzzy solution at time t (Fig. 5.1).

Since the initial values are triangular numbers, one can find any α-cut of the solution by geometric similarity without having to do additional computations (Fig. 5.1). The formula for α-cuts of the solution set $\widetilde{X}$ can be obtained from (4.4):

$$X_\alpha = \left\{ \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} (15(t+2) + (1-a+b)e^{-t} + (14+4a-b)e^{2t})/3 \\ (\ \ 15t^2 \ \ + 4(1-a+b)e^{-t} + (14+4a-b)e^{2t})/3 \end{bmatrix} \right. \left| \begin{array}{c} -0.5(1-\alpha) \le a \le (1-\alpha) \\ -2(1-\alpha) \le b \le 3(1-\alpha) \end{array} \right\}$$

Solution curves make up nested surfaces in the xyt-coordinate space (Fig. 5.2). Cross-sections of these surfaces with a plane $t = t_*$ are nested parallelograms.

Example 2. Solve the initial value problem and find the α-cut of the solution for

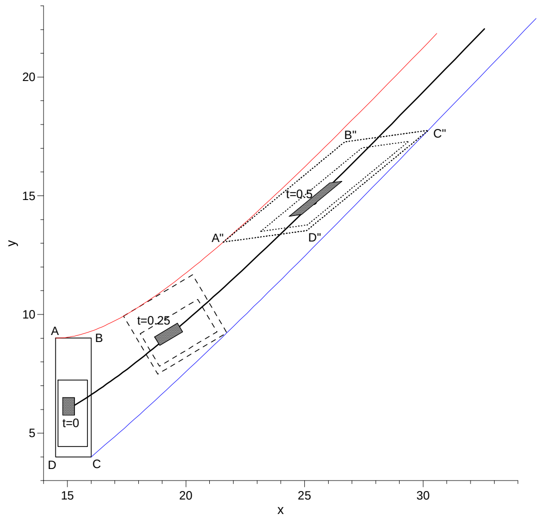

Figure 5.3: $ABCD$ is the boundary of the fuzzy region corresponding to the initial values. Dashed parallelograms are the boundaries of the fuzzy region, corresponding to the solution, and its $\alpha = 0.5$-cut at time $t = 0.25$. Dotted parallelograms are the same, but for time $t = 0.5$. The shaded regions denote $\alpha = 1$ cuts. The thick line is the crisp solution.

$\alpha = 0.5$.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 5t^2 - 15t - 25 \\ 10t^2 - 10t - 40 \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} (14.5 + 0.2r, \ 16 - 0.6r^2) \\ (4 + 1.75r^2, \quad 9 - 2.5\sqrt{r}) \end{bmatrix} = \begin{bmatrix} (\underline{a}(r), \ \overline{a}(r)) \\ (\underline{b}(r), \ \overline{b}(r)) \end{bmatrix}$$

Solution: We note that the given system is same as in Example 1, but with different initial values, which are represented in parametric form.

For $r = 1$ we have

$$\begin{bmatrix} (\underline{a}(1), \ \overline{a}(1)) \\ (\underline{b}(1), \ \overline{b}(1)) \end{bmatrix} = \begin{bmatrix} (14.7, \ 15.4) \\ (5.75, \ 6.50) \end{bmatrix}$$

Therefore, we can arbitrarily choose the first and the second components of $\mathbf{b}_{cr}$ from the intervals $[14.7, \ 15.4]$ and $[5.75, \ 6.50]$, respectively. In the computations, we take $\mathbf{b}_{cr} = (15, 6)^T$, as in Example 1. Hence, the formulas found in Example 1 are still valid.

To determine α-cut of the solution, firstly we need to calculate the lower and upper limits of the corresponding α-cut of the initial values. For $r = 0.5$ we have

$$\begin{bmatrix} (\underline{a}(0.5), \ \overline{a}(0.5)) \\ (\underline{b}(0.5), \ \overline{b}(0.5)) \end{bmatrix} \approx \begin{bmatrix} (14.600, \ 15.8500) \\ (4.4375, \ 7.23223) \end{bmatrix}$$

The results of the computation are shown in Fig. 5.3. The values of the solution on the shaded regions have the possibility 1.

In Fig. 5.3, the parallelograms, which are boundaries of α-cuts, are not similar to the parallelogram corresponding to the solution. This is because the initial values are in parametric form rather than triangular fuzzy numbers.

The formula for α-cuts of the solution set $\widetilde{X}$ can be obtained from (4.6):

$$X_\alpha = \left\{ \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} (15(t+2) + (1-a+b)e^{-t} + (14+4a-b)e^{2t})/3 \\ (\quad 15t^2 \quad + 4(1-a+b)e^{-t} + (14+4a-b)e^{2t})/3 \end{bmatrix} \;\middle|\; \begin{matrix} 0.2\alpha - 0.5 \le a \le 1 - 0.6\alpha^2 \\ 1.75\alpha^2 - 2 \le b \le 3 - 2.5\sqrt{\alpha} \end{matrix} \right\}$$

The following example is taken from [8].

Example 3. (Arms race model). Solve the initial value problem:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} (70,\ 100,\ 130) \\ (70,\ 100,\ 130) \end{bmatrix}$$

Solution: We first determine the crisp solution. Consider the following:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

The solution is:

$$\begin{cases} x_{cr}(t) = \frac{4}{3} + 98\frac{3}{5}e^{-t} + \frac{1}{15}e^{-6t} \\ y_{cr}(t) = \frac{3}{2} + 98\frac{3}{5}e^{-t} - \frac{1}{10}e^{-6t} \end{cases}$$

Secondly, we consider:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} (-30,\ 0,\ 30) \\ (-30,\ 0,\ 30) \end{bmatrix}$$

We use the notation introduced in Example 1. Then,

$$A_0(-30,30),\ B_0(30,30),\ C_0(30,-30),\ D_0(-30,-30)$$

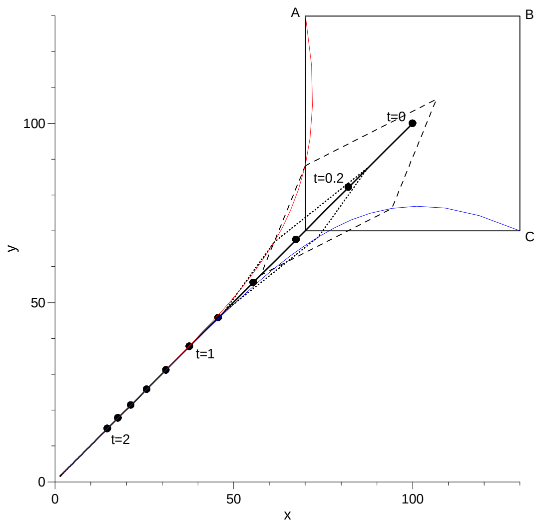

Figure 5.4: Rectangle $ABCD$ is the boundary of the fuzzy region corresponding to the initial values. Dashed parallelogram is the boundary of the fuzzy region, corresponding to the solution at time $t = 0.2$. Dotted parallelogram is the same, but for time $t = 0.4$. The thick line is the crisp solution.

The solution of the homogeneous system corresponding to initial point $P(a, b)$ is given by:

$$\begin{cases} x_P(t) = c_1 e^{-t} + 2c_2 e^{-6t} \\ y_P(t) = c_1 e^{-t} - 3c_2 e^{-6t} \end{cases}$$

where $c_1 = (3a + 2b)/5$ and $c_2 = (a - b)/5$.

As shown in Fig. 5.4, the fuzzy region corresponding to the solution at time t gets smaller as t increases, and shrinks down to a point as t approaches infinity.

The α-cuts of the solution set $\widetilde{X}$ can be expressed by the following formula:

$$X_\alpha = \left\{ \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \frac{4}{3} + (98.6 + 0.6a + 0.4b)e^{-t} + (\ \frac{1}{15} + 0.4a - 0.4b)e^{-6t} \\ \frac{3}{2} + (98.6 + 0.6a + 0.4b)e^{-t} + (-\frac{1}{10} - 0.6a + 0.6b)e^{-6t} \end{bmatrix} \middle| \begin{array}{l} -30(1-\alpha) \le a \le 30(1-\alpha) \\ -30(1-\alpha) \le b \le 30(1-\alpha) \end{array} \right\}$$

6 Conclusion

In this paper, we dealt with systems of linear differential equations with crisp real coefficients and fuzzy initial condition. We proposed a geometric approach to solve the problem. Instead of looking for the solution as a vector of fuzzy functions, we determined the solution as a fuzzy set of vector-functions, each of which satisfies FLSDE with some possibility. We showed that at a given time the solution forms an n dimensional parallelepiped in the coordinate space. We suggested an efficient method to compute the fuzzy solution set. We illustrated the results with numerical examples.

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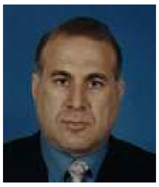

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