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Existence of Solutions of Multi-Point Boundary Value Problems for Fractional Differential Systems with Impulse Effects

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Abstract: In this article, we present a new method for converting the considered impulsive systems to integral systems. As application of this method, we establish existence results for solutions of boundary value problems for nonlinear impulsive fractional differential systems. Our analysis relies on the well known Schauder's fixed point theorem. The mistakes in [Solvability of multi-point boundary value problem of nonlinear impulsive fractional differential equation at resonance, Electron. J. Qual. Theory Differ. Equ. 89(2011), 1-19] and [Existence result for boundary value problem of nonlinear impulsive fractional differential equation at resonance, J, Appl, Math, Comput., 39(2012) 421-443] are corrected, see Remark 2.1.

Keywords: Impulsive multi-term fractional differential system, boundary value problem, Schauder's fixed point theorem.

1 Introduction

Fractional differential equations have many applications in modeling of physical and chemical processes [1,2]. In its turn, mathematical aspects of fractional differential equations and methods of their solutions were discussed by many authors, see the text books [3,4,5]. Existence of solutions or positive solutions of boundary value problems (BVPs for short) of fractional differential equations have been studied by many authors see the recent published papers [6,7,8,9,10,11,12,13, 14,15,16,17,18,19]. In recent years, some authors have studied solvability or existence of solutions or positive solutions of BVPs for fractional differential systems see [20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35]. However, studies on solvability of BVPs for impulsive fractional differential systems have not been studied well.

In [36], Bai studied the existence of solutions of the following boundary value problem for impulsive fractional differential equation

$$\begin{cases} D_{0^+}^{\alpha}x(t)=f(t,x(t),D_{0^+}^{\alpha-1}x(t)),t\in(0,1),t\neq t_i,i=1,2,\cdots,k,\\ \lim\limits_{t\to 0}t^{2-\alpha}x(t)=\sum\limits_{i=1}^{m}a_ix(\xi_i),\ x(1)=\sum\limits_{i=1}^{n}b_ix(\eta_i),\\ \Delta u(t_i)=I_i(u(t_i),D_{0^+}^{\alpha-1}x(t_i)),\ \Delta D_{0^+}^{\alpha-1}x(t_i)=J_i(u(t_i),D_{0^+}^{\alpha-1}x(t_i)),i=1,2,\cdots,k,\end{cases}\tag{1.1}$$

where $\alpha\in(1,2)$, $D_{0^+}^{*}$ is the Riemann-Liouville fractional derivative or order $*$, m,n,k are positive integers, $0<\xi_1<\cdots<\xi_m<1$, $0<\eta_1<\cdots<\eta_n<1$, $0<t_1<\cdots<t_k<1$, $f,g:[0,1]\times\mathbb{R}^2\mapsto\mathbb{R}$, $I_i,J_i:\mathbb{R}^2\mapsto\mathbb{R}$ are continuous functions, $\Delta w(t_i)=w(t_i^+)-w(t_i^-)$, $\Delta D_{0^+}^{\alpha-1}w(t_i)=D_{0^+}^{\alpha-1}w(t_i^+)-D_{0^+}^{\alpha-1}w(t_i^-)$, $w(t_i^+)$ and $w(t_i^-)$ denote the right and left limits of $w(t)$ at $t=t_i$, respectively, $a_i,b_i\in\mathbb{R}$ satisfy $\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha-2}=\sum\limits_{i=1}^{n}b_i\eta_i^{\alpha-2}=1$, $\sum\limits_{i=1}^{m}a_i\xi_i^{\alpha-1}=0$, $\sum\limits_{i=1}^{n}b_i\eta_i^{\alpha-1}=1$. The following claim was made in [36]:

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Claim 1. (see (2.8) on page 6 in [36]) the solution of

$$D_{0+}^{\alpha}x=z(t),0<t<1,\ \lim_{t\to 0}t^{2-\alpha}x(t)=\sum_{i=1}^{m}a_ix(\xi_i),\ x(1)=\sum_{i=1}^{n}b_ix(\eta_i),$$

$$\Delta x(t_i)=c_i,\ \Delta D_{0^+}^{\alpha-1}x(t_i)=d_i,i=1,2,\cdots,k$$

is as follows:

$$x(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}z(s)ds+\left[h_1+\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{i}d_j\right]t^{\alpha-1}$$
$$+\left[h_2+\sum_{j=1}^{i}c_jt_j^{2-\alpha}-\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{i}d_jt_j\right]t^{\alpha-2},t\in(t_i,t_{i+1}],i=0,1,2,\cdots,k, \tag{1.2}$$

where $h_1,h_2\in\mathbb{R}$ are constants.

In [37], Bai studied the existence of solutions of the following boundary value problem for impulsive fractional differential equation

$$\begin{cases}D_{0+}^{\alpha}x(t)=f(t,x(t),D_{0^+}^{\alpha-1}x(t)),t\in(0,1),t\neq t_i,i=1,2,\cdots,k,\\ D_{0^+}^{\alpha-1}x(0)=\sum_{i=1}^{m}a_iD_{0^+}^{\alpha-1}x(\xi_i),\ x(1)=\sum_{i=1}^{n}b_i\eta_i^{2-\alpha}x(\eta_i),\\ \Delta u(t_i)=I_i(u(t_i),D_{0^+}^{\alpha-1}x(t_i)),\ \Delta D_{0^+}^{\alpha-1}x(t_i)=J_i(u(t_i),D_{0^+}^{\alpha-1}x(t_i)),i=1,2,\cdots,k,\end{cases} \tag{1.3}$$

where $\alpha\in(1,2)$, D_{0+}^{*} is the Riemann-Liouville fractional derivative or order $*$, m,n,k are positive integers, $0<\xi_1<\cdots<\xi_m<1$, $0<\eta_1<\cdots<\eta_n<1$, $0<t_1<\cdots<t_k<1$, $f,g:[0,1]\times\mathbb{R}^2\mapsto\mathbb{R}$, $I_i,J_i:\mathbb{R}^2\mapsto\mathbb{R}$ are continuous functions, $\Delta w(t_i)=w(t_i^+)-w(t_i^-)$, $\Delta D_{0^+}^{\alpha-1}w(t_i)=D_{0^+}^{\alpha-1}w(t_i^+)-D_{0^+}^{\alpha-1}w(t_i^-)$, $w(t_i^+)$ and $w(t_i^-)$ denote the right and left limits of $w(t)$ at $t=t_i$, respectively, $a_i,b_i\in\mathbb{R}$ satisfy $\sum_{i=1}^{m}a_i=\sum_{i=1}^{n}b_i=1$. The following claim was made (see (2.7) on page 426 in [37]):

Claim 2. the solution of

$$D_{0+}^{\alpha}x(t)=z(t),t\in(0,1),t\neq t_i,i=1,2,\cdots,k,$$

$$D_{0^+}^{\alpha-1}x(0)=\sum_{i=1}^{m}a_iD_{0^+}^{\alpha-1}x(\xi_i),\ x(1)=\sum_{i=1}^{n}b_i\eta_i^{2-\alpha}x(\eta_i),$$

$$\Delta u(t_i)=c_i,\ \Delta D_{0^+}^{\alpha-1}x(t_i)=d_i,i=1,2,\cdots,k$$

is as follows:

$$x(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}z(s)ds+\left[h_1+\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{i}d_j\right]t^{\alpha-1}$$
$$+\left[h_2+\sum_{j=1}^{i}c_jt_j^{2-\alpha}+\frac{1}{\Gamma(\alpha)}\sum_{j=1}^{i}d_jt_j\right]t^{\alpha-2},t\in(t_i,t_{i+1}],i\in\mathbb{N}_0^m, \tag{1.4}$$

where $h_1,h_2\in\mathbb{R}$ are constants.

We find that both Claim 1 and Claim 2 are wrong. In fact, for ease expression, from (1.3) and (1.4), we re-write x by

$$x(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}z(s)ds+\sum_{j=0}^{i}H_jt^{\alpha-1}+\sum_{j=0}^{i}G_jt^{\alpha-2},t\in(t_i,t_{i+1}],i\in\mathbb{N}_0^m,$$

where H_i, G_i denote some constants. By direct computation, we have for $t \in (t_i, t_{i+1}]$ by Definition 2.2 in Section 2 that

$$D_{0^+}^{\alpha}x(t) = \frac{\left[\int_0^t (t-s)^{1-\alpha}x(s)ds\right]''}{\Gamma(2-\alpha)} = \frac{\left[\sum_{\chi=0}^{i-1}\int_{t_\chi}^{t_{\chi+1}}(t-s)^{1-\alpha}x(s)ds+\int_{t_i}^t (t-s)^{1-\alpha}x(s)ds\right]''}{\Gamma(2-\alpha)}$$

$$= \frac{\left[\sum_{\chi=0}^{i-1}\int_{t_\chi}^{t_{\chi+1}}(t-s)^{1-\alpha}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}z(u)du+H_\chi s^{\alpha-1}+G_\chi s^{\alpha-2}\right)ds\right]''}{\Gamma(2-\alpha)}$$

$$+\frac{\left[\int_{t_i}^t (t-s)^{1-\alpha}\left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}z(u)du+H_i s^{\alpha-1}+G_i s^{\alpha-2}\right)ds\right]''}{\Gamma(2-\alpha)}$$

$$= \frac{\left[\sum_{\chi=0}^{i-1}\int_{t_\chi}^{t_{\chi+1}}(t-s)^{1-\alpha}H_\chi s^{\alpha-1}ds\right]''+\left[\sum_{\chi=0}^{i-1}\int_{t_\chi}^{t_{\chi+1}}(t-s)^{1-\alpha}G_\chi s^{\alpha-2}ds\right]''}{\Gamma(2-\alpha)}$$

$$+\frac{\left[\int_0^t (t-s)^{1-\alpha}\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}z(u)duds\right]''+\left[H_i\int_{t_i}^t (t-s)^{1-\alpha}s^{\alpha-1}ds\right]''+G_i\left[\int_{t_i}^t (t-s)^{1-\alpha}s^{\alpha-2}ds\right]''}{\Gamma(2-\alpha)}$$

$$= \frac{\left[\sum_{\chi=0}^{i-1}H_\chi t\int_{\frac{t_\chi}{t}}^{\frac{t_{\chi+1}}{t}}(1-w)^{1-\alpha}w^{\alpha-1}dw\right]''+\left[\sum_{\chi=0}^{i-1}G_\chi\int_{\frac{t_\chi}{t}}^{\frac{t_{\chi+1}}{t}}(1-w)^{1-\alpha}w^{\alpha-2}dw\right]''}{\Gamma(2-\alpha)}$$

$$+\frac{\left[\int_0^t\int_u^t (t-s)^{1-\alpha}\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}dsz(u)du\right]''+\left[H_i t\int_{\frac{t_i}{t}}^1 (1-w)^{1-\alpha}w^{\alpha-1}dw\right]''+G_i\left[\int_{\frac{t_i}{t}}^1 (1-w)^{1-\alpha}w^{\alpha-2}dw\right]''}{\Gamma(2-\alpha)}$$

$$= \frac{\left[\sum_{\chi=0}^{i-1}H_\chi t\int_{\frac{t_\chi}{t}}^{\frac{t_{\chi+1}}{t}}(1-w)^{1-\alpha}w^{\alpha-1}dw\right]''+\left[\sum_{\chi=0}^{i-1}G_\chi\int_{\frac{t_\chi}{t}}^{\frac{t_{\chi+1}}{t}}(1-w)^{1-\alpha}w^{\alpha-2}dw\right]''}{\Gamma(2-\alpha)}$$

$$+\frac{\left[\int_0^t (t-u)\int_0^1 (1-w)^{1-\alpha}\frac{w^{\alpha-1}}{\Gamma(\alpha)}dwz(u)du\right]''+\left[H_i t\int_{\frac{t_i}{t}}^1 (1-w)^{1-\alpha}w^{\alpha-1}dw\right]''+G_i\left[\int_{\frac{t_i}{t}}^1 (1-w)^{1-\alpha}w^{\alpha-2}dw\right]''}{\Gamma(2-\alpha)}$$

$$= \frac{\left[\sum_{\chi=0}^{i-1}H_\chi t\int_{\frac{t_\chi}{t}}^{\frac{t_{\chi+1}}{t}}(1-w)^{1-\alpha}w^{\alpha-1}dw\right]''+\left[\sum_{\chi=0}^{i-1}G_\chi\int_{\frac{t_\chi}{t}}^{\frac{t_{\chi+1}}{t}}(1-w)^{1-\alpha}w^{\alpha-2}dw\right]''}{\Gamma(2-\alpha)}$$

$$+\frac{\left[H_i t\int_{\frac{t_i}{t}}^1 (1-w)^{1-\alpha}w^{\alpha-1}dw\right]''+G_i\left[\int_{\frac{t_i}{t}}^1 (1-w)^{1-\alpha}w^{\alpha-2}dw\right]''}{\Gamma(2-\alpha)}+\left[\int_0^t (t-u)z(u)du\right]''$$

$$= \frac{\left[\sum_{\chi=0}^{i-1}H_\chi t\int_{\frac{t_\chi}{t}}^{\frac{t_{\chi+1}}{t}}(1-w)^{1-\alpha}w^{\alpha-1}dw+H_i t\int_{\frac{t_i}{t}}^1 (1-w)^{1-\alpha}w^{\alpha-1}dw\right]''}{\Gamma(2-\alpha)}$$

$$+\frac{\left[\sum_{\chi=0}^{i-1}G_\chi\int_{\frac{t_\chi}{t}}^{\frac{t_{\chi+1}}{t}}(1-w)^{1-\alpha}w^{\alpha-2}dw+G_i\int_{\frac{t_i}{t}}^1 (1-w)^{1-\alpha}w^{\alpha-2}dw\right]''}{\Gamma(2-\alpha)}+z(t)$$

$$= z(t), t \in (t_i, t_{i+1}] \text{ if and only if } H_0 = H_1 = \cdots = H_i, G_0 = G_1 = \cdots = G_i (i \in \mathbb{N}_0^m).$$

It means that $c_i = d_i = 0$ for all $i = 1, 2, \cdots, k$. Hence x given by (1.2) and (1.4) does not satisfies $D_{0^+}^{\alpha}x(t) = z(t), t \in (t_i, t_{i+1}] (i \in \mathbb{N}_0^m)$ if $c_i, d_i \neq 0$. Hence Claim 1 and Claim 2 are wrong.

In [38], the authors studied the following $2m$-point boundary value problem for a coupled system of impulsive fractional differential equations at resonance

$$\begin{cases} D_{0^+}^{\alpha}u = f(t,v(t),D_{0^+}^{p}v(t)), 0<t<1, \\ D_{0^+}^{\beta}v = g(t,u(t),D_{0^+}^{q}u(t)), 0<t<1, \\ \\ \Delta u(t_i) = A_i(v(t_i),D_{0^+}^{p}v(t_i)), \Delta D_{0^+}^{q}u(t_i) = B_i(v(t_i),D_{0^+}^{p}v(t_i)), i=1,2,\cdots,k, \\ \Delta v(t_i) = C_i(u(t_i),D_{0^+}^{q}u(t_i)), \Delta D_{0^+}^{p}v(t_i) = D_i(u(t_i),D_{0^+}^{q}u(t_i)), i=1,2,\cdots,k, \\ \\ D_{0^+}^{\alpha-1}u(0) = \sum\limits_{i=1}^{m} a_i D_{0^+}^{\alpha-1}u(\xi_i),\ u(1) = \sum\limits_{i=1}^{m} b_i\eta_i^{2-\alpha}u(\eta_i), \\ D_{0^+}^{\beta-1}v(0) = \sum\limits_{i=1}^{m} c_i D_{0^+}^{\beta-1}v(\zeta_i),\ v(1) = \sum\limits_{i=1}^{m} d_i\theta_i^{2-\beta}v(\theta_i), \end{cases} \tag{1.5}$$

where $\alpha,\beta\in(1,2)$, $\alpha-q\geq 1$, $\beta-p\geq 1$, $0<\xi_1<\cdots<\xi_m<1$, $0<\eta_1<\cdots<\eta_m<1$, $0<\zeta_1<\cdots<\zeta_m<1$ and $0<\theta_1<\cdots<\theta_m<1$, $f,g:[0,1]\times\mathbb{R}^2\mapsto\mathbb{R}$ satisfy Carathéodory conditions, $A_i,B_i,C_i,D_i:\mathbb{R}\times\mathbb{R}\mapsto\mathbb{R}$, $\Delta w(t_i) = w(t_i^+)-w(t_i^-)$, $\Delta D_{0^+}^{r}w(t_i) = D_{0^+}^{r}w(t_i^+) - D_{0^+}^{r}w(t_i^-)$ with $w\in\{u,v\}$ and $r\in\{p,q\}$, $w(t_i^+)$ and $w(t_i^-)$ denote the right and left limits of $w(t)$ at $t=t_i$, respectively, and the fractional derivative is understood in the Riemann-Liouville sense. $k,m,a_i,b_i,c_i,d_i(i=1,2,\cdots,m)$ are fixed constant satisfying $\sum\limits_{i=1}^{m}a_i = \sum\limits_{i=1}^{m}b_i = \sum\limits_{i=1}^{m}c_i = \sum\limits_{i=1}^{m}d_i = 1$ and $\sum\limits_{i=1}^{m}b_i\eta_i = \sum\limits_{i=1}^{m}d_i\theta_i = 1$. This system happens to be at resonance in the sense that the associated linear homogeneous coupled system

$$\begin{cases} D_{0+}^{\alpha}u = 0,\ D_{0+}^{\beta}v = 0, 0<t<1, \\ \\ D_{0^+}^{\alpha-1}u(0) = \sum\limits_{i=1}^{m} a_i D_{0^+}^{\alpha-1}u(\xi_i),\ u(1) = \sum\limits_{i=1}^{m} b_i\eta_i^{2-\alpha}u(\eta_i), \\ D_{0^+}^{\beta-1}v(0) = \sum\limits_{i=1}^{m} c_i D_{0^+}^{\beta-1}v(\zeta_i),\ v(1) = \sum\limits_{i=1}^{m} d_i\theta_i^{2-\beta}v(\eta_i) \end{cases} \tag{1.6}$$

has $(u(t),v(t)) = (h_1t^{\alpha-1}+h_2t^{\alpha-2}, h_3t^{\beta-1}+h_4t^{\beta-2})(h_i\in\mathbb{R}, i=1,2,3,4)$ as nontrivial solution [38].

We find that BVP(1.5) is unsuitably proposed. The reason is as follows:

(i) By Corollary 2.1 in Section 2 in this paper, the piecewise continuous solutions of $D_{0^+}^{\alpha}u = h(t), t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^m$ is given by

$$x(t) = \sum_{j=0}^{i} c_j(t-t_j)^{\alpha-1} + \sum_{j=0}^{i} d_j(t-t_j)^{\alpha-2} + \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)}h(s)ds, t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^m \tag{1.7}$$

and for $q\in(0,\alpha-1)$ we have by Definition 2.2 in Section 2 for $t\in(t_i,t_{i+1}]$ that

$$\begin{aligned} D_{0^+}^{q}x(t) = {} & \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}\sum_{j=0}^{i} c_j(t-t_j)^{\alpha-q-1} + \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)}\sum_{j=0}^{i} d_j(t-t_j)^{\alpha-q-2} \\ & + \int_0^t \frac{(t-s)^{\alpha-q-2}}{\Gamma(\alpha-q)}h(s)ds, t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^m. \end{aligned} \tag{1.8}$$

From (1.7) and(1.8), we know that both x and $D_{0^+}^{q}x$ may be not right continuous at $t=t_i$, i.e., $\lim\limits_{t\to t_i^+}x(t)$ and $\lim\limits_{t\to t_i^+}D_{0^+}^{q}x(t)$ may be infinite. So the operators $\Delta x(t_i)$ and $\Delta D_{0^+}^{q}x(t_i)$ are unsuitable.

(ii) Even we consider the right continuous solutions of (1.5) on $(0,1]$, we get $d_i=0$ for all $i\in\mathbb{N}_1^m$ in (1.7). Then

$$\begin{aligned} x(t) &= \sum_{j=0}^{i} c_j(t-t_j)^{\alpha-1} + d_0t^{\alpha-2} + \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha)}h(s)ds, t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^m, \\ D_{0^+}^{q}x(t) &= \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}\sum_{j=0}^{i} c_j(t-t_j)^{\alpha-q-1} + \int_0^t \frac{(t-s)^{\alpha-q-2}}{\Gamma(\alpha-q)}h(s)ds, t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^m. \end{aligned} \tag{1.9}$$

It follows that $\Delta x(t_i) = \Delta D_{0^+}^{q}x(t_i) = 0$. The impulse functions A_i,B_i,C_i,D_i are redundant.

It is interesting to propose a suitable model of boundary value problems for impulsive fractional differential equations, to find a correct way to convert boundary value problems for impulsive fractional differential equations to integral equations and establish existence results for solutions of these kinds of problems.

Motivated by [36,37,38], in this paper we propose and study the following periodic type boundary value problems consisting of the following fractional differential equations

$$\begin{cases} D_{0^+}^{\alpha_1}u_1(t)-\lambda_1u_1(t)=p_1(t)f_1\left(t,u_2(t),D_{0^+}^{\alpha_2-1}u_2(t)\right),a.e.,t\in(t_k,t_{k+1}],k\in\mathbb{N}_0^m,\\ D_{0^+}^{\alpha_2}u_2(t)-\lambda_2u_2(t)=p_2(t)f_2\left(t,u_1(t),D_{0^+}^{\alpha_1-1}u_1(t)\right),a.e.,t\in(t_k,t_{k+1}],k\in\mathbb{N}_0^m,\end{cases} \tag{1.10}$$

subjected to the impulse effects

$$\begin{cases} \Delta I_{0^+}^{2-\alpha_1}u_1(t_k)=I_1(t_k,u_2(t_k),D_{0^+}^{\alpha_2-1}u_2(t_k)),k\in\mathbb{N}_1^m,\\ \Delta D_{0^+}^{\alpha_2-1}u_1(t_k)=J_1(t_k,u_2(t_k),D_{0^+}^{\alpha_2-1}u_2(t_k)),k\in\mathbb{N}_1^m,\\ \Delta I_{0^+}^{2-\alpha_2}u_2(t_k)=I_2(t_k,u_1(t_k),D_{0^+}^{\alpha_1-1}u_1(t_k)),k\in\mathbb{N}_1^m,\\ \Delta D_{0^+}^{\alpha_2-1}u_2(t_k)=J_4(t_k,u_1(t_k),D_{0^+}^{\alpha_1-1}u_1(t_k)),k\in\mathbb{N}_1^m,\end{cases} \tag{1.11}$$

and the multi-point boundary conditions

$$\begin{cases} I_{0^+}^{2-\alpha_1}u_1(0)=\sum_{i=1}^{m}a_{1i}I_{0^+}^{2-\alpha_1}u_1(\xi_i),\ D_{0^+}^{\alpha_1-1}u_1(1)=\sum_{i=0}^{m-1}b_{1i}D_{0^+}^{\alpha_1-1}u_1(\eta_i),\\ I_{0^+}^{2-\alpha_2}u_2(0)=\sum_{i=1}^{m}a_{2i}I_{0^+}^{2-\alpha_2}u_2(\xi_i),\ D_{0^+}^{\alpha_2-1}u_2(1)=\sum_{i=0}^{m-1}b_{2i}D_{0^+}^{\alpha_2-1}u_2(\eta_i)\end{cases} \tag{1.12}$$

where

(a) $\alpha_i\in(1,2)$, $\lambda_i\in\mathbb{R}(i=1,2)$, I_{0+}^{*} is the Riemann-Liouville fractional integral, see Definition 2.1, D_{0+}^{*} is the Riemann-Liouville type fractional derivative of order $*>0$ with the starting point 0, see Definition 2.2,

(b) m is a positive integer, $\mathbb{N}_0^m=\{0,1,2,\cdots,m\}$ and $\mathbb{N}_1^m=\{1,2,\cdots,m\}$, $0=t_0<t_1<t_2<\cdots<t_m<t_{m+1}=1$, $\xi_i\in(t_i,t_{i+1}]$ and $\eta_i\in(t_{i-1},t_i]$ for $i\in\mathbb{N}_1^m$, $a_{ij},b_{ij}\in\mathbb{R}(i\in\mathbb{N}_1^2,j\in\mathbb{N}_1^m)$ are constants.

(c) $p_i\in L^1(0,1)$,

(d) $f_1:(0,1)\times\mathbb{R}^2\mapsto\mathbb{R}$ is a α_2-Carathéodory function, f_2 a α_1-Carathódory function, see Definition 2.3, $I_1,J_1:\{t_i\}\times\mathbb{R}^2\mapsto\mathbb{R}$ are discrete α_2- Carathéodory functions, I_2,J_2 discrete α_1-Carathéodory functions, see Definition 2.4.

A pair of functions $u_1,u_2:(0,1]\mapsto\mathbb{R}$ is called a solution of BVP(1.10)-(1.12) if

$$u_i|_{(t_k,t_{k+1}]},\ D_{0^+}^{\alpha_i-1}u_i|_{(t_k,t_{k+1}]}\in C^0(t_k,t_{k+1}],k\in\mathbb{N}_0^m,i\in\mathbb{N}_1^2,$$

$$\lim_{t\to t_k^+}(t-t_k)^{2-\alpha_i}u_i(t),\ \lim_{t\to t_k^+}D_{0^+}^{\alpha_i-1}u_i(t)\text{ are finite},k\in\mathbb{N}_0^m,i\in\mathbb{N}_1^2$$

u_1,u_2satisfy all equations in (1.10)-(1.12) are satisfied.

The first purpose of this paper is to provide a new method to convert boundary value problems for impulsive fractional differential equations to integral systems. Then we apply the method to establish existence results for solutions of BVP(1.10)-(1.12) by using the Schauder's fixed point theorem [18] under some suitable assumptions.

The main features of our paper are as follows. Firstly, compared with known papers [20,36,37,21,26,32,35], we construct a new Banach space and establish existence results for solutions of BVP(1.10)-(1.12) (Theorem 3.1 and Theorem 3.2). Secondly, the boundary conditions in BVP(1.10)-(1.12) are different from known ones and new impulse effects models are proposed. Thirdly, the boundary conditions and impulse effects in BVP(1.10)-(1.12) imply that solutions obtained in this paper are continuous on $(t_i,t_{i+1}](i\in\mathbb{N}_0^m)$ but they may be unbounded on $(t_k,t_{k+1}](k\in\mathbb{N}_0^m)$. Fourthly, both $p_if_i:(t,x,y)\mapsto p_i(t)f_i(t,x,y)$ may be singular at $t=0,1$ while in known papers mentioned nonlinearities are supposed to be continuous. Fifthly, resonant conditions of BVP(1.10)-(1.12) are different from those defined in [36,37,38], the mistakes in these known papers are corrected (see Remark 2.1). Finally, the results in [17,41] are generalized.

The remainder of the paper is organized as follows: In Section 2, we present some preliminary results. In Section 3, the existence results for solutions of BVP(1.10)-(1.12) (see Theorem 3.1 and Theorem 3.2).

2 Preliminaries

In this section, we present some necessary definitions from the fractional calculus theory which can be found in the literatures [3,4,1,5,2]. Let $a<b$. Denote $L^1(a,b)$ the set of all integrable functions on (a,b), $C^0(a,b]$ the set of all continuous functions on $(a,b]$. For $\phi\in L^1(a,b)$, denote $||\phi||_1=\int_a^b|\phi(s)|ds$. For $\phi\in C^0[a,b]$, denote $||\phi||_0=\max_{t\in[a,b]}|\phi(t)|$. Let the Gamma and Beta functions $\Gamma(\alpha)$, $\mathbf{B}(p,q)$ and the Mitag-Leffler function $E_{\alpha,\delta}(x)$ be defined by

$$\Gamma(\alpha)=\int_0^{+\infty}x^{\alpha-1}e^{-x}dx,\ \ \mathbf{B}(p,q)=\int_0^1x^{p-1}(1-x)^{q-1}dx,\ \ \mathbf{E}_{\alpha,\delta}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(k\alpha+\delta)},\alpha,p,q,\delta>0.$$

Definition 2.1[1]. The left Riemann-Liouville fractional integral of order $\alpha>0$ of a function $h:(a,+\infty)\mapsto\mathbb{R}$ is given by $I_{a^+}^{\alpha}h(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}h(s)ds,t>a$ provided that the right-hand side exists.

Definition 2.2[1]. The left Riemann-Liouville fractional derivative of order $\alpha>0$ of a function $h:(a,+\infty)\mapsto\mathbb{R}$ is given by $D_{a^+}^{\alpha}h(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t\frac{h(s)}{(t-s)^{\alpha-n+1}}ds,t>a$ where $n-1<\alpha<n$, provided that the right-hand side exists.

Definition 2.3. Let $a\in(1,2)$. $h:(0,1)\times\mathbb{R}^2\mapsto\mathbb{R}$ is called a $a-$Carathéodory function if

(i) $t\mapsto h\left(t,(t-t_i)^{a-2}x_1,x_2\right)$ is integrable function on (t_i,t_{i+1}) for every $(x_1,x_2)\in\mathbb{R}^2$,

(ii) $(x_1,x_2)\mapsto h\left(t,(t-t_i)^{a-2}x_1,x_2\right)$ is continuous on $\mathbb{R}^2$ for each $t\in(t_i,t_{i+1}](i\in\mathbb{N}_0^m)$,

(iii) for each $r>0$, there exists $M_r>0$ such that $|x_i|\le r(i=1,2)$ imply that

$$\left|h\left(t,(t-t_i)^{a-2}x_1,x_2\right)\right|\le M_r,t\in(t_i,t_{i+1}),i\in\mathbb{N}_0^m.$$

Definition 2.4. Let $a\in(1,2)$. $I:\{t_i:i\in\mathbb{N}_1^m\}\times\mathbb{R}^2\mapsto\mathbb{R}$ is a discrete $a-$Carathéodory function if

(i)$(x_1,x_2)\mapsto I\left(t_i,(t_i-t_{i-1})^{a-2}x_1,x_2\right)$ is continuous on $\mathbb{R}^2$ for each $i\in\mathbb{N}_1^m$,

(ii) for each $r>0$, there exists $M_{I,r}>0$ such that $|x_i|\le r(i=1,2)$ imply that

$$\left|I\left(t_i,(t_i-t_{i-1})^{a-2}x_1,x_2\right)\right|\le M_{I,r},i\in\mathbb{N}_1^m.$$

Definition 2.5. Let n be a positive integer, $\alpha\in(n-1,n)$, $\lambda\in\mathbb{R}$, $h\in L^1(0,1)$. $x:(0,1]\mapsto\mathbb{R}$ is called a piecewise continuous solution of

$$D_{0^+}^{\alpha}x(t)-\lambda x(t)=h(t),a.e.,t\in(t_k,t_{k+1}],k\in\mathbb{N}_0^m \tag{2.1}$$

if $x|_{(t_k,t_{k+1}]}$ and $D_{0^+}^{\alpha-\chi}x|_{(t_k,t_{k+1}]}(\chi\in\mathbb{N}_1^{n-1})$ are continuous, the limits $\lim_{t\to t_k^+}(t-t_k)^{2-\alpha}x(t)(k\in\mathbb{N}_0^m)$ and $\lim_{t\to t_k^+}D_{0^+}^{\alpha-\chi}x(t)(k\in\mathbb{N}_0^m,\chi\in\mathbb{N}_1^{n-1}))$ exists (finite) and x satisfies (2.1).

Lemma 2.1. Suppose that $h\in L^1(0,1)$. Then x is a piecewise continuous solution of (2.1) if and only if there exist constants $c_{i,j}\in\mathbb{R}(i\in\mathbb{N}_1^n,j\in\mathbb{N}_0^m)$ such that

$$\begin{aligned}x(t)&=\int_0^t(t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})h(s)ds\\&+\sum_{j=0}^{k}\sum_{i=1}^{n}c_{i,j}(t-t_j)^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda(t-t_j)^{\alpha}),t\in(t_k,t_{k+1}],k\in\mathbb{N}_0^m.\end{aligned} \tag{2.2}$$

Furthermore we have

$$\begin{aligned}I_{0^+}^{n-\alpha}x(t)&=\int_0^t(t-s)^{n-1}\mathbf{E}_{\alpha,n}(\lambda(t-s)^{\alpha})h(s)ds\\&+\sum_{j=0}^{k}\sum_{i=1}^{n}c_{i,j}(t-t_j)^{n-i}\mathbf{E}_{\alpha,n-i+1}(\lambda(t-t_j)^{\alpha}),t\in(t_k,t_{k+1}],k\in\mathbb{N}_0^m\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}D_{0^+}^{\alpha-\chi}x(t)&=\sum_{j=0}^{k}\sum_{i=1}^{\kappa}c_{i,j}(t-t_j)^{\kappa-i}\mathbf{E}_{\alpha,\kappa-i+1}(\lambda(t-t_j)^{\alpha})\\&+\lambda\sum_{j=0}^{k}\sum_{i=\kappa+1}^{n}c_{i,j}(t-t_j)^{\alpha+\kappa-i}\mathbf{E}_{\alpha,\alpha+\kappa-i+1}(\lambda(t-t_j)^{\alpha})\\&+\int_0^t(t-u)^{\kappa-1}\mathbf{E}_{\alpha,\kappa}(\lambda(t-u)^{\alpha})h(u)du,t\in(t_k,t_{k+1}],k\in\mathbb{N}_0^m,\chi\in\mathbb{N}_1^{n-1}.\end{aligned} \tag{2.4}$$

Proof. We divide the proof into two steps:

Step 1. We prove that x is a piecewise continuous solution of (2.1) and satisfies (2.3) and(2.4) if x is given by (2.2). Since $\alpha \in (n-1,n), h \in L^1(0,1)$, we know that

$$t \mapsto \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^\alpha)h(s)ds,\ \ t \mapsto \int_0^t (t-u)^{\kappa-1}\mathbf{E}_{\alpha,\kappa}(\lambda(t-u)^\alpha)h(u)du(\kappa \in \mathbb{N}_1^{n-1})$$

are continuous on $[0,1]$.

By Definition 2.1 and (2.2), we have for $t \in (t_k,t_{k+1}]$ that

$$I_{0^+}^{n-\alpha}x(t) = \frac{\int_0^t (t-s)^{n-\alpha-1}x(s)ds}{\Gamma(n-\alpha)} = \frac{\sum\limits_{\tau=0}^{k-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}x(s)ds+\int_{t_k}^t(t-s)^{n-\alpha-1}x(s)ds}{\Gamma(n-\alpha)}$$

$$= \frac{\sum\limits_{\tau=0}^{k-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}\left[\int_0^s(s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)h(u)du+\sum\limits_{j=0}^{\tau}\sum\limits_{i=1}^{n}c_{i,j}(s-t_j)^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda(s-t_j)^\alpha)\right]ds}{\Gamma(n-\alpha)}$$

$$+\frac{\int_{t_k}^t(t-s)^{n-\alpha-1}\left[\int_0^s(s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)h(u)du+\sum\limits_{j=0}^{k}\sum\limits_{i=1}^{n}c_{i,j}(s-t_j)^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda(s-t_j)^\alpha)\right]ds}{\Gamma(n-\alpha)}$$

$$= \frac{\sum\limits_{\tau=0}^{k-1}\sum\limits_{j=0}^{\tau}\sum\limits_{i=1}^{n}c_{i,j}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}(s-t_j)^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda(s-t_j)^\alpha)ds}{\Gamma(n-\alpha)}$$

$$+\frac{\sum\limits_{j=0}^{k}\sum\limits_{i=1}^{n}c_{i,j}\int_{t_k}^t(t-s)^{n-\alpha-1}(s-t_j)^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda(s-t_j)^\alpha)ds}{\Gamma(n-\alpha)}$$

$$+\frac{\int_0^t(t-s)^{n-\alpha-1}\int_0^s(s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)h(u)duds}{\Gamma(n-\alpha)}$$

by changing the order of the sum and the integral

$$= \frac{\sum\limits_{j=0}^{k-1}\sum\limits_{\tau=j}^{i-1}\sum\limits_{i=1}^{n}c_{i,j}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}(s-t_j)^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda(s-t_j)^\alpha)ds}{\Gamma(n-\alpha)}$$

$$+\frac{\sum\limits_{j=0}^{k}\sum\limits_{i=1}^{n}c_{i,j}\int_{t_k}^t(t-s)^{n-\alpha-1}(s-t_j)^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda(s-t_j)^\alpha)ds}{\Gamma(n-\alpha)}$$

$$+\frac{\int_0^t\int_u^t(t-s)^{n-\alpha-1}(s-u)^{\alpha-1}\sum\limits_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha)}(s-u)^{\chi\alpha}dsh(u)du}{\Gamma(n-\alpha)}$$

$$= \frac{\sum\limits_{j=0}^{k}\sum\limits_{i=1}^{n}c_{i,j}\sum\limits_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha-i+1)}\int_{t_j}^t(t-s)^{n-\alpha-1}(s-t_j)^{\chi\alpha+\alpha-i}ds}{\Gamma(n-\alpha)}$$

$$+\frac{\sum\limits_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha)}\int_0^t\int_u^t(t-s)^{n-\alpha-1}(s-u)^{\chi\alpha+\alpha-1}dsh(u)du}{\Gamma(n-\alpha)} \text{ by } \frac{s-t_j}{t-t_j}=w,\ \frac{s-u}{t-u}=w$$

$$= \frac{\sum\limits_{j=0}^{k}\sum\limits_{i=1}^{n}c_{i,j}\sum\limits_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha-i+1)}(t-t_j)^{\chi\alpha+n-i}\int_0^1(1-w)^{n-\alpha-1}w^{\chi\alpha+\alpha-i}dw}{\Gamma(n-\alpha)}$$

$$+\frac{\sum\limits_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha)}\int_0^t(t-u)^{\chi\alpha+n-1}\int_0^1(1-w)^{n-\alpha-1}w^{\chi\alpha+\alpha-1}dwh(u)du}{\Gamma(n-\alpha)}$$

$$= \sum_{j=0}^{k}\sum_{i=1}^{n} c_{i,j} \sum_{\chi=0}^{\infty} \frac{\lambda^\chi}{\Gamma(\chi\alpha+n-i+1)}(t-t_j)^{\chi\alpha+n-i} + \sum_{\chi=0}^{\infty} \frac{\lambda^\chi}{\Gamma(\chi\alpha+n)} \int_0^t (t-u)^{\chi\alpha+n-1}h(u)du$$

$$= \int_0^t (t-s)^{n-1}\mathbf{E}_{\alpha,n}(\lambda(t-s)^\alpha)h(s)ds + \sum_{j=0}^{k}\sum_{i=1}^{n} c_{i,j}(t-t_j)^{n-i}\mathbf{E}_{\alpha,n-i+1}(\lambda(t-t_j)^\alpha).$$

Then (2.3) is proved.

From $\alpha \in (n-1,n)$ and $\kappa \in \mathbb{N}_0^{n-1}$, we know $\alpha-\kappa \in (n-1-\kappa, n-\kappa)$. For $t \in (t_k, t_{k+1}]$, by Definition 2.2, we have

$$D_{0^+}^{\alpha-\chi}x(t) = \frac{\left[\int_0^t (t-s)^{n-\alpha-1}x(s)ds\right]^{(n-\kappa)}}{\Gamma(n-\alpha)} = \frac{\left[\sum_{\tau=0}^{k-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}x(s)ds+\int_{t_k}^t (t-s)^{n-\alpha-1}x(s)ds\right]^{(n-\kappa)}}{\Gamma(n-\alpha)}$$

$$= \frac{\left[\sum_{\tau=0}^{k-1}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}\left(\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)h(u)du+\sum_{j=0}^{\tau}\sum_{i=1}^{n} c_{i,j}(s-t_j)^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda(s-t_j)^\alpha)\right)ds\right]^{(n-\kappa)}}{\Gamma(n-\alpha)}$$

$$+ \frac{\left[\int_{t_k}^t (t-s)^{n-\alpha-1}\left(\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)h(u)du+\sum_{j=0}^{k}\sum_{i=1}^{n} c_{i,j}(s-t_j)^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda(s-t_j)^\alpha)\right)ds\right]^{(n-\kappa)}}{\Gamma(n-\alpha)}$$

$$= \frac{\left[\sum_{\tau=0}^{k-1}\sum_{j=0}^{\tau}\sum_{i=1}^{n} c_{i,j}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}(s-t_j)^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda(s-t_j)^\alpha)ds\right]^{(n-\kappa)}}{\Gamma(n-\alpha)}$$

$$+ \frac{\left[\sum_{j=0}^{k}\sum_{i=1}^{n} c_{i,j}\int_{t_k}^t (t-s)^{n-\alpha-1}(s-t_j)^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda(s-t_j)^\alpha)ds\right]^{(n-\kappa)}}{\Gamma(n-\alpha)}$$

$$+ \frac{\left[\int_0^t (t-s)^{n-\alpha-1}\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)h(u)duds\right]^{(n-\kappa)}}{\Gamma(n-\alpha)}$$

by changing the order of sum and integral

$$= \frac{\left[\sum_{j=0}^{k-1}\sum_{\tau=j}^{k-1}\sum_{i=1}^{n} c_{i,j}\int_{t_\tau}^{t_{\tau+1}}(t-s)^{n-\alpha-1}(s-t_j)^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda(s-t_j)^\alpha)ds\right]^{(n-\kappa)}}{\Gamma(n-\alpha)}$$

$$+ \frac{\left[\sum_{j=0}^{k}\sum_{i=1}^{n} c_{i,j}\int_{t_k}^t (t-s)^{n-\alpha-1}(s-t_j)^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda(s-t_j)^\alpha)ds\right]^{(n-\kappa)}}{\Gamma(n-\alpha)}$$

$$+ \frac{\left[\int_0^t \int_u^t (t-s)^{n-\alpha-1}(s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^\alpha)dsh(u)du\right]^{(n-\kappa)}}{\Gamma(n-\alpha)}$$

$$= \frac{\left[\sum_{j=0}^{k}\sum_{i=1}^{n} c_{i,j}\int_{t_j}^t (t-s)^{n-\alpha-1}(s-t_j)^{\alpha-i}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha-i+1)}(s-t_j)^{\chi\alpha}ds\right]^{(n-\kappa)}}{\Gamma(n-\alpha)}$$

$$+ \frac{\left[\int_0^t \int_u^t (t-s)^{n-\alpha-1}(s-u)^{\alpha-1}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi}{\Gamma(\chi\alpha+\alpha)}(s-u)^{\chi\alpha}dsh(u)du\right]^{(n-\kappa)}}{\Gamma(n-\alpha)} \text{ by } \frac{s-t_j}{t-t_j}=w, \frac{s-u}{t-u}=w$$

$$= \frac{\left[\sum_{j=0}^{k}\sum_{i=1}^{n} c_{i,j}\sum_{\chi=0}^{\infty}\frac{\lambda^\chi(t-t_j)^{\chi\alpha+n-i}}{\Gamma(\chi\alpha+\alpha-i+1)}\int_0^1 (1-w)^{n-\alpha-1}w^{\chi\alpha+\alpha-i}dw\right]^{(n-\kappa)}}{\Gamma(n-\alpha)}$$

$$+ \frac{\left[\int_0^t \sum_{\chi=0}^{\infty}\frac{\lambda^\chi(t-u)^{\chi\alpha+n-1}}{\Gamma(\chi\alpha+\alpha)}\int_0^1 (1-w)^{n-\alpha-1}w^{\chi\alpha+\alpha-1}dwh(u)du\right]^{(n-\kappa)}}{\Gamma(n-\alpha)}$$

$$= \left[\sum_{j=0}^{k}\sum_{i=1}^{n} c_{i,j} \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi}(t-t_j)^{\chi\alpha+n-i}}{\Gamma(\chi\alpha+n-i+1)}\right]^{(n-\kappa)} + \left[\int_0^t \sum_{\chi=0}^{\infty} \frac{\lambda^{\chi}(t-u)^{\chi\alpha+n-1}}{\Gamma(\chi\alpha+n)} h(u)du\right]^{(n-\kappa)}.$$

If $\kappa \geq 1$, we get

$$D_{0^+}^{\alpha-\chi}x(t) = \sum_{j=0}^{k}\sum_{i=1}^{\kappa} c_{i,j}\frac{(t-t_j)^{\kappa-i}}{\Gamma(\kappa-i+1)} + \sum_{j=0}^{k}\sum_{i=1}^{n} c_{i,j}\sum_{\chi=1}^{\infty}\frac{\lambda^{\chi}(t-t_j)^{\chi\alpha+\kappa-i}}{\Gamma(\chi\alpha+\kappa-i+1)} + \int_0^t \sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}(t-u)^{\chi\alpha+\kappa-1}}{\Gamma(\chi\alpha+\kappa)}h(u)du$$

$$= \sum_{j=0}^{k}\sum_{i=1}^{\kappa} c_{i,j}(t-t_j)^{\kappa-i}\mathbf{E}_{\alpha,\kappa-i+1}(\lambda(t-t_j)^{\alpha})$$

$$+\lambda\sum_{j=0}^{k}\sum_{i=\kappa+1}^{n} c_{i,j}(t-t_j)^{\alpha+\kappa-i}\mathbf{E}_{\alpha,\alpha+\kappa-i+1}(\lambda(t-t_j)^{\alpha})$$

$$+\int_0^t (t-u)^{\kappa-1}\mathbf{E}_{\alpha,\kappa}(\lambda(t-u)^{\alpha})h(u)du.$$

Then (2.4) is proved.

If $\kappa = 0$, then we get

$$D_{0^+}^{\alpha}x(t) = \left[\sum_{j=0}^{k}\sum_{i=1}^{n} c_{i,j}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}(t-t_j)^{\chi\alpha+n-i}}{\Gamma(\chi\alpha+n-i+1)}\right]^{(n)} + \left[\int_0^t\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}(t-u)^{\chi\alpha+n-1}}{\Gamma(\chi\alpha+n)}h(u)du\right]^{(n)}$$

$$= h(t), t \in (t_k, t_{k+1}], k \in \mathbb{N}_0^m.$$

This is (2.1). Furthermore, we see from (2.3) and (2.4) that x satisfies that $I_{0^+}^{n-\alpha}x|_{(t_k,t_{k+1}]}$ and $D_{0^+}^{\alpha-\chi}x|_{(t_k,t_{k+1}]}(\chi \in \mathbb{N}_1^{n-1})$ are continuous, the limits

$$\lim_{t\to t_k^+}(t-t_k)^{2-\alpha}x(t)(k\in\mathbb{N}_0^m) \text{ and } \lim_{t\to t_k^+}D_{0^+}^{\alpha-\chi}x(t)(k\in\mathbb{N}_0^m, \chi\in\mathbb{N}_1^{n-1}))$$

exists (finite). So x is a piecewise continuous solution of (2.1). Step 1 is completed.

Step 2. We prove that x satisfies (2.2) if x is a piecewise continuous solution of (2.1).

Since $\alpha \in (n-1,n)$ and $h \in L^1(0,1)$, by (5.3) in [39], we have for $t \in (t_0,t_1]$ that there exist constants $c_{i,0} \in \mathbb{R}(i \in \mathbb{N}_1^n)$ such that

$$x(t) = \sum_{i=1}^{n} c_{i,0}t^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda t^{\alpha}) + \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})h(s)ds, t \in (0,t_1].$$

So we get the expression of x on $(0,t_1]$. This implies that (2.2) holds for $k=0$. We will apply the mathematical induction method to prove that (2.2) holds for all $k \in \mathbb{N}_0^m$. Suppose that (2.2) holds for $k = 0,1,2,\cdots,\nu$, i.e., there exist constants $c_{i,j} \in \mathbb{R}(i \in \mathbb{N}_1^n, j \in \mathbb{N}_0^{\nu})$

$$x(t) = \sum_{j=0}^{k}\sum_{i=1}^{n} c_{i,j}(t-t_j)^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda(t-t_j)^{\alpha})$$

$$+\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})h(s)ds, t \in (t_k,t_{k+1}], k = 0,1,\cdots,\nu.$$

In order to get the expression of x on $(t_{\nu+1},t_{\nu+2}]$, we suppose that

$$x(t) = \sum_{j=0}^{k}\sum_{i=1}^{n} c_{i,j}(t-t_j)^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda(t-t_j)^{\alpha})$$

$$+\int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})h(s)ds + \Phi(t), t \in (t_{\nu+1},t_{\nu+2}].$$

Then for $t \in (t_{v+1}, t_{v+2}]$, we have by Definition 2.2 that

$$D_{0^+}^{\alpha} x(t) = \frac{\left[\int_0^t (t-s)^{n-\alpha-1} x(s)ds\right]^{(n)}}{\Gamma(n-\alpha)} = \frac{\left[\sum_{\tau=0}^{v} \int_{t_\tau}^{t_{\tau+1}} (t-s)^{n-\alpha-1} x(s)ds + \int_{t_{v+1}}^{t} (t-s)^{n-\alpha-1} x(s)ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$= \frac{\left[\sum_{\tau=0}^{v} \int_{t_\tau}^{t_{\tau+1}} (t-s)^{n-\alpha-1} \left(\sum_{j=0}^{\tau}\sum_{i=1}^{n} c_{i,j}(s-t_j)^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda(s-t_j)^{\alpha}) + \int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})h(u)du\right)ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\int_{t_{v+1}}^{t} (t-s)^{n-\alpha-1} \left(\sum_{j=0}^{v+1}\sum_{i=1}^{n} c_{i,j}(s-t_j)^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda(s-t_j)^{\alpha}) + \int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})h(u)du + \Phi(s)\right)ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$= D_{t_{v+1}^+}^{\alpha}\Phi(t) + \frac{\left[\sum_{\tau=0}^{v}\sum_{j=0}^{\tau}\sum_{i=1}^{n} c_{i,j}\int_{t_\tau}^{t_{\tau+1}} (t-s)^{n-\alpha-1}(s-t_j)^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda(s-t_j)^{\alpha})ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\sum_{j=0}^{v+1}\sum_{i=1}^{n} c_{i,j}\int_{t_{v+1}}^{t} (t-s)^{n-\alpha-1}(s-t_j)^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda(s-t_j)^{\alpha})ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\int_0^t (t-s)^{n-\alpha-1}\left(\int_0^s (s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})h(u)du\right)ds\right]^{(n)}}{\Gamma(n-\alpha)} \text{ change the order of sum and integral}$$

$$= D_{t_{v+1}^+}^{\alpha}\Phi(t) + \frac{\left[\sum_{j=0}^{v}\sum_{\tau=j}^{v}\sum_{i=1}^{n} c_{i,j}\int_{t_\tau}^{t_{\tau+1}} (t-s)^{n-\alpha-1}(s-t_j)^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda(s-t_j)^{\alpha})ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\sum_{j=0}^{v+1}\sum_{i=1}^{n} c_{i,j}\int_{t_{v+1}}^{t} (t-s)^{n-\alpha-1}(s-t_j)^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda(s-t_j)^{\alpha})ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\int_0^t \int_u^t (t-s)^{n-\alpha-1}(s-u)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(s-u)^{\alpha})ds h(u)du\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$= D_{t_{v+1}^+}^{\alpha}\Phi(t) + \frac{\left[\sum_{j=0}^{v+1}\sum_{i=1}^{n} c_{i,j}\int_{t_j}^{t} (t-s)^{n-\alpha-1}(s-t_j)^{\alpha-i}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\alpha+\alpha-i+1)}(s-t_j)^{\chi\alpha}ds\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\int_0^t \int_u^t (t-s)^{n-\alpha-1}(s-u)^{\alpha-1}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\alpha+\alpha)}(s-u)^{\chi\alpha}ds h(u)du\right]^{(n)}}{\Gamma(n-\alpha)} \text{ use } \frac{s-t_j}{t-t_j} = w, \frac{s-u}{t-u} = w$$

$$= D_{t_{v+1}^+}^{\alpha}\Phi(t) + \frac{\left[\sum_{j=0}^{v+1}\sum_{i=1}^{n} c_{i,j}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\alpha+\alpha-i+1)}(t-t_j)^{\chi\alpha+n-i}\int_0^1 (1-w)^{n-\alpha-1}w^{\alpha-i+\chi\alpha}dw\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$+\frac{\left[\int_0^t \sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\alpha+\alpha)}(t-u)^{\chi\alpha+n-1}\int_0^1 (1-w)^{n-\alpha-1}w^{\chi\alpha+\alpha-1}dw h(u)du\right]^{(n)}}{\Gamma(n-\alpha)}$$

$$= D_{t_{v+1}^+}^{\alpha}\Phi(t) + \left[\sum_{j=0}^{v+1}\sum_{i=1}^{n} c_{i,j}\sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\alpha+n-i+1)}(t-t_j)^{\chi\alpha+n-i}\right]^{(n)}$$

$$+\left[\int_0^t \sum_{\chi=0}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\alpha+n)}(t-u)^{\chi\alpha+n-1}h(u)du\right]^{(n)}$$

$$= D_{t_{v+1}^+}^{\alpha}\Phi(t) + \sum_{j=0}^{v+1}\sum_{i=1}^{n} c_{i,j}\sum_{\chi=1}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\alpha-i+1)}(t-t_j)^{\chi\alpha-i}$$

$$+h(t) + \int_0^t \sum_{\chi=1}^{\infty}\frac{\lambda^{\chi}}{\Gamma(\chi\alpha)}(t-u)^{\chi\alpha-1}h(u)du$$

$$= D_{t_{v+1}^+}^{\alpha}\Phi(t) + h(t), t \in (t_{v+1}, t_{v+2}].$$

It follows that

$$\lambda x(t) + h(t) = D_{t_{v+1}^+}^{\alpha}\Phi(t) + h(t) + \lambda x(t) - \lambda\Phi(t), t \in (t_{v+1}, t_{v+2}].$$

Then $D_{t_{v+1}^+}^{\alpha}\Phi(t) - \lambda\Phi(t) = 0$ on $(t_{v+1}, t_{v+2}]$. By (5.3) in [39], we know that there exist constants $c_{i,v+1} \in \mathbb{R}$ such that $\Phi(t) = \sum_{i=1}^{n} c_{i,v+1}(t-t_{v+1})^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda(t-t_{v+1})^{\alpha})$ on $(t_{v+1}, t_{v+2}]$. Then the expression of x on $(t_{v+1}, t_{v+2}]$ is as follows

$$x(t) = \sum_{j=0}^{v+1}\sum_{i=1}^{n} c_{i,j}(t-t_j)^{\alpha-i}\mathbf{E}_{\alpha,\alpha-i+1}(\lambda(t-t_j)^{\alpha})$$

$$+ \int_0^t (t-s)^{\alpha-1}\mathbf{E}_{\alpha,\alpha}(\lambda(t-s)^{\alpha})h(s)ds, t \in (t_{v+1}, t_{v+2}].$$

So (2.2) holds for $k = v+1$. By the mathematical induction method, (2.2) is proved. The proof of Lemma 2.1 is complete. □

Corollary 2.1. Suppose that $h \in L^1(0,1)$. Then x is a piecewise continuous solution of

$$D_{0^+}^{\alpha}x(t) = h(t), t \in (t_i, t_{i+1}], i \in \mathbb{N}_0^m$$

if and only if there exist constants $c_{i,j} \in \mathbb{R}(i \in \mathbb{N}_1^n, j \in \mathbb{N}_0^m)$ such that

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + \sum_{j=0}^{k}\sum_{i=1}^{n} c_{i,j}\frac{(t-t_j)^{\alpha-i}}{\Gamma(\alpha-i+1)}, t \in (t_k, t_{k+1}], k \in \mathbb{N}_0^m.$$

Furthermore we have

$$I_{0^+}^{n-\alpha}x(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}h(s)ds + \sum_{j=0}^{k}\sum_{i=1}^{n} c_{i,j}\frac{(t-t_j)^{n-i}}{(n-i)!}, t \in (t_k, t_{k+1}], k \in \mathbb{N}_0^m$$

and

$$D_{0^+}^{\alpha-\chi}x(t) = \sum_{j=0}^{k}\sum_{i=1}^{\kappa} c_{i,j}\frac{(t-t_j)^{\kappa-i}}{(\kappa-i)!} + \lambda\sum_{j=0}^{k}\sum_{i=\kappa+1}^{n} c_{i,j}\frac{(t-t_j)^{\alpha+\kappa-i}}{\Gamma(\alpha+\kappa-i+1)}$$

$$+ \int_0^t \frac{(t-u)^{\kappa-1}}{(\kappa-1)!}h(u)du, t \in (t_k, t_{k+1}], k \in \mathbb{N}_0^m, \chi \in \mathbb{N}_1^{n-1}.$$

Proof. It follows from Lemma 2.1 directly when one chooses $\lambda = 0$ in (2.1). The proof is omitted. □

Remark 2.1. Different from (1.1) and (1.3), we consider the following boundary value problems for impulsive fractional differential equations

$$\begin{cases} D_{0^+}^{\alpha}x(t) = z(t), a.e., t \in (t_i, t_{i+1}], i \in \mathbb{N}_0^m, \\ \lim_{t\to 0} t^{2-\alpha}x(t) = \sum_{i=1}^{m} a_i x(\xi_i),\ x(1) = \sum_{i=1}^{n} b_i x(\eta_i), \\ \lim_{t\to t_i^+}(t-t_i)^{2-\alpha}x(t) = c_i,\ \Delta D_{0^+}^{\alpha-1}x(t_i) = d_i, i \in \mathbb{N}_1^m, \end{cases} \tag{2.5}$$

and

$$\begin{cases} D_{0^+}^{\alpha}x(t) = z(t), a.e., t \in (t_i, t_{i+1}], i \in \mathbb{N}_0^m, \\ D_{0^+}^{\alpha-1}x(0) = \sum_{i=1}^{m} a_i D_{0^+}^{\alpha-1}x(\xi_i),\ x(1) = \sum_{i=1}^{n} b_i \eta_i^{2-\alpha}x(\eta_i), \\ \lim_{t\to t_i^+}(t-t_i)^{2-\alpha}x(t) = c_i,\ \Delta D_{0^+}^{\alpha-1}x(t_i) = d_i, i \in \mathbb{N}_1^m, \end{cases} \tag{2.6}$$

where $\alpha\in(1,2)$, $0=t_0<t_1<\cdots<t_m<t_{m+1}=1$, $\xi_i\in(t_i,t_{i+1}]$, $\eta_i\in(t_{i-1},t_i](i\in\mathbb{N}_1^m)$, $a_i,b_i\in\mathbb{R}$ are constants, z satisfies some suitable assumptions. Then BVP(2.5) is equivalent to the integral equation

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}z(s)ds+A_0t^{\alpha-1}+B_0t^{\alpha-2}\\ &+\sum_{j=1}^{i}\frac{d_j}{\Gamma(\alpha)}(t-t_j)^{\alpha-1}+\sum_{j=1}^{i}c_j(t-t_j)^{\alpha-2}, t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^m, \end{aligned} \tag{2.7}$$

with A_0,B_0 satisfying

$$\begin{aligned} &\left(1-\sum_{i=1}^{m}a_i\xi_i^{\alpha-2}\right)B_0-\sum_{i=1}^{m}a_i\xi_i^{\alpha-1}A_0\\ &=\sum_{i=1}^{m}a_i\left(\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-1}}{\Gamma(\alpha)}z(s)ds+\sum_{j=1}^{i}\frac{d_j}{\Gamma(\alpha)}(\xi_i-t_j)^{\alpha-1}+\sum_{j=1}^{i}c_j(\xi_i-t_j)^{\alpha-2}\right),\\ &\left(1-\sum_{i=1}^{m}b_i\eta_i^{\alpha-2}\right)B_0+\left(1-\sum_{i=1}^{m}b_i\eta_i^{\alpha-1}\right)A_0\\ &=-\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}z(s)ds-\sum_{j=1}^{m}\frac{d_j}{\Gamma(\alpha)}(1-t_j)^{\alpha-1}-\sum_{j=1}^{m}c_j(1-t_j)^{\alpha-2}\\ &+\sum_{i=1}^{m}b_i\left(\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}z(s)ds+\sum_{j=1}^{i-1}\frac{d_j}{\Gamma(\alpha)}(\eta_i-t_j)^{\alpha-1}+\sum_{j=0}^{i-1}c_j(\eta_i-t_j)^{\alpha-2}\right). \end{aligned} \tag{2.8}$$

BVP(2.6) is equivalent to the integral equation (2.7) with A_0,B_0 satisfying

$$\begin{aligned} &\Gamma(\alpha)\left(1-\sum_{i=1}^{m}a_i\right)A_0=\sum_{i=1}^{m}a_i\left[\int_0^{\xi_i}z(s)ds+\Gamma(\alpha)\sum_{j=1}^{i}\frac{d_j}{\Gamma(\alpha)}\right],\\ &\left(1-\sum_{i=1}^{m}b_i\eta_i\right)A_0+\left(1-\sum_{i=1}^{m}b_i\right)B_0\\ &=-\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}z(s)ds-\sum_{j=1}^{m}\frac{d_j}{\Gamma(\alpha)}(1-t_j)^{\alpha-1}-\sum_{j=0}^{m}c_j(1-t_j)^{\alpha-2}\\ &+\sum_{i=1}^{m}b_i\eta_i^{2-\alpha}\left[\int_0^{\eta_i}\frac{(\eta_i-s)^{\alpha-1}}{\Gamma(\alpha)}z(s)ds+\sum_{j=1}^{i-1}\frac{d_j}{\gamma(\alpha)}(\eta_i-t_j)^{\alpha-1}+\sum_{j=0}^{i-1}c_j(\eta_i-t_j)^{\alpha-2}\right]. \end{aligned} \tag{2.9}$$

Proof. From Corollary 2.1 and $D_{0^+}^{\alpha}x(t)=f(t,x(t),D_{0^+}^{\alpha-1}x(t)), a.e., t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^m$, we know that there exist constants $A_i,B_i(i\in\mathbb{N}_0^m)$ such that

$$x(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}z(s)ds+\sum_{j=0}^{i}A_j(t-t_j)^{\alpha-1}+\sum_{j=0}^{i}B_j(t-t_j)^{\alpha-2}, t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^m \tag{2.10}$$

and

$$D_{0^+}^{\alpha-1}x(t)=\int_0^t z(s)ds+\Gamma(\alpha)\sum_{j=0}^{i}A_j, t\in(t_i,t_{i+1}], i\in\mathbb{N}_0^m. \tag{2.11}$$

(i) From $\lim_{t\to t_i^+}(t-t_i)^{2-\alpha}x(t)=c_i$, $\Delta D_{0^+}^{\alpha-1}x(t_i)=d_i, i\in\mathbb{N}_1^m$ and (2.10) and(2.11), we get $B_i=c_i$ and $A_i=\frac{d_i}{\Gamma(\alpha)}(i\in\mathbb{N}_1^m)$.

(ii) From $\lim_{t\to 0} t^{2-\alpha}x(t) = \sum_{i=1}^{m} a_i x(\xi_i)$, $x(1) = \sum_{i=1}^{n} b_i x(\eta_i)$ and (2.10) and the results in (i), we get

$$B_0 = \sum_{i=1}^{m} a_i \left[\int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} z(s)ds + \sum_{j=0}^{i} A_j(\xi_i - t_j)^{\alpha-1} + \sum_{j=0}^{i} B_j(\xi_i - t_j)^{\alpha-2} \right],$$

$$\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} z(s)ds + \sum_{j=0}^{m} A_j(1-t_j)^{\alpha-1} + \sum_{j=0}^{m} B_j(1-t_j)^{\alpha-2}$$

$$= \sum_{i=1}^{m} b_i \left[\int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-1}}{\Gamma(\alpha)} z(s)ds + \sum_{j=0}^{i-1} A_j(\eta_i - t_j)^{\alpha-1} + \sum_{j=0}^{i-1} B_j(\eta_i - t_j)^{\alpha-2} \right].$$

It is easy to convert this system to (2.8). One sees that (2.8) has solution A_0, B_0 if and only if

$$\text{Rank} \begin{pmatrix} 1 - \sum_{i=1}^{m} a_i \xi_i^{\alpha-2} & -\sum_{i=1}^{m} a_i \xi_i^{\alpha-1} \\ 1 - \sum_{i=1}^{m} b_i \eta_i^{\alpha-2} & 1 - \sum_{i=1}^{m} b_i \eta_i^{\alpha-1} \end{pmatrix}$$

$$= \text{Rank} \begin{pmatrix} 1 - \sum_{i=1}^{m} a_i \xi_i^{\alpha-2} & -\sum_{i=1}^{m} a_i \xi_i^{\alpha-1} & M \\ 1 - \sum_{i=1}^{m} b_i \eta_i^{\alpha-2} & 1 - \sum_{i=1}^{m} b_i \eta_i^{\alpha-1} & N \end{pmatrix},$$

where M, N are defined by

$$M = \sum_{i=1}^{m} a_i \left(\int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-1}}{\Gamma(\alpha)} z(s)ds + \sum_{j=1}^{i} \frac{d_j}{\Gamma(\alpha)} (\xi_i - t_j)^{\alpha-1} + \sum_{j=1}^{i} c_j(\xi_i - t_j)^{\alpha-2} \right),$$

$$N = \sum_{i=1}^{m} b_i \left(\int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-1}}{\Gamma(\alpha)} z(s)ds + \sum_{j=1}^{i-1} \frac{d_j}{\Gamma(\alpha)} (\eta_i - t_j)^{\alpha-1} + \sum_{j=0}^{i-1} c_j(\eta_i - t_j)^{\alpha-2} \right)$$

$$- \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} z(s)ds - \sum_{j=1}^{m} \frac{d_j}{\Gamma(\alpha)} (1-t_j)^{\alpha-1} - \sum_{j=1}^{m} c_j(1-t_j)^{\alpha-2}.$$

By substituting A_i, B_i into (2.10), we know that

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} z(s)ds + A_0 t^{\alpha-1} + B_0 t^{\alpha-2}$$

$$+ \sum_{j=1}^{i} \frac{d_j}{\Gamma(\alpha)} (t-t_j)^{\alpha-1} + \sum_{j=1}^{i} c_j(t-t_j)^{\alpha-2}, t \in (t_i, t_{i+1}], i \in \mathbb{N}_0^m,$$

where A_0, B_0 are the solutions of (2.8). Then (2.7)is proved.

Using (2.10), (2.11) and $D_{0^+}^{\alpha-1}x(0) = \sum_{i=1}^{m} a_i D_{0^+}^{\alpha-1} x(\xi_i)$, $x(1) = \sum_{i=1}^{n} b_i \eta_i^{2-\alpha} x(\eta_i)$. Similarly we can proof BVP(2.6) is equivalent to (2.7) with A_0, B_0 satisfying (2.9). The proof is complete. □

Lemma 2.2(Schauder's fixed point theorem) [18]. Let X be a Banach space and $T : X \mapsto X$ be a completely continuous operator. Suppose Ω is a nonempty open convex bounded subset of X and $T(\overline{\Omega}) \subseteq \overline{\Omega}$. Then there exists $x \in \overline{\Omega}$ such that $x = Tx$.

Let $\alpha \in (1,2)$. Choose

$$X_\alpha = \left\{ u:(0,1]\mapsto \mathbb{R} \;\; \begin{array}{c} u|_{(t_k,t_{k+1}]},\ D^{\alpha-1}u|_{(t_k,t_{k+1}]} \in C^0(t_k,t_{k+1}], k\in \mathbb{N}_0^m, \\ \text{the following limits exist:} \\ \lim_{t\to t_k^+}(t-t_k)^{2-\alpha}u(t),\ \lim_{t\to t_k^+} D_{0^+}^{\alpha-1}u(t), k\in \mathbb{N}_0^m \end{array} \right\}.$$

For $u \in X_\alpha$, define

$$||u|| =: ||u||_\alpha = \max\left\{ \sup_{t\in(t_k,t_{k+1}]} (t-t_k)^{2-\alpha}|u(t)|,\ \sup_{t\in(t_k,t_{k+1}]} |D_{0^+}^{\alpha-1}u(t)| \ : k\in \mathbb{N}_0^m \right\}.$$

Then X_α is a Banach space with the norm defined.

Denote $E = X_{\alpha_1}\times X_{\alpha_2}$. Define $||(x_1,x_2)|| = \max\{||x||_{\alpha_i} : i\in \mathbb{N}_1^2\}$. Then E is a Banach space.

Let $x_i \in X_{\alpha_i} (i\in \mathbb{N}_1^4)$. For ease expression, denote

$$f_{1x_2}(t) = f_1\left(t, x_2(t), D_{0^+}^{\alpha_2-1}x_2(t)\right),\ \ f_{2x_1}(t) = f_2(t, x_1(t), D_{0^+}^{\alpha_1-1}x_1(t)),$$

$$I_{1x_2}(t_k) = I_1(t_k, x_2(t_k), D_{0^+}^{\alpha_2-1}x_2(t_k)), k\in \mathbb{N}_1^m,$$

$$I_{2x_1}(t_k) = I_2(t_k, x_1(t_k), D_{0^+}^{lpha_1-1}x_1(t_k)), k\in \mathbb{N}_1^m,$$

$$J_{1x_2}(t_k) = J_1(t_k, x_2(t_k), D_{0^+}^{\alpha_2-1}x_2(t_k)), k\in \mathbb{N}_1^m,$$

$$J_{2x_1}(t_k) = J_2(t_k, x_1(t_k), D_{0^+}^{\alpha_1-1}x_1(t_k)), k\in \mathbb{N}_1^m.$$

Denote for $s = 1,2$ that

$$A_{11} = -\sum_{i=1}^m a_{1i}\xi_i,\ A_{12} = 1-\sum_{i=1}^m a_{1i},\ A_{21} = \mathbf{E}_{\alpha_1,1}(\lambda_1) - \sum_{i=0}^{m-1} b_{1i}\mathbf{E}_{\alpha_1,1}(\lambda_1\eta_i^{\alpha_1}),$$

$$A_{22} = \lambda_1\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1) - \lambda_1\sum_{i=0}^{m-1} b_{1i}\eta_i^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1\eta_i^{\alpha_1}),$$

$$B_{11} = -\sum_{i=1}^m a_{2i}\xi_i,\ B_{12} = 1-\sum_{i=1}^m a_{2i},\ B_{21} = \mathbf{E}_{\alpha_2,1}(\lambda_2) - \sum_{i=0}^{m-1} b_{2i}\mathbf{E}_{\alpha_2,1}(\lambda_2\eta_i^{\alpha_2}),$$

$$B_{22} = \lambda_2\mathbf{E}_{\alpha_2,\alpha_2}(\lambda_2) - \lambda_2\sum_{i=0}^{m-1} b_{2i}\eta_i^{\alpha_2-1}\mathbf{E}_{\alpha_2,\alpha_2}(\lambda_2\eta_i^{\alpha_2}),$$

$$\prod_1 = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix},\ \prod_2 = \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{vmatrix}.$$

Lemma 2.3. Suppose that (a)-(d) hold and $\prod_s \neq 0 (s=1,2)$. Then BVP(1.10)-(1.12) is equivalent to the following integral system

$$\begin{aligned} x_1(t) &= \int_0^t (t-s)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1}p_1(s)f_{1x_2}(s)ds + c_{0,1}t^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1 t^{\alpha_1}) \\ &+ \sum_{j=1}^k J_{1x_2}(t_j)(t-t_j)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(t-t_j)^{\alpha_1}) + d_{0,1}t^{\alpha_1-2}\mathbf{E}_{\alpha_1,\alpha_1-1}(\lambda_1 t^{\alpha_1}) \\ &+ \sum_{j=1}^k I_{1x_2}(t_j)(t-t_j)^{\alpha_1-2}\mathbf{E}_{\alpha_1,\alpha_1-1}(\lambda_1(t-t_j)^{\alpha_1}), t\in(t_k,t_{k+1}], k\in \mathbb{N}_0^m \end{aligned} \quad (2.13)$$

and

$$x_2(t) = \int_0^t (t-s)^{\alpha_2-1}\mathbf{E}_{\alpha_2,\alpha_2}(\lambda_2(t-s)^{\alpha_2}p_2(s)f_{2x_1}(s)ds + c_{0,2}t^{\alpha_2-1}\mathbf{E}_{\alpha_2,\alpha_2}(\lambda_2 t^{\alpha_2})$$
$$+\sum_{j=2}^{k} J_{2x_1}(t_j)(t-t_j)^{\alpha_2-1}\mathbf{E}_{\alpha_2,\alpha_2}(\lambda_2(t-t_j)^{\alpha_2}) + d_{0,2}t^{\alpha_2-2}\mathbf{E}_{\alpha_2,\alpha_2-1}(\lambda_2 t^{\alpha_2}) \tag{2.14}$$
$$+\sum_{j=1}^{k} I_{2x_1}(t_j)(t-t_j)^{\alpha_2-2}\mathbf{E}_{\alpha_2,\alpha_2-1}(\lambda_2(t-t_j)^{\alpha_2}), t\in(t_k,t_{k+1}], k\in \mathbb{N}_0^m,$$

where

$$c_{0,1} = \frac{1}{\Pi_1}\Bigg[A_{22}\Bigg(\sum_{i=1}^{m} a_{1i}\Bigg(\int_0^{\xi_i}(\xi_i-s)\mathbf{E}_{\alpha_1,2}(\lambda_1(\xi_i-s)^{\alpha_1})p_1(s)f_{1x_2}(s)ds$$
$$+\sum_{j=1}^{i} J_{1x_2}(t_j)(\xi_i-t_j)\mathbf{E}_{\alpha_1,2}(\lambda_1(\xi_i-t_j)^{\alpha_1}) + \sum_{j=1}^{i} I_{1x_2}(t_j)\mathbf{E}_{\alpha_1,1}(\lambda_1(\xi_i-t_j)^{\alpha_1})\Bigg)\Bigg)$$
$$-A_{12}\Bigg(\sum_{j=1}^{m-1}\sum_{i=j}^{m-1} b_{1i}\mathbf{E}_{\alpha_1,1}(\lambda_1(\eta_i-t_j)^{\alpha_1})J_{1x_2}(t_j) - \sum_{j=1}^{m}\mathbf{E}_{\alpha_1,1}(\lambda_1(1-t_j)^{\alpha_1})J_{1x_2}(t_j)$$
$$+\lambda_1\sum_{j=1}^{m-1}\sum_{i=j}^{m-1} b_{1i}(\eta_i-t_j)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(\eta_i-t_j)^{\alpha_1})I_{1x_2}(t_j) \tag{2.15}$$
$$-\lambda_1\sum_{j=1}^{m}(1-t_j)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(1-t_j)^{\alpha_1})I_{1x_2}(t_j)$$
$$-\int_0^1 \mathbf{E}_{\alpha_1,1}(\lambda_1(1-u)^{\alpha_1})p_1(u)f_{1x_2}(u)du$$
$$+\int_0^1 \sum_{u<\eta_i\le\eta_{m-1}} b_{1i}\mathbf{E}_{\alpha_1,1}(\lambda_1(\eta_i-u)^{\alpha_1})p_1(u)f_{1x_2}(u)du\Bigg)\Bigg]$$

$$c_{0,2} = \frac{1}{\Pi_2}\Bigg[B_{22}\Bigg(\sum_{i=1}^{m} a_{2i}\Bigg(\int_0^{\xi_i}(\xi_i-s)\mathbf{E}_{\alpha_2,2}(\lambda_2(\xi_i-s)^{\alpha_2})p_2(s)f_{2x_1}(s)ds$$
$$+\sum_{j=1}^{i} J_{2x_1}(t_j)(\xi_i-t_j)\mathbf{E}_{\alpha_2,2}(\lambda_2(\xi_i-t_j)^{\alpha_2}) + \sum_{j=1}^{i} I_{2x_1}(t_j)\mathbf{E}_{\alpha_2,1}(\lambda_2(\xi_i-t_j)^{\alpha_2})\Bigg)\Bigg)$$
$$-B_{12}\Bigg(\sum_{j=1}^{m-1}\sum_{i=j}^{m-1} b_{2i}\mathbf{E}_{\alpha_2,1}(\lambda_2(\eta_i-t_j)^{\alpha_2})J_{2x_1}(t_j) - \sum_{j=1}^{m}\mathbf{E}_{\alpha_2,1}(\lambda_2(1-t_j)^{\alpha_2})J_{2x_1}(t_j)$$
$$+\lambda_2\sum_{j=1}^{m-1}\sum_{i=j}^{m-1} b_{2i}(\eta_i-t_j)^{\alpha_2-1}\mathbf{E}_{\alpha_2,\alpha_2}(\lambda_2(\eta_i-t_j)^{\alpha_2})I_{2x_1}(t_j) \tag{2.16}$$
$$-\lambda_2\sum_{j=1}^{m}(1-t_j)^{\alpha_2-1}\mathbf{E}_{\alpha_2,\alpha_2}(\lambda_2(1-t_j)^{\alpha_2})I_{2x_1}(t_j)$$
$$-\int_0^1 \mathbf{E}_{\alpha_2,1}(\lambda_2(1-u)^{\alpha_2})p_2(u)f_{2x_1}(u)du$$
$$+\int_0^1 \sum_{u<\eta_i\le\eta_{m-1}} b_{2i}\mathbf{E}_{\alpha_2,1}(\lambda_2(\eta_i-u)^{\alpha_2})p_2(u)f_{2x_1}(u)du\Bigg)\Bigg]$$

$$
\begin{aligned}
d_{0,1} = \frac{1}{\Pi_1}\Bigg[& A_{11}\Bigg(\sum_{j=1}^{m-1}\sum_{i=j}^{m-1} b_{1i}\mathbf{E}_{\alpha_1,1}(\lambda_1(\eta_i-t_j)^{\alpha_1})J_{1x_2}(t_j) \\
& -\sum_{j=1}^{m}\mathbf{E}_{\alpha_1,1}(\lambda_1(1-t_j)^{\alpha_1})J_{1x_2}(t_j)-\int_0^1\mathbf{E}_{\alpha_1,1}(\lambda_1(1-u)^{\alpha_1})p_1(u)f_{1x_2}(u)du \\
& +\lambda_1\sum_{j=1}^{m-1}\sum_{i=j}^{m-1} b_{1i}(\eta_i-t_j)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(\eta_i-t_j)^{\alpha_1})I_{1x_2}(t_j) \\
& +\int_0^1\sum_{u<\eta_i\le\eta_{m-1}} b_{1i}\mathbf{E}_{\alpha_1,1}(\lambda_1(\eta_i-u)^{\alpha_1})p_1(u)f_{1x_2}(u)du \\
& -\lambda_1\sum_{j=1}^{m}(1-t_j)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(1-t_j)^{\alpha_1})I_{1x_2}(t_j)\Bigg) \\
& -A_{21}\Bigg(\sum_{i=1}^{m} a_{1i}\Bigg(\int_0^{\xi_i}(\xi_i-s)\mathbf{E}_{\alpha_1,2}(\lambda_1(\xi_i-s)^{\alpha_1})p_1(s)f_{1x_2}(s)ds \\
& +\sum_{j=1}^{i} J_{1x_2}(t_j)(\xi_i-t_j)\mathbf{E}_{\alpha_1,2}(\lambda_1(\xi_i-t_j)^{\alpha_1})+\sum_{j=1}^{i} I_{1x_2}(t_j)\mathbf{E}_{\alpha_1,1}(\lambda_1(\xi_i-t_j)^{\alpha_1})\Bigg)\Bigg)\Bigg],
\end{aligned}
\tag{2.17}
$$

$$
\begin{aligned}
d_{0,2} = \frac{1}{\Pi_2}\Bigg[& B_{11}\Bigg(\sum_{j=1}^{m-1}\sum_{i=j}^{m-1} b_{2i}\mathbf{E}_{\alpha_2,1}(\lambda_2(\eta_i-t_j)^{\alpha_2})J_{2x_1}(t_j) \\
& -\sum_{j=1}^{m}\mathbf{E}_{\alpha_2,1}(\lambda_2(1-t_j)^{\alpha_2})J_{2x_1}(t_j)-\int_0^1\mathbf{E}_{\alpha_2,1}(\lambda_2(1-u)^{\alpha_2})p_2(u)f_{2x_1}(u)du \\
& +\lambda_2\sum_{j=1}^{m-1}\sum_{i=j}^{m-1} b_{2i}(\eta_i-t_j)^{\alpha_2-1}\mathbf{E}_{\alpha_2,\alpha_2}(\lambda_2(\eta_i-t_j)^{\alpha_2})I_{2x_1}(t_j) \\
& +\int_0^1\sum_{u<\eta_i\le\eta_{m-1}} b_{2i}\mathbf{E}_{\alpha_2,1}(\lambda_2(\eta_i-u)^{\alpha_2})p_2(u)f_{2x_1}(u)du \\
& -\lambda_2\sum_{j=1}^{m}(1-t_j)^{\alpha_2-1}\mathbf{E}_{\alpha_2,\alpha_2}(\lambda_2(1-t_j)^{\alpha_2})I_{2x_1}(t_j)\Bigg) \\
& -B_{21}\Bigg(\sum_{i=1}^{m} a_{2i}\Bigg(\int_0^{\xi_i}(\xi_i-s)\mathbf{E}_{\alpha_2,2}(\lambda_2(\xi_i-s)^{\alpha_2})p_2(s)f_{2x_1}(s)ds \\
& +\sum_{j=1}^{i} J_{2x_1}(t_j)(\xi_i-t_j)\mathbf{E}_{\alpha_2,2}(\lambda_2(\xi_i-t_j)^{\alpha_2})+\sum_{j=1}^{i} I_{2x_1}(t_j)\mathbf{E}_{\alpha_2,1}(\lambda_2(\xi_i-t_j)^{\alpha_2})\Bigg)\Bigg)\Bigg].
\end{aligned}
\tag{2.18}
$$

Proof. From Lemma 2.1, we have that there exist constants $c_{j,\chi}, d_{j,\chi} \in \mathbb{R}(\chi \in \mathbb{N}_1^2, j \in \mathbb{N}_0^m)$ such that

$$
\begin{aligned}
x_1(t) = & \int_0^t (t-s)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1}p_1(s)f_{1x_2}(s)ds \\
& +\sum_{j=0}^{k} c_{j,1}(t-t_j)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(t-t_j)^{\alpha_1}) \\
& +\sum_{j=0}^{k} d_{j,1}(t-t_j)^{\alpha_1-2}\mathbf{E}_{\alpha_1,\alpha_1-1}(\lambda_1(t-t_j)^{\alpha_1}), t\in(t_k,t_{k+1}], k\in\mathbb{N}_0^m.
\end{aligned}
\tag{2.19}
$$

Furthermore we have

$$I_{0^+}^{2-\alpha_1}x(t) = \int_0^t (t-s)\mathbf{E}_{\alpha_1,2}(\lambda_1(t-s)^{\alpha_1}p_1(s)f_{1x_2}(s)ds$$
$$+\sum_{j=0}^{k} c_{j,1}(t-t_j)\mathbf{E}_{\alpha_1,2}(\lambda_1(t-t_j)^{\alpha_1}) \tag{2.20}$$
$$+\sum_{j=0}^{k} d_{j,1}\mathbf{E}_{\alpha_1,1}(\lambda_1(t-t_j)^{\alpha_1}), t\in(t_k,t_{k+1}], k\in \mathbb{N}_0^m$$

and

$$D_{0^+}^{\alpha-1}x_1(t) = \sum_{j=0}^{k} c_{j,1}\mathbf{E}_{\alpha_1,1}(\lambda_1(t-t_j)^{\alpha_1})$$
$$+\lambda_1\sum_{j=0}^{k} d_{j,1}(t-t_j)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(t-t_j)^{\alpha_1}) \tag{2.21}$$
$$+\int_0^t \mathbf{E}_{\alpha_1,1}(\lambda_1(t-u)^{\alpha_1})p_1(u)f_{1x_2}(u)du, t\in(t_k,t_{k+1}], k\in\mathbb{N}_0^m, \chi\in\mathbb{N}_1^{n-1}.$$

(i) From (2.20) and (2.21) and

$$\Delta I_{0^+}^{2-\alpha_1}x_1(t_k) = I_{1x_2}(t_k),\ \Delta D_{0^+}^{\alpha_1-1}x_1(t_k) = J_{1x_2}(t_k),\ k\in\mathbb{N}_1^m,$$

we get

$$d_{k,1} = I_{1x_2}(t_k),\ c_{k,1} = J_{1x_2}(t_k),\ k\in\mathbb{N}_1^m. \tag{2.22}$$

(ii) From (2.20), (2.21), (2.22) and

$$I_{0^+}^{2-\alpha_1}u_1(0) = \sum_{i=1}^{m} a_{1i}I_{0^+}^{2-\alpha_1}u_1(\xi_i),\ \ D_{0^+}^{\alpha_1-1}u_1(1) = \sum_{i=0}^{m-1} b_{1i}D_{0^+}^{\alpha_1-1}u_1(\eta_i),$$

we have

$$-\sum_{i=1}^{m} a_{1i}\xi_i c_{0,1} + \left(1-\sum_{i=1}^{m} a_{1i}\right)d_{0,1} = \sum_{i=1}^{m} a_{1i}\left[\int_0^{\xi_i}(\xi_i-s)\mathbf{E}_{\alpha_1,2}(\lambda_1(\xi_i-s)^{\alpha_1})p_1(s)f_{1x_2}(s)ds\right.$$
$$\left.+\sum_{j=1}^{i} J_{1x_2}(t_j)(\xi_i-t_j)\mathbf{E}_{\alpha_1,2}(\lambda_1(\xi_i-t_j)^{\alpha_1}) + \sum_{j=1}^{i} I_{1x_2}(t_j)\mathbf{E}_{\alpha_1,1}(\lambda_1(\xi_i-t_j)^{\alpha_1})\right], \tag{2.23}$$

and

$$\sum_{j=0}^{m} c_{j,1}\mathbf{E}_{\alpha_1,1}(\lambda_1(1-t_j)^{\alpha_1}) + \lambda_1\sum_{j=0}^{m} d_{j,1}(1-t_j)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(1-t_j)^{\alpha_1})$$
$$+\int_0^1 \mathbf{E}_{\alpha_1,1}(\lambda_1(1-u)^{\alpha_1})p_1(u)f_{1x_2}(u)du = \sum_{i=0}^{m-1} b_{1i}\left[\sum_{j=0}^{i} c_{j,1}\mathbf{E}_{\alpha_1,1}(\lambda_1(\eta_i-t_j)^{\alpha_1})\right.$$
$$\left.+\lambda_1\sum_{j=0}^{i} d_{j,1}(\eta_i-t_j)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(\eta_i-t_j)^{\alpha_1}) + \int_0^{\eta_i}\mathbf{E}_{\alpha_1,1}(\lambda_1(\eta_i-u)^{\alpha_1})p_1(u)f_{1x_2}(u)du\right].$$

It follows that

$$\left[\mathbf{E}_{\alpha_1,1}(\lambda_1)-\sum_{i=0}^{m-1}b_{1i}\mathbf{E}_{\alpha_1,1}(\lambda_1\eta_i^{\alpha_1})\right]c_{0,1}$$
$$+\left[\lambda_1\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1)-\lambda_1\sum_{i=0}^{m-1}b_{1i}\eta_i^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1\eta_i^{\alpha_1})\right]d_{0,1}$$
$$=\sum_{j=1}^{m-1}\sum_{i=j}^{m-1}b_{1i}\mathbf{E}_{\alpha_1,1}(\lambda_1(\eta_i-t_j)^{\alpha_1})J_{1x_2}(t_j)-\sum_{j=1}^{m}\mathbf{E}_{\alpha_1,1}(\lambda_1(1-t_j)^{\alpha_1})J_{1x_2}(t_j)$$
$$+\lambda_1\sum_{j=1}^{m-1}\sum_{i=j}^{m-1}b_{1i}(\eta_i-t_j)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(\eta_i-t_j)^{\alpha_1})I_{1x_2}(t_j) \tag{2.24}$$
$$-\lambda_1\sum_{j=1}^{m}(1-t_j)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(1-t_j)^{\alpha_1})I_{1x_2}(t_j)$$
$$+\int_0^1\sum_{u<\eta_i\le\eta_{m-1}}b_{1i}\mathbf{E}_{\alpha_1,1}(\lambda_1(\eta_i-u)^{\alpha_1})p_1(u)f_{1x_2}(u)du$$
$$-\int_0^1\mathbf{E}_{\alpha_1,1}(\lambda_1(1-u)^{\alpha_1})p_1(u)f_{1x_2}(u)du.$$

It follows from (2.23) and (2.24) that $c_{0,1},d_{0,1}$ satisfy (2.15) and (2.16). Substituting $c_{0,1},d_{0,1}$ into (2.19), we get that x_1 satisfies (2.13). Similarly we can get the expression of x_2 defined by (2.14) and $c_{0,2},d_{0,2}$ satisfy (2.17) and (2.18).

Now we suppose that x_1 satisfies (2.13) and x_2 satisfies (2.14). We can prove that (x_1,x_2) is a solution of BVP(1.10)-(1.12) by direct computation. The proof is completed. □

For $(x_1,x_2)\in E$, let $c_{0,1},d_{0,1},c_{0,2},d_{0,2}$ be defined by (2.15)-(2.18), define $T(x_1,x_2)$ by $T(x_1,x_2)(t)=((T_1x_2)(t),(T_2x_1)(t))$ with

$$(T_1x_2)(t)=\int_0^t(t-s)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1})p_1(s)f_{1x_2}(s)ds+c_{0,1}t^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1t^{\alpha_1})$$
$$+\sum_{j=1}^{k}J_{1x_2}(t_j)(t-t_j)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(t-t_j)^{\alpha_1})+d_{0,1}t^{\alpha_1-2}\mathbf{E}_{\alpha_1,\alpha_1-1}(\lambda_1t^{\alpha_1})$$
$$+\sum_{j=1}^{k}I_{1x_2}(t_j)(t-t_j)^{\alpha_1-2}\mathbf{E}_{\alpha_1,\alpha_1-1}(\lambda_1(t-t_j)^{\alpha_1}),t\in(t_k,t_{k+1}],k\in\mathbb{N}_0^m$$

and

$$(T_2x_1)(t)=\int_0^t(t-s)^{\alpha_2-1}\mathbf{E}_{\alpha_2,\alpha_2}(\lambda_2(t-s)^{\alpha_2})p_2(s)f_{2x_1}(s)ds+c_{0,2}t^{\alpha_2-1}\mathbf{E}_{\alpha_2,\alpha_2}(\lambda_2t^{\alpha_2})$$
$$+\sum_{j=2}^{k}J_{2x_1}(t_j)(t-t_j)^{\alpha_2-1}\mathbf{E}_{\alpha_2,\alpha_2}(\lambda_2(t-t_j)^{\alpha_2})+d_{0,2}t^{\alpha_2-2}\mathbf{E}_{\alpha_2,\alpha_2-1}(\lambda_2t^{\alpha_2})$$
$$+\sum_{j=1}^{k}I_{2x_1}(t_j)(t-t_j)^{\alpha_2-2}\mathbf{E}_{\alpha_2,\alpha_2-1}(\lambda_2(t-t_j)^{\alpha_2}),t\in(t_k,t_{k+1}],k\in\mathbb{N}_0^m,$$

Lemma 2.4. Suppose that (a)-(d) hold and $\prod_1\neq0,\prod_2\neq0$. Then $T:E\mapsto E$ is well defined and is completely continuous, (x,y) is a solution of BVP(1.10)-(1.12) if and only if $(x,y)=T(x,y)$.
Proof. The proof is standard and we omit the details. The readers should refer [15]. □

3 Main Results

In this section, we prove the existence of solutions of BVP(1.10)-(1.12) under the assumptions in Lemma 2.4 which imply that BVP(1.10)-(1.12) is non-resonant. Suppose that $\sigma_j,\tau_j\geq0(j=1,2)$ are constants. We need the following assumptions:

(H1) there exist nonnegative constants $A_j, B_j (j=1,2)$ and two functions ϕ_0, ψ_0 such that $p_1\psi_0$ is a α_1-integrable function and $p_2\phi_0$ a α_2-integrable function and

$$\left|f_1\left(t, \frac{y_1}{(t-t_i)^{2-\alpha_2}}, y_2\right) - \psi_0(t)\right| \le \sum_{j=1}^{2} A_j |y_j|^{\sigma_j},\ t \in (t_i, t_{i+1}),\ y_j \in \mathbb{R}(j=1,2), i \in \mathbb{N}_0^m,$$

$$\left|f_2\left(t, \frac{x_1}{(t-t_i)^{2-\alpha_1}}, x_2\right) - \phi_0(t)\right| \le \sum_{j=1}^{2} B_j |x_j|^{\tau_j},\ t \in (t_i, t_{i+1}),\ x_j \in \mathbb{R}(j=1,2), i \in \mathbb{N}_0^m,$$

(H2) there exist constants $I_{1i}, J_{1i}, I_{2i}, J_{2i} (i \in \mathbb{N}_1^m)$, $C_j, D_j, E_j, F_j \ge 0 (j=1,2)$ such that

$$\left|I_1\left(t_i, \frac{y_1}{(t_i-t_{i-1})^{2-\alpha_2}}, y_2\right) - I_{1i}\right| \le \sum_{j=1}^{2} C_j |y_j|^{\sigma_j}, i \in \mathbb{N}_1^m,$$

$$\left|J_1\left(t, \frac{y_1}{(t_i-t_{i-1})^{2-\alpha_2}}, y_2\right) - J_{1i}\right| \le \sum_{j=1}^{2} D_j |y_j|^{\sigma_j}, i \in \mathbb{N}_1^m,$$

$$\left|I_2\left(t_i, \frac{x_1}{(t_i-t_{i-1})^{2-\alpha_1}}, x_2\right) - I_{2i}\right| \le \sum_{j=1}^{2} E_j |x_j|^{\tau_j}, i \in \mathbb{N}_1^m,$$

$$\left|J_2\left(t, \frac{x_1}{(t_i-t_{i-1})^{2-\alpha_1}}, x_2\right) - J_{2i}\right| \le \sum_{j=1}^{2} F_j |x_j|^{\tau_j}, i \in \mathbb{N}_1^m.$$

Denote

$$\overline{c}_{0,1} = \frac{1}{\Pi_1}\Bigg[A_{22}\Bigg(\sum_{i=1}^{m} a_{1i}\Bigg(\int_0^{\xi_i}(\xi_i-s)\mathbf{E}_{\alpha_1,2}(\lambda_1(\xi_i-s)^{\alpha_1})p_1(s)\psi_0(s)ds$$

$$+\sum_{j=1}^{i} J_{1j}(\xi_i-t_j)\mathbf{E}_{\alpha_1,2}(\lambda_1(\xi_i-t_j)^{\alpha_1}) + \sum_{j=1}^{i} I_{1j}\mathbf{E}_{\alpha_1,1}(\lambda_1(\xi_i-t_j)^{\alpha_1})\Bigg)\Bigg)$$

$$-A_{12}\Bigg(\sum_{j=1}^{m-1}\sum_{i=j}^{m-1} b_{1i}\mathbf{E}_{\alpha_1,1}(\lambda_1(\eta_i-t_j)^{\alpha_1})J_{1j} - \sum_{j=1}^{m}\mathbf{E}_{\alpha_1,1}(\lambda_1(1-t_j)^{\alpha_1})J_{1j}$$

$$+\lambda_1\sum_{j=1}^{m-1}\sum_{i=j}^{m-1} b_{1i}(\eta_i-t_j)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(\eta_i-t_j)^{\alpha_1})I_{1j}$$

$$-\lambda_1\sum_{j=1}^{m}(1-t_j)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(1-t_j)^{\alpha_1})I_{1j}$$

$$-\int_0^1 \mathbf{E}_{\alpha_1,1}(\lambda_1(1-s)^{\alpha_1})p_1(u)\psi_0(s)ds$$

$$+\int_0^1 \sum_{s<\eta_i\le\eta_{m-1}} b_{1i}\mathbf{E}_{\alpha_1,1}(\lambda_1(\eta_i-s)^{\alpha_1})p_1(s)\psi_0(s)ds\Bigg)\Bigg],$$

$$\overline{c}_{0,2} = \frac{1}{\Pi_2}\Bigg[B_{22}\Bigg(\sum_{i=1}^{m} a_{2i}\Bigg(\int_0^{\xi_i}(\xi_i-s)\mathbf{E}_{\alpha_2,2}(\lambda_2(\xi_i-s)^{\alpha_2})p_2(s)\phi_0(s)ds$$

$$+\sum_{j=1}^{i} J_{2j}(\xi_i-t_j)\mathbf{E}_{\alpha_2,2}(\lambda_2(\xi_i-t_j)^{\alpha_2}) + \sum_{j=1}^{i} I_{2j}\mathbf{E}_{\alpha_2,1}(\lambda_2(\xi_i-t_j)^{\alpha_2})\Bigg)\Bigg)$$

$$-B_{12}\Bigg(\sum_{j=1}^{m-1}\sum_{i=j}^{m-1} b_{2i}\mathbf{E}_{\alpha_2,1}(\lambda_2(\eta_i-t_j)^{\alpha_2})J_{2j} - \sum_{j=1}^{m}\mathbf{E}_{\alpha_2,1}(\lambda_2(1-t_j)^{\alpha_2})J_{2j}$$

$$+\lambda_2\sum_{j=1}^{m-1}\sum_{i=j}^{m-1}b_{2i}(\eta_i-t_j)^{\alpha_2-1}\mathbf{E}_{\alpha_2,\alpha_2}(\lambda_2(\eta_i-t_j)^{\alpha_2})I_{2j}$$

$$-\lambda_2\sum_{j=1}^{m}(1-t_j)^{\alpha_2-1}\mathbf{E}_{\alpha_2,\alpha_2}(\lambda_2(1-t_j)^{\alpha_2})I_{2j}$$

$$-\int_0^1\mathbf{E}_{\alpha_2,1}(\lambda_2(1-u)^{\alpha_2})p_2(u)\phi_0(u)du$$

$$+\int_0^1\sum_{u<\eta_i\le\eta_{m-1}}b_{2i}\mathbf{E}_{\alpha_2,1}(\lambda_2(\eta_i-u)^{\alpha_2})p_2(u)\phi_0(u)du\Bigg)\Bigg],$$

$$\overline{d}_{0,1}=\frac{1}{\Pi_1}\Bigg[A_{11}\Bigg(\sum_{j=1}^{m-1}\sum_{i=j}^{m-1}b_{1i}\mathbf{E}_{\alpha_1,1}(\lambda_1(\eta_i-t_j)^{\alpha_1})J_{1j}$$

$$-\sum_{j=1}^{m}\mathbf{E}_{\alpha_1,1}(\lambda_1(1-t_j)^{\alpha_1})J_{1j}-\int_0^1\mathbf{E}_{\alpha_1,1}(\lambda_1(1-u)^{\alpha_1})p_1(u)\psi_0(u)du$$

$$+\lambda_1\sum_{j=1}^{m-1}\sum_{i=j}^{m-1}b_{1i}(\eta_i-t_j)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(\eta_i-t_j)^{\alpha_1})I_{1j}$$

$$+\int_0^1\sum_{u<\eta_i\le\eta_{m-1}}b_{1i}\mathbf{E}_{\alpha_1,1}(\lambda_1(\eta_i-u)^{\alpha_1})p_1(u)\psi_0(u)du$$

$$-\lambda_1\sum_{j=1}^{m}(1-t_j)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(1-t_j)^{\alpha_1})I_{1j}\Bigg)$$

$$-A_{21}\Bigg(\sum_{i=1}^{m}a_{1i}\Bigg(\int_0^{\xi_i}(\xi_i-s)\mathbf{E}_{\alpha_1,2}(\lambda_1(\xi_i-s)^{\alpha_1})p_1(s)\psi_0(s)ds$$

$$+\sum_{j=1}^{i}J_{1j}(\xi_i-t_j)\mathbf{E}_{\alpha_1,2}(\lambda_1(\xi_i-t_j)^{\alpha_1})+\sum_{j=1}^{i}I_{1j}\mathbf{E}_{\alpha_1,1}(\lambda_1(\xi_i-t_j)^{\alpha_1})\Bigg)\Bigg)\Bigg],$$

$$\overline{d}_{0,2}=\frac{1}{\Pi_2}\Bigg[B_{11}\Bigg(\sum_{j=1}^{m-1}\sum_{i=j}^{m-1}b_{2i}\mathbf{E}_{\alpha_2,1}(\lambda_2(\eta_i-t_j)^{\alpha_2})J_{2j}$$

$$-\sum_{j=1}^{m}\mathbf{E}_{\alpha_2,1}(\lambda_2(1-t_j)^{\alpha_2})J_{2j}-\int_0^1\mathbf{E}_{\alpha_2,1}(\lambda_2(1-u)^{\alpha_2})p_2(u)\phi_0(u)du$$

$$+\lambda_2\sum_{j=1}^{m-1}\sum_{i=j}^{m-1}b_{2i}(\eta_i-t_j)^{\alpha_2-1}\mathbf{E}_{\alpha_2,\alpha_2}(\lambda_2(\eta_i-t_j)^{\alpha_2})I_{2j}$$

$$+\int_0^1\sum_{u<\eta_i\le\eta_{m-1}}b_{2i}\mathbf{E}_{\alpha_2,1}(\lambda_2(\eta_i-u)^{\alpha_2})p_2(u)\phi_0(u)du$$

$$-\lambda_2\sum_{j=1}^{m}(1-t_j)^{\alpha_2-1}\mathbf{E}_{\alpha_2,\alpha_2}(\lambda_2(1-t_j)^{\alpha_2})I_{2j}\Bigg)$$

$$-B_{21}\Bigg(\sum_{i=1}^{m}a_{2i}\Bigg(\int_0^{\xi_i}(\xi_i-s)\mathbf{E}_{\alpha_2,2}(\lambda_2(\xi_i-s)^{\alpha_2})p_2(s)\phi_0(s)ds$$

$$+\sum_{j=1}^{i}J_{2j}(\xi_i-t_j)\mathbf{E}_{\alpha_2,2}(\lambda_2(\xi_i-t_j)^{\alpha_2})+\sum_{j=1}^{i}I_{2j}\mathbf{E}_{\alpha_2,1}(\lambda_2(\xi_i-t_j)^{\alpha_2})\Bigg)\Bigg)\Bigg].$$

$$\Phi(t)=\int_0^t(t-s)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(t-s)^{\alpha_1}p_1(s)\psi_0(s)ds+\overline{c}_{0,1}t^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1t^{\alpha_1})$$

$$+\sum_{j=1}^{k}J_{1j}(t-t_j)^{\alpha_1-1}\mathbf{E}_{\alpha_1,\alpha_1}(\lambda_1(t-t_j)^{\alpha_1})+\overline{d}_{0,1}t^{\alpha_1-2}\mathbf{E}_{\alpha_1,\alpha_1-1}(\lambda_1t^{\alpha_1})$$

$$+\sum_{j=1}^{k}I_{1j}(t-t_j)^{\alpha_1-2}\mathbf{E}_{\alpha_1,\alpha_1-1}(\lambda_1(t-t_j)^{\alpha_1}),t\in(t_k,t_{k+1}],k\in\mathbb{N}_0^m$$

and

$$\Psi(t)=\int_0^t(t-s)^{\alpha_2-1}\mathbf{E}_{\alpha_2,\alpha_2}(\lambda_2(t-s)^{\alpha_2}p_2(s)\phi_0(s)ds+\overline{c}_{0,2}t^{\alpha_2-1}\mathbf{E}_{\alpha_2,\alpha_2}(\lambda_2t^{\alpha_2})$$

$$+\sum_{j=2}^{k}J_{2j}(t-t_j)^{\alpha_2-1}\mathbf{E}_{\alpha_2,\alpha_2}(\lambda_2(t-t_j)^{\alpha_2})+\overline{d}_{0,2}t^{\alpha_2-2}\mathbf{E}_{\alpha_2,\alpha_2-1}(\lambda_2t^{\alpha_2})$$

$$+\sum_{j=1}^{k}I_{2j}(t-t_j)^{\alpha_2-2}\mathbf{E}_{\alpha_2,\alpha_2-1}(\lambda_2(t-t_j)^{\alpha_2}),t\in(t_k,t_{k+1}],k\in\mathbb{N}_0^m,$$

Denote

$$P_j=\Bigg[\frac{\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)+\mathbf{E}_{\alpha_1,1}(|\lambda_1|)|A_{22}|\sum_{i=1}^{m}|a_{1i}|\mathbf{E}_{\alpha_1,2}(|\lambda_1|)||p_1||_1}{|\Pi_1|}+\frac{\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)+\mathbf{E}_{\alpha_1,1}(|\lambda_1|)|A_{12}|||p_1||_1\mathbf{E}_{\alpha_1,1}(|\lambda_1|)}{|\Pi_1|}$$

$$+\frac{\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)+\mathbf{E}_{\alpha_1,1}(|\lambda_1|)|A_{12}|||p_1||_1\sum_{i=0}^{m-1}|b_{1i}|\mathbf{E}_{\alpha_1,1}(|\lambda_1|)}{|\Pi_1|}+\frac{\mathbf{E}_{\alpha_1,\alpha_1-1}(|\lambda_1|)+|\lambda_1|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)|A_{11}|||p_1||_1\mathbf{E}_{\alpha_1,1}(|\lambda_1|)}{|\Pi_1|}$$

$$+\frac{\mathbf{E}_{\alpha_1,\alpha_1-1}(|\lambda_1|)+|\lambda_1|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)|A_{11}|||p_1||_1\sum_{i=0}^{m-1}|b_{1i}|\mathbf{E}_{\alpha_1,1}(|\lambda_1|)}{|\Pi_1|}+||p_1||_1\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)+||p_1||_1\mathbf{E}_{\alpha_1,1}(|\lambda_1|)$$

$$+\frac{\mathbf{E}_{\alpha_1,\alpha_1-1}(|\lambda_1|)+|\lambda_1|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)|A_{21}|\sum_{i=1}^{m}|a_{1i}|||p_1||_1\mathbf{E}_{\alpha_1,2}(|\lambda_1|)}{|\Pi_1|}\Bigg]A_j$$

$$+\Bigg[\frac{\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)+\mathbf{E}_{\alpha_1,1}(|\lambda_1|)m|A_{22}|\sum_{i=1}^{m}|a_{1i}|\mathbf{E}_{\alpha_1,1}(|\lambda_1|)}{|\Pi_1|}+\frac{\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)+\mathbf{E}_{\alpha_1,1}(|\lambda_1|)m|A_{12}||\lambda_1|\sum_{i=0}^{m-1}|b_{1i}|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Pi_1|}$$

$$+\frac{\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)+\mathbf{E}_{\alpha_1,1}(|\lambda_1|)m|A_{12}||\lambda_1|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Pi_1|}+\frac{\mathbf{E}_{\alpha_1,\alpha_1-1}(|\lambda_1|)+|\lambda_1|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)m|A_{11}||\lambda_1|\sum_{i=0}^{m-1}|b_{1i}|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Pi_1|}$$

$$+\frac{\mathbf{E}_{\alpha_1,\alpha_1-1}(|\lambda_1|)+|\lambda_1|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)|A_{11}||\lambda_1|m\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Pi_1|}$$

$$+m\mathbf{E}_{\alpha_1,\alpha_1-1}(|\lambda_1|)C_j+m|\lambda_1|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)$$

$$+\frac{\mathbf{E}_{\alpha_1,\alpha_1-1}(|\lambda_1|)+|\lambda_1|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)m|A_{21}|\sum_{i=1}^{m}|a_{1i}|\mathbf{E}_{\alpha_1,1}(|\lambda_1|)}{|\Pi_1|}\Bigg]C_j$$

$$+\Bigg[\frac{\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)+\mathbf{E}_{\alpha_1,1}(|\lambda_1|)m|A_{22}|\sum_{i=1}^{m}|a_{1i}|\mathbf{E}_{\alpha_1,2}(|\lambda_1|)}{|\Pi_1|}+\frac{\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)+\mathbf{E}_{\alpha_1,1}(|\lambda_1|)m|A_{12}|\sum_{i=0}^{m-1}|b_{1i}|\mathbf{E}_{\alpha_1,1}(|\lambda_1|)}{|\Pi_1|}$$

$$+\frac{\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)+\mathbf{E}_{\alpha_1,1}(|\lambda_1|)m|A_{12}|\mathbf{E}_{\alpha_1,1}(|\lambda_1|)}{|\Pi_1|}+\frac{\mathbf{E}_{\alpha_1,\alpha_1-1}(|\lambda_1|)+|\lambda_1|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)m|A_{11}|\sum_{i=0}^{m-1}|b_{1i}|\mathbf{E}_{\alpha_1,1}(|\lambda_1|)}{|\Pi_1|}$$

$$+\frac{\mathbf{E}_{\alpha_1,\alpha_1-1}(|\lambda_1|)+|\lambda_1|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)m|A_{11}|\mathbf{E}_{\alpha_1,1}(|\lambda_1|)}{|\prod_1|}+m\mathbf{E}_{\alpha_1,1}(|\lambda_1|)+m\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)$$

$$\left.+\frac{\mathbf{E}_{\alpha_1,\alpha_1-1}(|\lambda_1|)+|\lambda_1|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)m|A_{21}|\sum_{i=1}^{m}|a_{1i}|\mathbf{E}_{\alpha_1,2}(|\lambda_1|)}{|\prod_1|}\right]D_j,\ j=1,2,$$

$$Q_j=\left[\frac{\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)+\mathbf{E}_{\alpha_2,1}(|\lambda_2|)|B_{22}|\sum_{i=1}^{m}|a_{2i}|\mathbf{E}_{\alpha_2,2}(|\lambda_2|)||p_2||_1}{|\prod_2|}+\frac{\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)+\mathbf{E}_{\alpha_2,1}(|\lambda_2|)|B_{12}|||p_2||_1\mathbf{E}_{\alpha_2,1}(|\lambda_2|)}{|\prod_2|}\right.$$

$$+\frac{\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)+\mathbf{E}_{\alpha_2,1}(|\lambda_2|)|B_{12}|||p_2||_1\sum_{i=0}^{m-1}|b_{2i}|\mathbf{E}_{\alpha_2,1}(|\lambda_2|)}{|\prod_2|}+\frac{\mathbf{E}_{\alpha_2,\alpha_2-1}(|\lambda_2|)+|\lambda_2|\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)|B_{11}|||p_2||_1\mathbf{E}_{\alpha_2,1}(|\lambda_2|)}{|\prod_2|}$$

$$+\frac{\mathbf{E}_{\alpha_2,\alpha_2-1}(|\lambda_2|)+|\lambda_2|\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)|B_{11}|||p_2||_1\sum_{i=0}^{m-1}|b_{2i}|\mathbf{E}_{\alpha_2,1}(|\lambda_2|)}{|\prod_2|}+||p_2||_1\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)+||p_2||_1\mathbf{E}_{\alpha_2,1}(|\lambda_2|)$$

$$\left.+\frac{\mathbf{E}_{\alpha_2,\alpha_2-1}(|\lambda_2|)+|\lambda_2|\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)|B_{21}|\sum_{i=1}^{m}|a_{2i}|||p_2||_1\mathbf{E}_{\alpha_2,2}(|\lambda_2|)}{|\prod_2|}\right]B_j$$

$$+\left[\frac{\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)+\mathbf{E}_{\alpha_2,1}(|\lambda_2|)m|B_{22}|\sum_{i=1}^{m}|a_{2i}|\mathbf{E}_{\alpha_2,1}(|\lambda_2|)}{|\prod_2|}+\frac{\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)+\mathbf{E}_{\alpha_2,1}(|\lambda_2|)m|B_{12}||\lambda_2|\sum_{i=0}^{m-1}|b_{2i}|\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)}{|\prod_2|}\right.$$

$$+\frac{\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)+\mathbf{E}_{\alpha_2,1}(|\lambda_2|)m|B_{12}||\lambda_2|\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)}{|\prod_2|}+\frac{\mathbf{E}_{\alpha_2,\alpha_2-1}(|\lambda_2|)+|\lambda_2|\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)m|B_{11}||\lambda_2|\sum_{i=0}^{m-1}|b_{2i}|\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)}{|\prod_2|}$$

$$+\frac{\mathbf{E}_{\alpha_2,\alpha_2-1}(|\lambda_2|)+|\lambda_2|\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)|B_{11}||\lambda_2|m\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)}{|\prod_2|}+m\mathbf{E}_{\alpha_2,\alpha_2-1}(|\lambda_2|)+m|\lambda_2|\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)$$

$$\left.+\frac{\mathbf{E}_{\alpha_2,\alpha_2-1}(|\lambda_2|)+|\lambda_2|\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)m|B_{21}|\sum_{i=1}^{m}|a_{2i}|\mathbf{E}_{\alpha_2,1}(|\lambda_2|)}{|\prod_2|}\right]E_j$$

$$+\left[\frac{\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)+\mathbf{E}_{\alpha_2,1}(|\lambda_2|)m|B_{22}|\sum_{i=1}^{m}|a_{2i}|\mathbf{E}_{\alpha_2,2}(|\lambda_2|)}{|\prod_2|}+\frac{\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)+\mathbf{E}_{\alpha_2,1}(|\lambda_2|)m|B_{12}|\sum_{i=0}^{m-1}|b_{2i}|\mathbf{E}_{\alpha_2,1}(|\lambda_2|)}{|\prod_2|}\right.$$

$$+\frac{\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)+\mathbf{E}_{\alpha_2,1}(|\lambda_2|)m|B_{12}|\mathbf{E}_{\alpha_2,1}(|\lambda_2|)}{|\prod_2|}+\frac{\mathbf{E}_{\alpha_2,\alpha_2-1}(|\lambda_2|)+|\lambda_2|\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)m|B_{11}|\sum_{i=0}^{m-1}|b_{2i}|\mathbf{E}_{\alpha_2,1}(|\lambda_2|)}{|\prod_2|}$$

$$+\frac{\mathbf{E}_{\alpha_2,\alpha_2-1}(|\lambda_2|)+|\lambda_2|\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)m|B_{11}|\mathbf{E}_{\alpha_2,1}(|\lambda_2|)}{|\prod_2|}+m\mathbf{E}_{\alpha_2,1}(|\lambda_2|)+m\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)$$

$$\left.+\frac{\mathbf{E}_{\alpha_2,\alpha_2-1}(|\lambda_2|)+|\lambda_2|\mathbf{E}_{\alpha_2,\alpha_2}(|\lambda_2|)m|B_{21}|\sum_{i=1}^{m}|a_{2i}|\mathbf{E}_{\alpha_2,2}(|\lambda_2|)}{|\prod_2|}\right]F_j,\ j=1,2.$$

Theorem 3.1. Let $\sigma=\max\{\sigma_i(i=1,2,3)\}$ and $\tau=\max\{\tau_i(i=1,2,3)\}$. Suppose that (a)-(d) and (H1)-(H2) hold, $\prod_1\neq 0,\prod_2\neq 0$. Then BVP(1.10)-(1.12) has at least one solution if

(i) $\sigma\tau\in[0,1)$ or

(ii) $\sigma\tau=1$ with

$$\sum_{j=1}^{3}Q_j||\Phi||^{\tau_j-\tau}<\left(\frac{1}{\sum_{j=1}^{3}P_j||\Psi||^{\sigma_j-\sigma}}\right)^{1/\sigma}\ \text{ or }\ \sum_{j=1}^{3}P_j||\Psi||^{\sigma_j-\sigma}<\left(\frac{1}{\sum_{j=1}^{3}Q_j||\Phi||^{\tau_j-\tau}}\right)^{1/\tau}.$$

or

(iii) $\sigma\tau > 1$ with

$$\frac{\sigma\tau-1}{||\Psi||}\sum_{j=1}^{3} Q_j||\Phi||^{\tau_j-\tau}\left[||\Phi||+\left(\frac{\sigma^2\tau||\Psi||}{\sigma(\sigma\tau-1)\sum_{j=1}^{3}P_j||\Psi||^{\sigma_j-\sigma}}\right)^{\tau}\right]^{\sigma}\leq 1,$$

$$\frac{\sigma\tau-1}{||\Phi||}\sum_{j=1}^{3} P_j||\Psi||^{\sigma_j-\sigma}\left[||\Psi||+\left(\frac{\tau^2\sigma||\Phi||}{\tau(\sigma\tau-1)\sum_{j=1}^{3}Q_j||\Phi||^{\tau_j-\tau}}\right)^{\sigma}\right]^{\tau}\leq 1.$$

Proof. Let $T : E \mapsto E$ be defined in Section 2. By Lemma 2.3, it suffices to get fixed point of T in E. From Lemma 2.4, T is completely continuous. We will seek fixed points of T in E. It is easy to see that $(\Phi,\Psi) \in E$. For $r_1, r_2 > 0$, denote $\Omega_{r_1,r_2} = \{(u_1,u_2) \in E : ||u_1-\Phi|| \leq r_1, ||u_2-\Psi|| \leq r_2\}$. One sees that $||u_1|| \leq ||u_1-\Phi||+||\Phi|| \leq r_1+||\Phi||$ and $||u_2|| \leq r_2+||\Psi||$ for all $(u_1,u_2) \in \Omega_{r_1,r_2}$.

Use (H1), for $(u_1,u_2) \in \Omega_{r_1,r_2}$, we have

$$|f_1(t,u_2(t),D_{0^+}^{\alpha_2-1}u_2(t))-\psi_0(t)|\left|f_1\left(t,\frac{(t-t_i)^{2-\alpha_2}u_2(t)}{(t-t_i)t^{2-\alpha_2}},D_{0^+}^{\alpha_2-1}u_2(t)\right)-\psi_0(t)\right|$$

$$\leq A_1\left|(t-t_i)^{2-\alpha_2}u_2(t)\right|^{\sigma_1}+A_2\left|D_{0^+}^{\alpha_2-1}u_2(t)\right|^{\sigma_2}\leq\sum_{j=1}^{2}A_j\left[r_2+||\Psi||\right]^{\sigma_j},t\in(t_i,t_{i+1}],i\in\mathbb{N}_0^m.$$

It follows that

$$|f_1(t,u_2(t),D_{0^+}^{\alpha_2-1}u_2(t))-\psi_0(t)|\leq\sum_{j=1}^{2}A_j\left[r_2+||\Psi||\right]^{\sigma_j}. \tag{3.1}$$

Similarly use (H1)-(H2), we can get for $(x,y) \in \Omega_r$ that

$$\begin{aligned}
&|f_2(t,u_1(t),D_{0^+}^{\alpha_1-1}u_1(t))-\phi_0(t)|\leq\sum_{j=1}^{2}B_j\left[r_1+||\Phi||\right]^{\tau_j},\\
&\left|I_1\left(t_i,u_2(t_i),D_{0^+}^{\alpha_2-1}u_2(t_i)\right)-I_{1i}\right|\leq\sum_{j=1}^{2}C_j[r_2+||\Psi||]^{\sigma_j},i\in\mathbb{N}_1^m,\\
&\left|J_1\left(t_i,u_2(t_i),D_{0^+}^{\alpha_2-1}u_2(t_i)\right)-J_{1i}\right|\leq\sum_{j=1}^{2}D_j[r_2+||\Psi||]^{\sigma_j},i\in\mathbb{N}_1^m,\\
&\left|I_2\left(t_i,u_1(t_i),D_{0^+}^{\alpha_1-1}u_1(t_i)\right)-I_{2i}\right|\leq\sum_{j=1}^{2}E_j[r_1+||\Phi||]^{\tau_j},i\in\mathbb{N}_1^m,\\
&\left|J_2\left(t_i,u_1(t_i),D_{0^+}^{\alpha_1-1}u_1(t_i)\right)-J_{2i}\right|\leq\sum_{j=1}^{2}F_j[r_1+||\Phi||]^{\tau_j},i\in\mathbb{N}_1^m.
\end{aligned}\tag{3.2}$$

Using Lemma 2.1, we can get the expressions of

$$D_{0^+}^{\alpha_1-1}\Phi(t),\ D_{0^+}^{\alpha_2-1}\Psi(t),\ D_{0^+}^{\alpha_1-1}(T_1u_2)(t),\ D_{0^+}^{\alpha_2-1}(T_2u_1)(t).$$

One gets

$$
\begin{aligned}
|c_{0,1}-\overline{c}_{0,1}| \leq \frac{1}{|\Pi_1|}\Bigg[&|A_{22}|\left(\sum_{i=1}^{m}|a_{1i}|\left(\mathbf{E}_{\alpha_1,2}(|\lambda_1|)||p_1||_1\sum_{j=1}^{2}A_j\,[r_2+||\Psi||]^{\sigma_j}\right.\right.\\
&\left.\left.+m\mathbf{E}_{\alpha_1,2}(|\lambda_1|)\sum_{j=1}^{2}D_j\,[r_2+||\Psi||]^{\sigma_j}+m\mathbf{E}_{\alpha_1,1}(|\lambda_1|)\sum_{j=1}^{2}C_j\,[r_2+||\Psi||]^{\sigma_j}\right)\right)\\
&+|A_{12}|\left(m\sum_{i=0}^{m-1}|b_{1i}|\mathbf{E}_{\alpha_1,1}(|\lambda_1|)\sum_{j=1}^{2}D_j\,[r_2+||\Psi||]^{\sigma_j}+m\mathbf{E}_{\alpha_1,1}(|\lambda_1|)\sum_{j=1}^{2}D_j\,[r_2+||\Psi||]^{\sigma_j}\right.\\
&+|\lambda_1|m\sum_{i=0}^{m-1}|b_{1i}|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)\sum_{j=1}^{2}C_j\,[r_2+||\Psi||]^{\sigma_j}\\
&+|\lambda_1|m\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)\sum_{j=1}^{2}C_j\,[r_2+||\Psi||]^{\sigma_j}+||p_1||_1\mathbf{E}_{\alpha_1,1}(|\lambda_1|)\sum_{j=1}^{2}A_j\,[r_2+||\Psi||]^{\sigma_j}\\
&\left.+||p_1||_1\sum_{i=0}^{m-1}|b_{1i}|\mathbf{E}_{\alpha_1,1}(|\lambda_1|)\sum_{j=1}^{2}A_j\,[r_2+||\Psi||]^{\sigma_j}\right)\Bigg],
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
|d_{0,1}-\overline{d}_{0,1}| \leq \frac{1}{|\Pi_1|}\Bigg[&|A_{11}|\left(m\sum_{i=0}^{m-1}|b_{1i}|\mathbf{E}_{\alpha_1,1}(|\lambda_1|)\sum_{j=1}^{2}D_j\,[r_2+||\Psi||]^{\sigma_j}\right.\\
&+m\mathbf{E}_{\alpha_1,1}(|\lambda_1|)\sum_{j=1}^{2}D_j\,[r_2+||\Psi||]^{\sigma_j}+||p_1||_1\mathbf{E}_{\alpha_1,1}(|\lambda_1|)\sum_{j=1}^{2}A_j\,[r_2+||\Psi||]^{\sigma_j}\\
&+m|\lambda_1|\sum_{i=0}^{m-1}|b_{1i}|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)\sum_{j=1}^{2}C_j\,[r_2+||\Psi||]^{\sigma_j}\\
&+||p_1||_1\sum_{i=0}^{m-1}|b_{1i}|\mathbf{E}_{\alpha_1,1}(|\lambda_1|)\sum_{j=1}^{2}A_j\,[r_2+||\Psi||]^{\sigma_j}\\
&\left.+|\lambda_1|m\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)\sum_{j=1}^{2}C_j\,[r_2+||\Psi||]^{\sigma_j}\right)\\
&+|A_{21}|\left(\sum_{i=1}^{m}|a_{1i}|\left(||p_1||_1\mathbf{E}_{\alpha_1,2}(|\lambda_1|)\sum_{j=1}^{2}A_j\,[r_2+||\Psi||]^{\sigma_j}\right.\right.\\
&\left.\left.+m\mathbf{E}_{\alpha_1,2}(|\lambda_1|)\sum_{j=1}^{2}D_j\,[r_2+||\Psi||]^{\sigma_j}+m\mathbf{E}_{\alpha_1,1}(|\lambda_1|)\sum_{j=1}^{2}C_j\,[r_2+||\Psi||]^{\sigma_j}\right)\right)\Bigg].
\end{aligned}
\tag{3.4}
$$

Use (c) and the definitions of T_1, Φ, (3.1), (3.2), we get for $t \in (t_i, t_{i+1}]$ that

$$
\begin{aligned}
&(t-t_i)^{2-\alpha_1}|(T_1u_2)(t)-\Phi(t)|\\
&\leq ||p_1||_1\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)\sum_{j=1}^{2}A_j\,[r_2+||\Psi||]^{\sigma_j}+|c_{0,1}-\overline{c}_{0,1}|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)\\
&+m\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)\sum_{j=1}^{2}D_j\,[r_2+||\Psi||]^{\sigma_j}+|d_{0,1}-\overline{d}_{0,1}|\mathbf{E}_{\alpha_1,\alpha_1-1}(|\lambda_1|)\\
&+m\mathbf{E}_{\alpha_1,\alpha_1-1}(|\lambda_1|)\sum_{j=1}^{2}C_j\,[r_2+||\Psi||]^{\sigma_j},t\in(t_i,t_{i+1}],i\in\mathbb{N}_0^m.
\end{aligned}
\tag{3.5}
$$

Similarly we have

$$
\begin{aligned}
&|D_{0^+}^{\alpha-1}(T_1 y)(t)-D_{0^+}^{\alpha-1}\Phi(t)| \leq |c_{0,1}-\overline{c}_{0,1}|\mathbf{E}_{\alpha_1,1}(|\lambda_1|)+|\lambda_1||d_{0,1}-\overline{d}_{0,1}|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)\\
&+m\mathbf{E}_{\alpha_1,1}(|\lambda_1|)\sum_{j=1}^{2} D_j\left[r_2+||\Psi||\right]^{\sigma_j}+m|\lambda_1|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)\sum_{j=1}^{2} C_j\left[r_2+||\Psi||\right]^{\sigma_j}\\
&+||p_1||_1\mathbf{E}_{\alpha_1,1}(|\lambda_1|)\sum_{j=1}^{2} A_j\left[r_2+||\Psi||\right]^{\sigma_j}.
\end{aligned}
\tag{3.6}
$$

It follows from (3.3)-(3.6) that

$$
\begin{aligned}
&||T_1u_2-\Phi|| \leq \max\left\{\sum_{t\in(t_i,t_{i+1}]}(t-t_i)^{2-\alpha_1}|(T_1u_2)(t)-\Phi(t)| : i\in\mathbb{N}_0^m\right\}\\
&+\max\left\{\sup_{t\in(t_i,t_{i+1}]}|D_{0^+}^{\alpha_1-1}(T_1u_2)(t)-D_{0^+}^{\alpha_1-1}\Phi(t)| : i\in\mathbb{N}_0^m\right\}\\
&\leq \frac{\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)+\mathbf{E}_{\alpha_1,1}(|\lambda_1|)|A_{22}|\sum_{i=1}^{m}|a_{1i}|\mathbf{E}_{\alpha_1,2}(|\lambda_1|)||p_1||_1}{|\Pi_1|}\sum_{j=1}^{2} A_j\left[r_2+||\Psi||\right]^{\sigma_j}\\
&+\frac{\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)+\mathbf{E}_{\alpha_1,1}(|\lambda_1|)|A_{12}|||p_1||_1\mathbf{E}_{\alpha_1,1}(|\lambda_1|)}{|\Pi_1|}\sum_{j=1}^{2} A_j\left[r_2+||\Psi||\right]^{\sigma_j}\\
&+\frac{\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)+\mathbf{E}_{\alpha_1,1}(|\lambda_1|)|A_{12}|||p_1||_1\sum_{i=0}^{m-1}|b_{1i}|\mathbf{E}_{\alpha_1,1}(|\lambda_1|)}{|\Pi_1|}\sum_{j=1}^{2} A_j\left[r_2+||\Psi||\right]^{\sigma_j}\\
&+\frac{\mathbf{E}_{\alpha_1,\alpha_1-1}(|\lambda_1|)+|\lambda_1|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)|A_{11}|||p_1||_1\mathbf{E}_{\alpha_1,1}(|\lambda_1|)}{|\Pi_1|}\sum_{j=1}^{2} A_j\left[r_2+||\Psi||\right]^{\sigma_j}\\
&+\frac{\mathbf{E}_{\alpha_1,\alpha_1-1}(|\lambda_1|)+|\lambda_1|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)|A_{11}|||p_1||_1\sum_{i=0}^{m-1}|b_{1i}|\mathbf{E}_{\alpha_1,1}(|\lambda_1|)}{|\Pi_1|}\sum_{j=1}^{2} A_j\left[r_2+||\Psi||\right]^{\sigma_j}\\
&+\frac{\mathbf{E}_{\alpha_1,\alpha_1-1}(|\lambda_1|)+|\lambda_1|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)|A_{21}|\sum_{i=1}^{m}|a_{1i}|||p_1||_1\mathbf{E}_{\alpha_1,2}(|\lambda_1|)}{|\Pi_1|}\sum_{j=1}^{2} A_j\left[r_2+||\Psi||\right]^{\sigma_j}\\
&+||p_1||_1\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)\sum_{j=1}^{2} A_j\left[r_2+||\Psi||\right]^{\sigma_j}+||p_1||_1\mathbf{E}_{\alpha_1,1}(|\lambda_1|)\sum_{j=1}^{2} A_j\left[r_2+||\Psi||\right]^{\sigma_j}\\
&+\frac{\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)+\mathbf{E}_{\alpha_1,1}(|\lambda_1|)m|A_{22}|\sum_{i=1}^{m}|a_{1i}|\mathbf{E}_{\alpha_1,1}(|\lambda_1|)}{|\Pi_1|}\sum_{j=1}^{2} C_j\left[r_2+||\Psi||\right]^{\sigma_j}\\
&+\frac{\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)+\mathbf{E}_{\alpha_1,1}(|\lambda_1|)m|A_{12}||\lambda_1|\sum_{i=0}^{m-1}|b_{1i}|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Pi_1|}\sum_{j=1}^{2} C_j\left[r_2+||\Psi||\right]^{\sigma_j}\\
&+\frac{\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)+\mathbf{E}_{\alpha_1,1}(|\lambda_1|)m|A_{12}||\lambda_1|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Pi_1|}\sum_{j=1}^{2} C_j\left[r_2+||\Psi||\right]^{\sigma_j}\\
&+\frac{\mathbf{E}_{\alpha_1,\alpha_1-1}(|\lambda_1|)+|\lambda_1|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)m|A_{11}||\lambda_1|\sum_{i=0}^{m-1}|b_{1i}|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Pi_1|}\sum_{j=1}^{2} C_j\left[r_2+||\Psi||\right]^{\sigma_j}
\end{aligned}
$$

$$+\frac{\mathbf{E}_{\alpha_1,\alpha_1-1}(|\lambda_1|)+|\lambda_1|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)|A_{11}||\lambda_1|m\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)}{|\Pi_1|}\sum_{j=1}^{2}C_j\left[r_2+||\Psi||\right]^{\sigma_j}$$

$$+\frac{\mathbf{E}_{\alpha_1,\alpha_1-1}(|\lambda_1|)+|\lambda_1|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)m|A_{21}|\sum_{i=1}^{m}|a_{1i}|\mathbf{E}_{\alpha_1,1}(|\lambda_1|)}{|\Pi_1|}\sum_{j=1}^{2}C_j\left[r_2+||\Psi||\right]^{\sigma_j}$$

$$+m\mathbf{E}_{\alpha_1,\alpha_1-1}(|\lambda_1|)\sum_{j=1}^{2}C_j\left[r_2+||\Psi||\right]^{\sigma_j}+m|\lambda_1|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)\sum_{j=1}^{2}C_j\left[r_2+||\Psi||\right]^{\sigma_j}$$

$$+\frac{\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)+\mathbf{E}_{\alpha_1,1}(|\lambda_1|)m|A_{22}|\sum_{i=1}^{m}|a_{1i}|\mathbf{E}_{\alpha_1,2}(|\lambda_1|)}{|\Pi_1|}\sum_{j=1}^{2}D_j\left[r_2+||\Psi||\right]^{\sigma_j}$$

$$+\frac{\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)+\mathbf{E}_{\alpha_1,1}(|\lambda_1|)m|A_{12}|\sum_{i=0}^{m-1}|b_{1i}|\mathbf{E}_{\alpha_1,1}(|\lambda_1|)}{|\Pi_1|}\sum_{j=1}^{2}D_j\left[r_2+||\Psi||\right]^{\sigma_j}$$

$$+\frac{\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)+\mathbf{E}_{\alpha_1,1}(|\lambda_1|)m|A_{12}|\mathbf{E}_{\alpha_1,1}(|\lambda_1|)}{|\Pi_1|}\sum_{j=1}^{2}D_j\left[r_2+||\Psi||\right]^{\sigma_j}$$

$$+\frac{\mathbf{E}_{\alpha_1,\alpha_1-1}(|\lambda_1|)+|\lambda_1|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)m|A_{11}|\sum_{i=0}^{m-1}|b_{1i}|\mathbf{E}_{\alpha_1,1}(|\lambda_1|)}{|\Pi_1|}\sum_{j=1}^{2}D_j\left[r_2+||\Psi||\right]^{\sigma_j}$$

$$+\frac{\mathbf{E}_{\alpha_1,\alpha_1-1}(|\lambda_1|)+|\lambda_1|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)m|A_{11}|\mathbf{E}_{\alpha_1,1}(|\lambda_1|)}{|\Pi_1|}\sum_{j=1}^{2}D_j\left[r_2+||\Psi||\right]^{\sigma_j}$$

$$+\frac{\mathbf{E}_{\alpha_1,\alpha_1-1}(|\lambda_1|)+|\lambda_1|\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)m|A_{21}|\sum_{i=1}^{m}|a_{1i}|\mathbf{E}_{\alpha_1,2}(|\lambda_1|)}{|\Pi_1|}\sum_{j=1}^{2}D_j\left[r_2+||\Psi||\right]^{\sigma_j}$$

$$+m\mathbf{E}_{\alpha_1,1}(|\lambda_1|)\sum_{j=1}^{2}D_j\left[r_2+||\Psi||\right]^{\sigma_j}+m\mathbf{E}_{\alpha_1,\alpha_1}(|\lambda_1|)\sum_{j=1}^{2}D_j\left[r_2+||\Psi||\right]^{\sigma_j}$$

$$=\sum_{j=1}^{2}P_j[r_2+||\Psi||]^{\sigma_j}.$$

It follows that

$$||T_1x-\Phi||\leq\sum_{j=1}^{3}P_j[r_2+||\Psi||]^{\sigma_j}\leq[r_2+||\Psi||]^{\sigma}\sum_{j=1}^{3}P_j||\Psi||^{\sigma_j-\sigma}. \tag{3.7}$$

Similarly we can get

$$||T_2x-\Psi||\leq\sum_{j=1}^{3}Q_j[r_1+||\Phi||]^{\tau_j}\leq[r_1+||\Phi||]^{\tau}\sum_{j=1}^{3}Q_j||\Phi||^{\tau_j-\tau}. \tag{3.8}$$

From (3.7), (3.8), we will seek $r_1,r_2>0$ such that

$$[r_2+||\Psi||]^{\sigma}\sum_{j=1}^{3}P_j||\Psi||^{\sigma_j-\sigma}\leq r_1,\ [r_1+||\Phi||]^{\tau}\sum_{j=1}^{3}Q_j||\Phi||^{\tau_j-\tau}\leq r_2. \tag{3.9}$$

Then one has $T\Omega_{r_1,r_2}\subseteq\Omega_{r_1,r_2}$. By Schauder's fixed point theorem, T has at least one fixed point $(u_1,u_2)\in\Omega_{r_1,r_2}$ which is a solution of BVP(1.10)-(1.12). It suffices to get positive solutions of the following inequality:

$$[r_1+||\Phi||]^{\tau}\sum_{j=1}^{3}Q_j||\Phi||^{\tau_j-\tau}\leq r_2\leq\left(\frac{r_1}{\sum_{j=1}^{3}P_j||\Psi||^{\sigma_j-\sigma}}\right)^{1/\sigma}-||\Psi|| \tag{3.10}$$

or

$$[r_2+||\Psi||]^\sigma \sum_{j=1}^{3} P_j||\Psi||^{\sigma_j-\sigma} \le r_1 \le \left(\frac{r_2}{\sum_{j=1}^{3} Q_j||\Phi||^{\tau_j-\tau}}\right)^{1/\tau} - ||\Phi||. \tag{3.11}$$

Case 1. $\sigma\tau<1$.
It is easy to see that

$$[r_1+||\Phi||]^\tau \sum_{j=1}^{3} Q_j||\Phi||^{\tau_j-\tau} \le \left(\frac{r_1}{\sum_{j=1}^{3} P_j||\Psi||^{\sigma_j-\sigma}}\right)^{1/\sigma} - ||\Psi||$$

has a positive solution $r_1>0$ sufficiently large. Choose r_2 satisfies

$$[r_1+||\Phi||]^\tau \sum_{j=1}^{3} Q_j||\Phi||^{\tau_j-\tau} \le r_2 \le \left(\frac{r_1}{\sum_{j=1}^{3} P_j||\Psi||^{\sigma_j-\sigma}}\right)^{1/\sigma} - ||\Psi||.$$

Then (3.10) has positive solutions $r_1>0$ and $r_2>0$. Then $T(u_1,u_2)\in\Omega_{r_1,r_2}$ for $(u_1,u_2)\in\Omega_{r_1,r_2}$. By Schauder's fixed point theorem (Lemma 2.2), T has at least one fixed point $(x,y)\in\Omega_{r_1,r_2}$. Then (u_1,u_2) is a solution of BVP(1.10)-(1.12).

Case 2.1. $\sigma\tau=1$ and $\sum_{j=1}^{3} Q_j||\Phi||^{\tau_j-\tau}\left(\sum_{j=1}^{3} P_j||\Psi||^{\sigma_j-\sigma}\right)^{1/\sigma}<1.$

Since

$$\lim_{r_1\to+\infty}\frac{[r_1+||\Phi||]^\tau \sum_{j=1}^{3} Q_j||\Phi||^{\tau_j-\tau}}{\left(\frac{r_1}{\sum_{j=1}^{3} P_j||\Psi||^{\sigma_j-\sigma}}\right)^{1/\sigma}-||\Psi||} = \frac{\sum_{j=1}^{3} Q_j||\Phi||^{\tau_j-\tau}}{\left(\frac{1}{\sum_{j=1}^{3} P_j||\Psi||^{\sigma_j-\sigma}}\right)^{1/\sigma}}<1,$$

we know that there exists $r_1>0$ sufficiently large such that

$$[r_1+||\Phi||]^\tau \sum_{j=1}^{3} Q_j||\Phi||^{\tau_j-\tau} \le \left(\frac{r_1}{\sum_{j=1}^{3} P_j||\Psi||^{\sigma_j-\sigma}}\right)^{1/\sigma} - ||\Psi||$$

Choose r_2 satisfies

$$[r_1+||\Phi||]^\tau \sum_{j=1}^{3} Q_j||\Phi||^{\tau_j-\tau} \le r_2 \le \left(\frac{r_1}{\sum_{j=1}^{3} P_j||\Psi||^{\sigma_j-\sigma}}\right)^{1/\sigma} - ||\Psi||.$$

Then (3.10) has a positive solution r_1,r_2. Then $T(u_1,u_2)\in\Omega_{r_1,r_2}$ for $(u_1,u_2)\in\Omega_{r_1,r_2}$. By Schauder's fixed point theorem (Lemma 2.3), T has at least one fixed point $(x,y)\in\Omega_{r_1,r_2}$. Then (u_1,u_2) is a solution of BVP(1.10)-(1.12).

Case 2.2. $\sigma\tau=1$ and $\sum_{j=1}^{3} P_j||\Psi||^{\sigma_j-\sigma}\left(\sum_{j=1}^{3} Q_j||\Phi||^{\tau_j-\tau}\right)^{1/\tau}<1.$

Similarly to Case 2.1, use (3.11), we get solutions of BVP(1.10)-(1.12) by using the Schauder's fixed point theorem (Lemma 2.2).

Case 3. $\sigma\tau>1$.
Choose

$$r_1=\left(\frac{\sigma\tau||\Psi||}{\sigma\tau-1}\right)^\tau \sum_{j=1}^{3} P_j||\Psi||^{\sigma_j-\sigma},\ \ r_2=\left(\frac{\tau\sigma||\Phi||}{\sigma\tau-1}\right)^\sigma \sum_{j=1}^{3} Q_j||\Phi||^{\tau_j-\tau}.$$

Since

$$\frac{\sigma\tau-1}{||\Psi||}\sum_{j=1}^{3}Q_j||\Phi||^{\tau_j-\tau}\left[||\Phi||+\left(\frac{\sigma\tau||\Psi||}{(\sigma\tau-1)\sum_{j=1}^{3}P_j||\Psi||^{\sigma_j-\sigma}}\right)^{\tau}\right]^{\sigma}\le 1,$$

$$\frac{\sigma\tau-1}{||\Phi||}\sum_{j=1}^{3}P_j||\Psi||^{\sigma_j-\sigma}\left[||\Psi||+\left(\frac{\tau\sigma||\Phi||}{(\sigma\tau-1)\sum_{j=1}^{3}Q_j||\Phi||^{\tau_j-\tau}}\right)^{\sigma}\right]^{\tau}\le 1,$$

we know that both (3.10) and (3.11) hold. Then we have $T(u_1,u_2)\in\Omega_{r_1,r_2}$ for $(u_1,u_2)\in\Omega_{r_1,r_2}$. By Schauder's fixed point theorem (Lemma 2.2), T has at least one fixed point $(x,y)\in\Omega_{r_1,r_2}$. Then (u_1,u_2) is a solution of BVP(1.10)-(1.12).

The proof of Theorem 3.1 is completed. □

Theorem 3.2. Suppose that (a)–(d) hold, $\prod_1\neq 0,\prod_2\neq 0$ and there exist constants $M_f,M_g,M_{I1},M_{J1},M_{I2},M_{J2}\geq 0$ such that

$$\left|f\left(t,\frac{y_1}{(t-t_i)^{2-\beta}},y_2\right)\right|\le M_f,\ t\in(t_i,t_{i+1}],\ y_j\in\mathbb{R}(j=1,2),i\in\mathbb{N}_0^m,$$

$$\left|g\left(t,\frac{x_1}{(t-t_i)^{2-\alpha}},x_2\right)\right|\le M_g,\ t\in(t_i,t_{i+1}],\ x_j\in\mathbb{R}(j=1,2),i\in\mathbb{N}_0^m,$$

$$\left|I_1\left(t_i,\frac{y_1}{(t_i-t_{i-1})^{2-\beta}},y_2\right)\right|\le M_{I1},i\in\mathbb{N}_1^m,$$

$$\left|J_1\left(t,\frac{y_1}{(t-t_i)^{2-\beta}},y_2\right)\right|\le M_{J1},i\in\mathbb{N}_1^m,$$

$$\left|I_2\left(t_i,\frac{x_1}{(t_i-t_{i-1})^{2-\alpha}},x_2\right)\right|\le M_{I2},i\in\mathbb{N}_1^m,$$

$$\left|J_2\left(t,\frac{x_1}{(t-t_i)^{2-\alpha}},x_2\right)\right|\le M_{J2},i\in\mathbb{N}_1^m.$$

Then BVP(1.10)-(1.12) has at least one solution in E.

Proof. In Theorem 3.1, choose $\phi_0(t)=\psi_0(t)=0$, $\sigma_1=\sigma_2=\tau_1=\tau_2=0$, $A_1=M_f,B_1=M_g$, $C_1=M_{I1},D_1=M_{J1}$, $E_1=M_{I2},F_1=M_{J2}$, and $A_2=B_2=C_2=D_2=E_2=F_2=0$. It is easy to see that (H1) and (H2) hold. We get Theorem 3.2 from Theorem 3.1. The proof is completed. □

4 Conclusion

In this paper, we establish sufficient conditions for the existence of solutions of impulsive initial value problems for singular fractional differential systems. We allow the nonlinearities $p_1(t)f_1(t,x,y)$ and $p_2(t)f_2(t,x,y)$ in fractional differential equations to be singular at $t=0,1$. Both f and g may be super-linear and sub-linear. The analysis relies on some well known fixed point theorems.

This paper contributes within the domain of impulsive fractional differential equations and adopts the traditional one, the Riemann-Liouville's integral, the Caputo's fractional derivative and the Riemann-Liouville fractional derivative definitions, which have many advantages and also have some shortcomings as discussed in some open literatures. These kinds of definitions have been replaced he some new ones such as the He's fractional derivative, the modified Riemann-Liouville derivative, the Hadamard fractional derivative, the Erdélyi-Kober fractional derivative, the Hilfer fractional derivative, and have been studied by many authors see for examples. The readers should study the similar problems for fractional differential systems involving with the other kinds of fractional derivative. Although the present study provides some insights in the equations encountered in the non-local existence solutions of anti-periodic boundary value problems, this existence theorem may be explored for other classes of boundary value problems for impulsive fractional differential systems, that is a subject for future study.

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