Applied Mathematics & Information Sciences **2**(3) (2008), 237–257
– An International Journal

The Rotating Inhomogeneous Elastic Cylinders of Variable-Thickness and Density

M. N. M. Allam[1], A. M. Zenkour[2]* and E. R. Elazab[3]

[1]Department of Mathematics, Faculty of Science, Mansoura University
Mansoura 35516, Egypt
E-mail address: mallam45@hotmail.com (M.N.M Allam)

[2]Department of Mathematics, Faculty of Education, Kafr El-Sheikh University
Kafr El-Sheikh 33516, Egypt
E-mail address: zenkour@gmail.com (A.M. Zenkour)

[3]Department of Mathematics, Faculty of Science, Al-Azhar University
Assiut 71524, Egypt

Received May 15, 2007; Accepted August 11, 2007

In this paper, an analytical solution is developed for the rotation problem of an inhomogeneous orthotropic cylinder with variable-thickness and density under plane strain assumption. The thickness of the cylinder and the elastic constants are taken as exponential functions in the radial direction but the density in a power law form. The cylinder may be solid or hollow with traction-free surface or clamped. On application of the boundary conditions, the stress and displacement for rotating homogeneous isotropic solid and hollow cylinders with uniform-thickness and density are obtained as special cases of the studied problem. Numerical results for stresses and displacement are presented in graphical forms. The effects of many parameters on stresses and displacement are investigated.

Keywords: Rotating, orthotropic cylinders, inhomogeneous, variable thickness and density.

1 Introduction

The rotation problem of elastic cylinder with variable-thickness and density is very important for numerous applications such as mechanical engineering, aircrafts, spacecrafts, satellites and biomechanics and the like. The plane strain problem of a rotating elastic

*Corresponding author

cylinder is solved in several investigations [13, 14, 16]. Chen [3] has formulated the stress distribution in an isotropic, inhomogeneous, thick cylinder under uniform pressure in terms of the stress function for both the plane-stress and plane-strain conditions. Wang and Lin [19] have analyzed the stresses in rotating homogeneous orthotropic cylindrical shells with uniform thickness under surface loading. The stress response of a rotating orthotropic hollow cylinder with the help of uniform internal pressure or a constant potential difference between its inner and outer surfaces or both is investigated by Horgan and Galic [7]. Also, they have studied the same problem of a solid cylinder with traction-free surface and zero applied electric charge.

Mukhopadhyay [11] has established the effects of inhomogeneity on the stresses in a rotation for an inhomogeneous aeolotropic cylindrical shell. Vasilenko and Klimenko [17] have analyzed the stress state of a rotating cylinder, inhomogeneous in the radial direction, having one plane of elastic symmetry and loaded with centrifugal forces. Klimenko [8] has obtained the solution of some problems concerning with the stress state of rotating anisotropic hollow cylinders that are inhomogeneous in the circumferential direction. Tarn [15] determined the thermoelastic deformation and stress for inhomogeneous hollow and solid cylinders subjected to an axial force, a torque at the ends and the surface loads that may vary circumferentially but not axially. In such study, exact solutions for thermoelastic response of rotating cylinders with variable density are obtained. The effect of inhomogeneity of elastic properties and density in the circumferential direction on the distribution of stress and displacement in orthotropic cylindrical panels using load in the axial direction is investigated by Grigorenko and Vasilenko [6]. Zenkour and Fares [23] have studied the bending, buckling and free vibration problems of inhomogeneous anisotropic composite laminated cylindrical shells with uniform thickness. Ding $et\ al.$ [4] have developed a solution of an inhomogeneous orthotropic elastic cylindrical shell for axisymmetric plane strain dynamic thermoelastic problems. In their paper, a special function is introduced to transform the inhomogeneous boundary conditions to the homogeneous ones. Oral and Anlas [12] have analyzed the effect of continuous inhomogeneity functionally graded material on the stress distribution for anisotropic cylindrical bodies. Klimenko [9] has presented a numerical-analytic solution of stress-strain problems for rotating inhomogeneous hollow cylinders under centrifugal loading. in their paper, elastic characteristics vary in both radial and circumferential directions. Liew $et\ al.$ [10] have presented the thermomechanical behavior of hollow circular cylinders of functionally graded material.

The importance of the present problem arises from the wide application of variable thickness structures in aerospace industry, underwater vehicles, machines and devices. The analytical solution of rotating cylinders becomes very complex when the thickness along the radius of the cylinder is variable, even for simple cases. Vasilenko and Sudavtsova [18] have used an approach to obtain stress-strain state of an inhomogeneous orthotropic hollow cylinder with variable-thickness and density in the circumferential direction. The stress problem for non-circular hollow cylinder with variable thickness under uniform and local loads is solved by Grigorenko and Rozhok [5]. Zenkour [20] has used the small parameter method and Le'vy-type approach to obtain an exact solution for the bending of rectangular plates with uniform, linear and quadratic thickness variations. Also, Zenkour [21] has established the effect of thickness variability on the stresses in a rotating orthotropic cylinder containing an isotropic core and a rigid core. In addition, Zenkour [22] has analytically investigated the behavior of composite circular cylinders subjected to internal and external surface loading. The cylinder consists of a number of homogeneous ply groups of axially variable thickness. Recently, Allam $et\ al.$ [2] have determined the stress concentrations around a triangular hole in a fiber-reinforced viscoelastic composite plate under uniform tension or pure bending.

In the present paper, exact elastic solutions for rotating solid and hollow cylinders with variable-thickness and density subjected to different boundary conditions are obtained. The material of the cylinders is assumed to be orthotropic and inhomogeneous . The thickness of the cylinder, the elastic constants and the material density are functions in the radial coordinate. Special cases of the studied problem for uniform thickness homogeneous cylinders are established. The effects due to many parameters on the displacement and stresses of rotating solid and hollow cylinders are investigated.

2 Formulation of the Problem

Consider an elastic cylinder made of an inhomogeneous orthotropic material and rotating about its axis. The cylindrical coordinates (r, θ, z) are chosen such that the axial coordinate z coinciding with the axis of rotation, r is the radial coordinate. Assuming the cylinder is symmetric with respect to the $z-$axis, we have only the radial displacement u which is independent of the circumferential coordinate θ. Furthermore, in the planes perpendicular to the $z-$axis in plane strain, u is a function of r alone.

Consequently, the Cauchy's relations under considerations can be written as follows form:

$$\varepsilon_{rr} = \frac{du}{dr}, \qquad \varepsilon_{\theta\theta} = \frac{u}{r}, \tag{2.1a}$$

$$\varepsilon_{zz} = \varepsilon_{r\theta} = \varepsilon_{rz} = \varepsilon_{\theta z} = 0, \tag{2.1b}$$

where ε_{ij} are the strain components.

From the generalized Hooke's law and using the geometric relations (2.1), we can obtain the stress components for an orthotropic cylinder in the following form:

$$\sigma_{rr} = c_{11}\frac{du}{dr} + c_{12}\frac{u}{r}, \tag{2.2a}$$

$$\sigma_{\theta\theta} = c_{12}\frac{du}{dr} + c_{22}\frac{u}{r}, \tag{2.2b}$$

$$_{zz} = c_{13}\frac{du}{dr} + c_{23}\frac{u}{r}\,; \tag{2.2c}$$

$$_{r} = \;_{rz} = \;_{z} = 0\,; \tag{2.2d}$$

where c_{ij} are the elastic constants.

Let us assume now that the thickness h of the cylinder vary in the radial direction in an exponential form given by:

$$h(r) = h_0\, e^{\;n(r/b)^k}\,; \tag{2.3}$$

where h_0 is the thickness at the axis of the cylinder, n and k are geometric parameters and b is the radius of the cylinder. The parameter k determines the shape of the thickness profile while n determines the thickness at the surface of the cylinder relative to h_0. For three different sets of geometric parameters n and k, the dimensionless thickness h/h_0 as a function of the dimensionless radius r/b is described by the profiles shown in Fig. 2.1-2.3.

Figure 2.1: Exponential cylinder profiles $k = 0.7;\ n = 2.3$.

In Fig. 2.1 the thickness profile is concave for large n and small k $(n = 2.3; k = 0.7)$ while the convex profile shown in Fig. 2.3 corresponds to small n and large k $(n = 0.7; k = 2.3)$. In addition, Fig. 2.2 represents equal value of n and k $(n = k = 2.3)$.

As the effect of thickness variation of rotating cylinders can be taken into account in their equation of equilibrium, the theory of the cylinders of variable thickness can give excellent results as that of the uniform thickness cylinders as long as they meet the assumption of plane strain. After considering this effect, the equation of equilibrium of rotating cylinder with variable thickness can be written as:

$$\frac{d}{dr}(hr\;_{rr}) \quad h \quad + h\;^2 r^2 = 0\,; \tag{2.4}$$

where is the constant angular velocity and is the density of the cylinder material.

$$+\left(\alpha_{11}\frac{dR(\bar{r})}{d\bar{r}}+\alpha_{12}\frac{R(\bar{r})}{\bar{r}}\right)\bigg], \tag{3.6a}$$

$$\sigma_{\theta\theta}(\bar{r})=\frac{e^{-n\bar{r}^k}}{b}\left[C_1\left(\alpha_{12}\frac{dP(\bar{r})}{d\bar{r}}+\alpha_{22}\frac{P(\bar{r})}{\bar{r}}\right)+C_2\left(\alpha_{12}\frac{dQ(\bar{r})}{d\bar{r}}+\alpha_{22}\frac{Q(\bar{r})}{\bar{r}}\right)\right.$$
$$\left.+\left(\alpha_{12}\frac{dR(\bar{r})}{d\bar{r}}+\alpha_{22}\frac{R(\bar{r})}{\bar{r}}\right)\right], \tag{3.6b}$$

$$\sigma_{zz}(\bar{r})=\frac{e^{-n\bar{r}^k}}{b}\left[C_1\left(\alpha_{13}\frac{dP(\bar{r})}{d\bar{r}}+\alpha_{23}\frac{P(\bar{r})}{\bar{r}}\right)+C_2\left(\alpha_{13}\frac{dQ(\bar{r})}{d\bar{r}}+\alpha_{23}\frac{Q(\bar{r})}{\bar{r}}\right)\right.$$
$$\left.+\left(\alpha_{13}\frac{dR(\bar{r})}{d\bar{r}}+\alpha_{23}\frac{R(\bar{r})}{\bar{r}}\right)\right], \tag{3.6c}$$

where the derivatives of P, Q and R are evaluated using the differentiation rule

$$\frac{d}{d\bar{r}}M(\xi,\eta,z)=\frac{\xi}{\eta}\frac{dz}{d\bar{r}}\ M(\xi+1,\eta+1,z). \tag{3.7}$$

Note that, if $n=m=0$ then $h(r)=h_0,\ c_{ij}=\alpha_{ij},\ \rho=\rho_0$ and the radial displacement given in (3.2) for the rotating uniform-thickness and density homogeneous orthotropic cylinder is reduced to

$$u(\bar{r})=C_1\bar{r}^{(\sqrt{\alpha_{22}/\alpha_{11}})}+C_2\bar{r}^{(-\sqrt{\alpha_{22}/\alpha_{11}})}+\frac{\rho_0\Omega^2b^3\bar{r}^3}{\alpha_{22}-9\alpha_{11}}, \tag{3.8a}$$

also, the corresponding stresses in this case are given by

$$\begin{aligned}\sigma_{rr}(\bar{r})=&\frac{1}{b}\left[C_1\left(\alpha_{11}\sqrt{\frac{\alpha_{22}}{\alpha_{11}}}+\alpha_{12}\right)\bar{r}^{(\sqrt{\alpha_{22}/\alpha_{11}}-1)}-C_2\left(\alpha_{11}\sqrt{\frac{\alpha_{22}}{\alpha_{11}}}-\alpha_{12}\right)\right.\\&\left.\times\,\bar{r}^{(-\sqrt{\alpha_{22}/\alpha_{11}}-1)}\right]+\left(\frac{3\alpha_{11}+\alpha_{12}}{\alpha_{22}-9\alpha_{11}}\right)\rho_0\Omega^2b^2\bar{r}^2,\end{aligned} \tag{3.8b}$$

$$\begin{aligned}\sigma_{\theta\theta}(\bar{r})=&\frac{1}{b}\left[C_1\left(\alpha_{12}\sqrt{\frac{\alpha_{22}}{\alpha_{11}}}+\alpha_{22}\right)\bar{r}^{(\sqrt{\alpha_{22}/\alpha_{11}}-1)}-C_2\left(\alpha_{12}\sqrt{\frac{\alpha_{22}}{\alpha_{11}}}-\alpha_{22}\right)\right.\\&\left.\times\,\bar{r}^{(-\sqrt{\alpha_{22}/\alpha_{11}}-1)}\right]+\left(\frac{3\alpha_{12}+\alpha_{22}}{\alpha_{22}-9\alpha_{11}}\right)\rho_0\Omega^2b^2\bar{r}^2,\end{aligned} \tag{3.8c}$$

$$\begin{aligned}\sigma_{zz}(\bar{r})=&\frac{1}{b}\left[C_1\left(\alpha_{13}\sqrt{\frac{\alpha_{22}}{\alpha_{11}}}+\alpha_{23}\right)\bar{r}^{(\sqrt{\alpha_{22}/\alpha_{11}}-1)}-C_2\left(\alpha_{13}\sqrt{\frac{\alpha_{22}}{\alpha_{11}}}-\alpha_{23}\right)\right.\\&\left.\times\,\bar{r}^{(-\sqrt{\alpha_{22}/\alpha_{11}}-1)}\right]+\left(\frac{3\alpha_{13}+\alpha_{23}}{\alpha_{22}-9\alpha_{11}}\right)\rho_0\Omega^2b^2\bar{r}^2.\end{aligned} \tag{3.8d}$$

In addition, for isotropic cylinder we have

$$\alpha_{11}=\alpha_{22}=\frac{E(1-\nu)}{(1+\nu)(1-2\nu)},\qquad \alpha_{12}=\alpha_{13}=\alpha_{23}=\frac{E\nu}{(1+\nu)(1-2\nu)}, \tag{3.9}$$

where E and ν are Young's modulus and Poisson's ratio of the cylinder material.

Using the formulas (3.9) we find that the solution (3.8) for the rotating uniform-thickness and density homogeneous isotropic cylinder takes the form:

$$u(\bar{r}) = C_1\bar{r} + \frac{C_2}{\bar{r}} - \frac{(1+\nu)(1-2\nu)}{8E(1-\nu)}\rho_0\Omega^2 b^3\bar{r}^3, \tag{3.10a}$$

$$\sigma_{rr}(\bar{r}) = \frac{E}{b(1+\nu)(1-2\nu)}\left[C_1 - C_2\frac{(1-2\nu)}{\bar{r}^2}\right] - \frac{(3-2\nu)}{8(1-\nu)}\rho_0\Omega^2 b^2\bar{r}^2, \tag{3.10b}$$

$$\sigma_{\theta\theta}(\bar{r}) = \frac{E}{b(1+\nu)(1-2\nu)}\left[C_1 + C_2\frac{(1-2\nu)}{\bar{r}^2}\right] - \frac{(1+2\nu)}{8(1-\nu)}\rho_0\Omega^2 b^2\bar{r}^2, \tag{3.10c}$$

$$\sigma_{zz}(\bar{r}) = \frac{2E\nu}{b(1+\nu)(1-2\nu)}\left[C_1 - \frac{(1+\nu)(1-2\nu)}{4E(1-\nu)}\rho_0\Omega^2 b^3\bar{r}^2\right]. \tag{3.10d}$$

The previous elastic solutions will be completed by calculating the integration constants C_i using the boundary conditions on the surface of the cylinder which may be free or clamped.

4 The Rotating Solid Cylinder

Here, we will obtain the elastic solutions for the rotating solid cylinder by the application of the boundary conditions. When the outer surface of the cylinder $(r = b)$ is free of any traction, hence the boundary condition is given by:

$$\sigma_{rr}(\bar{r}) = 0 \qquad at \qquad \bar{r} = 1. \tag{4.1}$$

Using the above condition and Eq. (3.6a), the constant C_1 is given by

$$C_1 = -\left(\frac{\alpha_{11}R'(1) + \alpha_{12}R(1)}{\alpha_{11}P'(1) + \alpha_{12}P(1)}\right), \tag{4.2}$$

where the prime $(')$ means differentiation with respect to $\bar{r}$. Since the stresses must be finite at the center of the cylinder $(r = 0)$, thus the constant C_2 must be set equal to zero.

The radial displacement and stresses for the rotating variable-thickness and density inhomogeneous orthotropic solid cylinder with free surface can be calculated from Eqs. (4.2), (3.2) and (3.6).

The solution (3.8) for the rotating uniform-thickness and density homogeneous orthotropic solid cylinder with free surface takes the form

$$u(\bar{r}) = \frac{\rho_0 \Omega^2 b^3}{\alpha_{22} - 9\alpha_{11}} \left[\bar{r}^3 - \left(\frac{3\alpha_{11} + \alpha_{12}}{\alpha_{11}\sqrt{\alpha_{22}/\alpha_{11}} + \alpha_{12}} \right) \bar{r}^{\sqrt{\alpha_{22}/\alpha_{11}}} \right], \tag{4.3a}$$

$$\sigma_{rr}(\bar{r}) = \frac{3\alpha_{11} + \alpha_{12}}{\alpha_{22} - 9\alpha_{11}} \left[\bar{r}^2 - \bar{r}^{(\sqrt{\alpha_{22}/\alpha_{11}} - 1)} \right] \rho_0 \Omega^2 b^2, \tag{4.3b}$$

$$\sigma_{\theta\theta}(\bar{r}) = \frac{3\alpha_{11} + \alpha_{12}}{\alpha_{22} - 9\alpha_{11}} \left[\left(\frac{3\alpha_{12} + \alpha_{22}}{3\alpha_{11} + \alpha_{12}} \right) \bar{r}^2 - \left(\frac{\alpha_{12}\sqrt{\alpha_{22}/\alpha_{11}} + \alpha_{22}}{\alpha_{11}\sqrt{\alpha_{22}/\alpha_{11}} + \alpha_{12}} \right) \times \bar{r}^{(\sqrt{\alpha_{22}/\alpha_{11}} - 1)} \right] \rho_0 \Omega^2 b^2, \tag{4.3c}$$

$$\sigma_{zz}(\bar{r}) = \frac{3\alpha_{11} + \alpha_{12}}{\alpha_{22} - 9\alpha_{11}} \left[\left(\frac{3\alpha_{13} + \alpha_{23}}{3\alpha_{11} + \alpha_{12}} \right) \bar{r}^2 - \left(\frac{\alpha_{13}\sqrt{\alpha_{22}/\alpha_{11}} + \alpha_{23}}{\alpha_{11}\sqrt{\alpha_{22}/\alpha_{11}} + \alpha_{12}} \right) \times \bar{r}^{(\sqrt{\alpha_{22}/\alpha_{11}} - 1)} \right] \rho_0 \Omega^2 b^2. \tag{4.3d}$$

Also, the radial displacement and stresses (3.10) for the rotating uniform-thickness and density homogeneous isotropic solid cylinder with free surface can be written as

$$u(\bar{r}) = \frac{(1+\nu)(1-2\nu)}{8E(1-\nu)} \left[(3-2\nu) - \bar{r}^2 \right] \rho_0 \Omega^2 b^3 \bar{r}, \tag{4.4a}$$

$$\sigma_{rr}(\bar{r}) = \frac{(3-2\nu)}{8(1-\nu)} \left[1 - \bar{r}^2 \right] \rho_0 \Omega^2 b^2, \tag{4.4b}$$

$$\sigma_{\theta\theta}(\bar{r}) = \frac{(3-2\nu)}{8(1-\nu)} \left[1 - \frac{(1+2\nu)}{(3-2\nu)} \bar{r}^2 \right] \rho_0 \Omega^2 b^2, \tag{4.4c}$$

$$\sigma_{zz}(\bar{r}) = \frac{\nu}{4(1-\nu)} \left[(3-2\nu) - 2\bar{r}^2 \right] \rho_0 \Omega^2 b^2, \tag{4.4d}$$

this is the well-known solution of the rotating uniform thickness cylinder [14].

On the other hand, when the outer surface of the cylinder $(r = b)$ is clamped, hence the boundary condition is given by:

$$u(\bar{r}) = 0 \qquad at \qquad \bar{r} = 1. \tag{4.5}$$

From Eqs. (4.5) and (3.2), the constant C_1 is given by

$$C_1 = -\frac{R(1)}{P(1)}. \tag{4.6}$$

Since the radial displacement must be finite at the axis of the cylinder $(r = 0)$, thus the constant C_2 must be vanished.

With the help of Eqs. (4.6), (3.2) and (3.6), we can obtain the radial displacement and stresses for the rotating inhomogeneous orthotropic solid cylinder with variable-thickness and density subjected to clamped surface.

The solution (3.8) for the rotating uniform-thickness and density homogeneous orthotropic solid cylinder with clamped surface takes the form:

$$u(\bar{r}) = \frac{\rho_0\Omega^2 b^3}{\alpha_{22} - 9\alpha_{11}} \left[\bar{r}^3 - \bar{r}^{\sqrt{\alpha_{22}/\alpha_{11}}}\right], \tag{4.7a}$$

$$\sigma_{rr}(\bar{r}) = \frac{\rho_0\Omega^2 b^2}{\alpha_{22} - 9\alpha_{11}} \left[(3\alpha_{11} + \alpha_{12})\bar{r}^2 - (\alpha_{11}\sqrt{\alpha_{22}/\alpha_{11}} + \alpha_{12})\bar{r}^{(\sqrt{\alpha_{22}/\alpha_{11}}-1)}\right], \tag{4.7b}$$

$$\sigma_{\theta\theta}(\bar{r}) = \frac{\rho_0\Omega^2 b^2}{\alpha_{22} - 9\alpha_{11}} \left[(3\alpha_{12} + \alpha_{22})\bar{r}^2 - (\alpha_{12}\sqrt{\alpha_{22}/\alpha_{11}} + \alpha_{22})\bar{r}^{(\sqrt{\alpha_{22}/\alpha_{11}}-1)}\right], \tag{4.7c}$$

$$\sigma_{zz}(\bar{r}) = \frac{\rho_0\Omega^2 b^2}{\alpha_{22} - 9\alpha_{11}} \left[(3\alpha_{13} + \alpha_{23})\bar{r}^2 - (\alpha_{13}\sqrt{\alpha_{22}/\alpha_{11}} + \alpha_{23})\bar{r}^{(\sqrt{\alpha_{22}/\alpha_{11}}-1)}\right]. \tag{4.7d}$$

Finally, the radial displacement and stresses (3.10) for the rotating homogeneous isotropic solid cylinder with uniform-thickness and density subjected to clamped surface can be written as

$$u(\bar{r}) = \frac{(1+\nu)(1-2\nu)}{8E(1-\nu)} \left[1 - \bar{r}^2\right] \rho_0\Omega^2 b^3 \bar{r}, \tag{4.8a}$$

$$\sigma_{rr}(\bar{r}) = \frac{\rho_0\Omega^2 b^2}{8(1-\nu)} \left[1 - (3-2\nu)\bar{r}^2\right], \tag{4.8b}$$

$$\sigma_{\theta\theta}(\bar{r}) = \frac{\rho_0\Omega^2 b^2}{8(1-\nu)} \left[1 - (1+2\nu)\bar{r}^2\right], \tag{4.8c}$$

$$\sigma_{zz}(\bar{r}) = \frac{\nu}{4(1-\nu)} \left[1 - 2\bar{r}^2\right] \rho_0\Omega^2 b^2. \tag{4.8d}$$

5 The Rotating Hollow Cylinder

In the present section, we will obtain the elastic solutions for the rotating hollow cylinder of internal radius a by the application of the boundary conditions. When the inner and outer surfaces $(r = a, r = b)$ of the cylinder are free of any traction, hence the boundary conditions are given by:

$$\sigma_{rr}(\bar{r}) = 0 \qquad at \qquad \bar{r} = a/b, \tag{5.1a}$$

$$\sigma_{rr}(\bar{r}) = 0 \qquad at \qquad \bar{r} = 1. \tag{5.1b}$$

From the conditions (5.1) and Eq. (3.7a), the constants C_1 and C_2 are given by

$$C_1 = \frac{S_{12}S_{23} - S_{13}S_{22}}{S_{11}S_{22} - S_{12}S_{21}}, \tag{5.2a}$$

$$C_2 = \frac{S_{13}S_{21} - S_{11}S_{23}}{S_{11}S_{22} - S_{12}S_{21}}, \tag{5.2b}$$

where

$$S_{11} = \alpha_{11} P'(a/b) + \alpha_{12} \frac{bP(a/b)}{a}, \qquad S_{21} = \alpha_{11} P'(1) + \alpha_{12} P(1),$$

$$S_{12} = \alpha_{11} Q'(a/b) + \alpha_{12} \frac{bQ(a/b)}{a}, \qquad S_{22} = \alpha_{11} Q'(1) + \alpha_{12} Q(1), \tag{5.3}$$

$$S_{13} = \alpha_{11} R'(a/b) + \alpha_{12} \frac{bR(a/b)}{a}, \qquad S_{23} = \alpha_{11} R'(1) + \alpha_{12} R(1).$$

Substituting from Eqs. (5.2) into Eqs. (3.2) and (3.6), we can get the radial displacement and stresses for the rotating inhomogeneous orthotropic hollow cylinder with variable-thickness and density subjected to free surfaces.

In addition, the solution for the rotating uniform-thickness and density homogeneous orthotropic hollow cylinder with free surfaces can be obtained from Eqs. (3.8) with the help of the following constants:

$$C_1 = \frac{(3\alpha_{11} + \alpha_{12}) \left((a/b)^{-\sqrt{\alpha_{22}/\alpha_{11}} - 3} - 1 \right) (a/b)^2 \rho_0 \Omega^2 b^3}{(\alpha_{22} - 9\alpha_{11})(\alpha_{11}\sqrt{\alpha_{22}/\alpha_{11}} + \alpha_{12}) \left((a/b)^{\sqrt{\alpha_{22}/\alpha_{11}} - 1} - (a/b)^{-\sqrt{\alpha_{22}/\alpha_{11}} - 1} \right)}, \tag{5.4a}$$

$$C_2 = \frac{(3\alpha_{11} + \alpha_{12}) \left((a/b)^{\sqrt{\alpha_{22}/\alpha_{11}} - 3} - 1 \right) (a/b)^2 \rho_0 \Omega^2 b^3}{(\alpha_{22} - 9\alpha_{11})(\alpha_{11}\sqrt{\alpha_{22}/\alpha_{11}} + \alpha_{12}) \left((a/b)^{\sqrt{\alpha_{22}/\alpha_{11}} - 1} - (a/b)^{-\sqrt{\alpha_{22}/\alpha_{11}} - 1} \right)}. \tag{5.4b}$$

Also, the radial displacement and stresses (3.10) for the rotating uniform-thickness and density homogeneous isotropic hollow cylinder with free surfaces become as

$$u(\bar{r}) = \frac{(1+\nu)(1-2\nu)}{8E(1-\nu)} \left[\frac{a^2+b^2}{b^2}(3-2\nu) + \frac{(3-2\nu)}{(1-2\nu)} \frac{a^2}{b^2 \bar{r}^2} - \bar{r}^2 \right] \rho_0 \Omega^2 b^3 \bar{r}, \tag{5.5a}$$

$$\sigma_{rr}(\bar{r}) = \frac{(3-2\nu)}{8(1-\nu)} \left[\frac{a^2+b^2}{b^2} - \frac{a^2}{b^2 \bar{r}^2} - \bar{r}^2 \right] \rho_0 \Omega^2 b^2, \tag{5.5b}$$

$$\sigma_{\theta\theta}(\bar{r}) = \frac{(3-2\nu)}{8(1-\nu)} \left[\frac{a^2+b^2}{b^2} + \frac{a^2}{b^2 \bar{r}^2} - \frac{(1+2\nu)}{(3-2\nu)} \bar{r}^2 \right] \rho_0 \Omega^2 b^2, \tag{5.5c}$$

$$\sigma_{zz}(\bar{r}) = \frac{\nu}{4(1-\nu)} \left[\frac{a^2+b^2}{b^2}(3-2\nu) - 2\bar{r}^2 \right] \rho_0 \Omega^2 b^2, \tag{5.5d}$$

this is the well-known solution of the rotating uniform thickness cylinder [14].

On the other hand, when the inner and outer surfaces $(r = a, r = b)$ of the cylinder are clamped, hence the boundary conditions are given by:

$$u(\bar{r}) = 0 \qquad at \qquad \bar{r} = a/b, \tag{5.6a}$$

$$u(\bar{r}) = 0 \qquad at \qquad \bar{r} = 1. \tag{5.6b}$$

With the aid of conditions (5.6) and Eq. (3.2) , the constants C_1 and C_2 are given by

$$C_1 = \frac{Q(a/b)R(1) - R(a/b)Q(1)}{P(a/b)Q(1) - Q(a/b)P(1)}, \tag{5.7a}$$

$$C_2 = \frac{R(a/b)P(1) - P(a/b)R(1)}{P(a/b)Q(1) - Q(a/b)P(1)}. \tag{5.7b}$$

The radial displacement and stresses for the rotating variable-thickness and density inhomogeneous orthotropic hollow cylinder with clamped surfaces can be obtained from Eqs. (5.7), (3.2) and (3.6).

Also, the solution (3.8) for the rotating uniform-thickness and density homogeneous orthotropic hollow cylinder with clamped surfaces can be calculated with the help of the following constants:

$$C_1 = \frac{\left((a/b)^{-\sqrt{\alpha_{22}/\alpha_{11}}-3} - 1\right)(a/b)^3 \rho_0 \Omega^2 b^3}{(\alpha_{22} - 9\alpha_{11})\left((a/b)^{\sqrt{\alpha_{22}/\alpha_{11}}} - (a/b)^{-\sqrt{\alpha_{22}/\alpha_{11}}}\right)}, \tag{5.8a}$$

$$C_2 = \frac{-\left((a/b)^{\sqrt{\alpha_{22}/\alpha_{11}}-3} - 1\right)(a/b)^3 \rho_0 \Omega^2 b^3}{(\alpha_{22} - 9\alpha_{11})\left((a/b)^{\sqrt{\alpha_{22}/\alpha_{11}}} - (a/b)^{-\sqrt{\alpha_{22}/\alpha_{11}}}\right)}. \tag{5.8b}$$

Finally, one can obtain easily the radial displacement and stresses (3.10) for the rotating uniform-thickness and density homogeneous isotropic hollow cylinder with clamped surfaces in the form:

$$u(\bar{r}) = \frac{(1+\nu)(1-2\nu)}{8E(1-\nu)}\left[\frac{a^2+b^2}{b^2} - \frac{a^2}{b^2\bar{r}^2} - \bar{r}^2\right]\rho_0\Omega^2 b^3 \bar{r}, \tag{5.9a}$$

$$\sigma_{rr}(\bar{r}) = \frac{\rho_0\Omega^2 b^2}{8(1-\nu)}\left[\frac{a^2+b^2}{b^2} + \frac{a^2(1-2\nu)}{b^2\bar{r}^2} - (3-2\nu)\bar{r}^2\right], \tag{5.9b}$$

$$\sigma_{\theta\theta}(\bar{r}) = \frac{\rho_0\Omega^2 b^2}{8(1-\nu)}\left[\frac{a^2+b^2}{b^2} - \frac{a^2(1-2\nu)}{b^2\bar{r}^2} - (1+2\nu)\bar{r}^2\right], \tag{5.9c}$$

$$\sigma_{zz}(\bar{r}) = \frac{\nu}{4(1-\nu)}\left[\frac{a^2+b^2}{b^2} - 2\bar{r}^2\right]\rho_0\Omega^2 b^2. \tag{5.9d}$$

6 Numerical Examples and Discussion

In this section, some numerical examples for the rotating inhomogeneous orthotropic solid and hollow cylinders with variable-thickness and density will be introduced. These examples will include some special cases of rotating homogeneous isotropic solid and hollow cylinders with uniform-thickness and density.

The numerical applications will be carried out for the radial displacement and stresses that being reported herein are in the following dimensionless form:

$$u_r = \frac{Eu(r)}{_0\ ^2b^3}; \qquad _r = \frac{_{rr}(r)}{_0\ ^2b^2}; \qquad = \frac{(r)}{_0\ ^2b^2}; \qquad _z = \frac{_{zz}(r)}{_0\ ^2b^2}:$$

The effect of the elastic properties of the cylinder material on the dimensionless radial displacement and stresses will be shown in Figs 6.1-6.16. Firstly, the effects of inhomogeneity and the orthotropy will be illustrated by putting $_{11} = \ _{22} = 1{:}0711409; \ _{12} = 0{:}27114093; \ _{13} = \ _{23} = 0{:}33557046$ and different values of n and m. The results are given for three profiles $(k = 0{:}7; n = 2{:}3; m = 0{:}25)$, $(k = n = 2{:}3; m = 0{:}5)$ and $(k = 2{:}3; n = 0{:}7; m = 1)$.

Figure 6.1: Dimensionless stresses and displacement for variable thickness – solid cylinder of free surface: $k = 0{:}7; \ n = 2{:}3; \ m = 0{:}25$:

Figure 6.2: Dimensionless stresses and displacement for variable thickness – solid cylinder of free surface: $k = \ n = 2{:}3; \ m = 0{:}5$:

[18] A. T. Vasilenko and G. K. Sudavtsova, Elastic equilibrium of circumferentially inhomogeneous orthotropic cylindrical shells of arbitrary thickness, *Int. Appl. Mech.* **37** (2001), 1046–1054.

[19] J. T. S. Wang and C. C. Lin, Stresses in rotating composite cylindrical shells, *Compos. Struct.* **25** (1993), 157–164.

[20] A. M. Zenkour, An exact solution for the bending of thin rectangular plates with uniform, linear and quadratic thickness variation, *Int. J. Mech. Sci.* **45** (2003), 295–315.

[21] A. M. Zenkour, Rotating variable-thickness orthotropic cylinder containing a solid core of uniform-thickness, *Arch. Appl. Mech.* **76** (2006), 89–102.

[22] A. M. Zenkour, Stresses in cross-ply laminated circular cylinders of axially variable thickness, *Acta Mech.* **187** (2006), 85–102.

[23] A. M. Zenkour and M. E. Fares, Bending, buckling and free vibration of non-homogeneous composite laminated cylindrical shells using a refined first-order theory, *Compos. Part B eng.* **32** (2001), 237–247.