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Review of quantum discord in bipartite and multipartite systems

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Abstract: Quantum information and computation, based on quantum correlations of quantum systems, is one of the promising new fields of physics and information science. Entanglement is the key resource of quantum information processing. However, it is shown recently that quantum entanglement is not the only kind of quantum correlations useful for quantum information processing. Quantum systems without any entanglement can still be helpful for quantum information processing since unentangled systems could have another kind of quantum correlations – quantum discord. Thus, quantum discord has drawn much attention in recent years. Here, we briefly review the concepts and properties of quantum discord. Several measures of discord including the original measure of discord, relative entropy-based discord, geometric discord, global discord, Gaussian discord, are reviewed and discussed.

Keywords: Quantum discord, quantum entanglement, quantum information

I. Introduction

Nowadays, quantum computation and information has attracted much attention of researchers from several scientific communities including physics, information science, and mathematics [1]. Entanglement is regarded as the key resource of quantum information processing. It plays an important role in many quantum protocols such as quantum teleportation, quantum key distribution, and quantum algorithm [2]. However, quantum entanglement is not the only kind of quantum correlations useful for quantum information processing [3–5]. It has shown both theoretically [6–13] and experimentally [14] that some tasks can be sped up over their classical counterparts using fully separable and highly mixed states.

Quantum discord first introduced in [15,16] is another kind of quantum correlations different from entanglement. In 2008, it was shown that separable states with quantum discord can be used to implement deterministic quantum computation with one qubit [14]. Later, other measures of quantum discord were proposed by several authors [17, 18].

There are, in general, two kinds of quantum discord: measurement-based discord and distance-based discord. The original definition of discord in [15,16] is the measurementbased discord. This kind of discord based on the fact that a local measurement performed upon a subsystem of a multipartite system disturbs the whole system. It is in general not possible to obtain all information contained in a subsystem by performing only local measurements on it. This is totally different from classical systems. Physically, quantum discord measures the amount of mutual information that is not locally accessible of multipartite systems. Distance-based discord is adopted in [17, 18]. This kind of discord is defined as the minimal distance of a quantum state and all states with zero discord. In [17], the authors employed relative entropy as a measure of distance of two states. A unified view of correlations is established with the help of quantum relative entropy in [17]. Comparing with the original definition of quantum discord, this kind of definition allows one to put all correlations (classical correlations, quantum discord, dissonance, and entanglement) on an equal footing. Unlike in [17], the authors of [18] adopted the square norm in the Hilbert-Schmidt space as a measure of distance between two states. Particularly, for arbitrary two-qubit systems, an analytical expression is obtained in [18]. It is similar to the geometry measure of quantum entanglement [19]. As a result, this kind of measure is also called the geometric measure of quantum discord (geometric discord). Other measures of quantum discord have also been proposed [20, 21].

Dynamics of quantum discord in several physical systems such as cavity QED [22–26], spin chains [27–30], and quantum dots [31] systems has been investigated extensively in the pase few years. The monogamy [32] and conservation law [33] of entanglement and discord were also discussed. Besides, the non-Markovian effects upon the dynamics of quantum discord was studied [34,35].

The organization of the review is as follows. In section 2, we first introduce the measurement-based discord or the original discord defined in [15,16]. Other measurement-based discord measures including global discord and Gaussian discord are also reviewed. We summarize and discuss the basic properties of them. In section 3, we review two important distance-based discord: relative entropy-based discord and square norm-based discord (or geometric discord). In section 4, we briefly review other measures of quantum correlations such as measurement-induced disturbance, quantum deficit, and locally inaccessible information. In section 5, the dynamics of quantum discord in several systems are reviewed and discussed.

2. Measurement-based discord

It is well-known that for a classical system, one can, in principle, obtain all information of the system without disturbing it. However, for quantum systems, this is in general impossible. Measurements can modify quantum systems and two equivalent expressions of mutual information in classical information theory are not the same for quantum systems. This is the idea of measurement-based discord as we will see more precisely later.

2.1. The original one

2.1.1. Definition of discord

Suppose we have two random variables $\mathcal{X}$ and $\mathcal{Y}$ whose ignorance is described by the Shannon entropy $H(\mathcal{X})$ and $H(\mathcal{Y})$, respectively. Here, the Shannon entropy is defined as follows $H(\mathcal{X}) = -\sum_{p_x\in\mathcal{X}} p_x \log p_x$. In other words, the Shannon entropy $H(\mathcal{X})$ of $\mathcal{X}$ quantifies how much information one can obtain after he learns the value of $\mathcal{X}$ averagely [1]. In classical information theory [1], there are two equivalent expressions of mutual information. The first one is defined as

$$\mathcal{I}(\mathcal{X}:\mathcal{Y}) = H(\mathcal{X}) + H(\mathcal{Y}) - H(\mathcal{X},\mathcal{Y}). \tag{1}$$

Here $H(\mathcal{X},\mathcal{Y}) = -\sum_{p_{(x,y)}\in(\mathcal{X},\mathcal{Y})} p_{(x,y)} \log p_{(x,y)}$, is the joint entropy of the pair $(\mathcal{X},\mathcal{Y})$. It accounts our total ignorance about the pair $(\mathcal{X},\mathcal{Y})$. The second one is

$$\mathcal{J}(\mathcal{X}:\mathcal{Y}) = H(\mathcal{X}) - H(\mathcal{X}|\mathcal{Y}), \tag{2}$$

with $H(\mathcal{X}|\mathcal{Y})$ being the conditional entropy of $\mathcal{X}$ given $\mathcal{Y}$. In classical information theory, $H(\mathcal{X}|\mathcal{Y}) = H(\mathcal{X},\mathcal{Y}) - H(\mathcal{Y})$, and two definitions of mutual information in Eqs. (1-2) are the same.

The situation is fundamentally different in quantum information theory since measurements can disturb quantum systems and two equivalent mutual information expressions $\mathcal{I}$ and $\mathcal{J}$ are mismatched for quantum systems. In quantum information theory the Shannon entropy, random variables $(\mathcal{X},\mathcal{Y})$, $\mathcal{X}$, and $\mathcal{Y}$ are replaced by the von Neumann entropy, ρ_{AB}, ρ_A, and ρ_B, respectively. Here, ρ_{AB} is the density matrix of the whole system AB; $\rho_A = Tr_B(\rho_{AB})$ and $\rho_B = Tr_A(\rho_{AB})$ are the reduced density matrix of subsystem A and subsystem B. The von Neumann entropy of a density matrix is $S(\rho) = -Tr(\rho \log \rho)$. Assume we perform a set of local projective measurements (von Neumann measurements) $\{\Pi_B^{(j)}\} = \{|j_B\rangle\langle j_B|\}$ on subsystem B. The measurements will disturb subsystem B and the whole system AB simultaneously. The state of the whole system related to the measurement $\Pi_B^{(j)}$ is

$$\rho_{AB|j} = \frac{1}{p_j}(I_A \otimes \Pi_B^{(j)})\rho_{AB}(I_A \otimes \Pi_B^{(j)}), \tag{3}$$

with I_A being the identity matrix of subsystem A. Here, we perform measurements on subsystem B only. Note that $p_j = Tr[(I_A\otimes\Pi_B^{(j)})\rho_{AB}(I_A\otimes\Pi_B^{(j)})]$ is the probability of obtaining the outcome j. The conditional entropy is $S(\rho_{AB}|\{\Pi_B^{(j)}\}) = \sum_j p_j S(\rho_{A|j})$, with $\rho_{A|j} = Tr_B(\rho_{AB|j})$ being the reduced density matrix of subsystem A after measurements. Thus, the two expressions of quantum mutual information are defined by

$$\mathcal{I}(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{A,B}), \tag{4}$$

$$\mathcal{J}(\rho_{AB}) = S(\rho_A) - S(\rho_{AB}|\{\Pi_B^{(j)}\}). \tag{5}$$

It has been noticed that [15,16] these two quantities are in general different.

The original measure of quantum discord is defined as

$$\overleftarrow{D}^{ori}_{AB}(\rho_{AB}) = \mathcal{I}(\rho_{AB}) - \max_{\Pi_B}\{\mathcal{J}(\rho_{AB})\}. \tag{6}$$

The maximization is taken over all possible measurements $\{\Pi_B^{(j)}\}$. It is introduced to eliminate the dependence of discord upon measurements. The first term is the total correlations (quantum and classical correlations) of the whole system. The second term of the above equation is all information can be obtained by performing local measurements on subsystem B only. It is often referred to as the classical correlations [16]. From the above discuss, we see that discord is the difference between the total correlations and classical correlations. It denotes quantum correlations that is not locally accessible [31]. Here, $\overleftarrow{D}^{ori}_{AB}$ stands for quantum discord of ρ_{AB} by performing measurements on subsystem B. One can also perform measurements on subsystem A and the corresponding discord is denoted by $\overrightarrow{D}^{ori}_{AB}$. Note that we can perform positive-operator-valued measurements too [21].

An equivalent definition of discord is as follows [36]

$$\overleftarrow{D}^{ori}_{AB}(\rho_{AB}) = \mathcal{I}(\rho_{AB}) - \max_{\Pi_B}\{\mathcal{I}(\Lambda_B(\rho_{AB}))\}, \tag{7}$$

with $\Lambda_B(\rho_{AB}) = \sum_j (I_A \otimes \Pi_B^{(j)})\rho_{AB}(I_A \otimes \Pi_B^{(j)}) = \sum_j p_j \rho_{A|j} \otimes |j_B\rangle\langle j_B|$. Using the basic property of von Neumann entropy (see Eq. (11.57) of [1]), one can prove that the above definition of discord is equivalent to the original one in Eq. (6). This definition of discord can be interpreted as the minimal loss of correlations measured by the quantum mutual information due to measurements.

As one can see clearly from Eqs. (6-7), the main obstacle in calculating discord is the complicated optimization process. Therefor, it is usually difficult to evaluate quantum discord for generic states. Even for the simplest case of two-qubit states, analytical expressions of discord have been obtained only for a certain class of highly symmetrical states, such as Bell-diagonal [37], rank-2 [38], and X states [39]. Recently, quantum discord of N-qubit X states is also calculated [40].

2.1.2. Basic properties of the original discord

Here, we summarize some basic properties of discord:
(1) Discord is nonnegative and almost all quantum states have nonzero discord , $\overleftarrow{D}^{ori}_{AB} \geq 0$ and $\overrightarrow{D}^{ori}_{AB} \geq 0$ [36,41].
(2) Discord is zero if and only if local measurements can not disturb quantum systems [15],
i.e., $\overleftarrow{D}^{ori}_{AB} = 0 \Leftrightarrow \Lambda_B(\rho_{AB}) = \rho_{AB}$ and $\overrightarrow{D}^{ori}_{AB} = 0 \Leftrightarrow \Lambda_A(\rho_{AB}) = \rho_{AB}$
(3) Discord is not symmetric, i.e., $\overleftarrow{D}^{ori}_{AB} \neq \overrightarrow{D}^{ori}_{AB}$ in general.
(4) Discord remains unchanged by performing local unitary transformation.
(5) Discord of a bipartite pure state coincides with the von Neumann entropy of entanglement.

Property (1) follows from the monotonicity of quantum relative entropy and the fact that quantum mutual information decreases under local measurements [36]. From Eq. (7), we see that quantum mutual information of a quantum state before measurements is always larger than or equal to that of a quantum state after measurements. It is shown numerically that almost all quantum states is discordant (with nonzero discord) in [41]. Now, we give a simple example to check properties (2 - 3). Consider the following two-qubit state $\rho_{AB} = \frac{1}{2}(|0\rangle\langle 0|\otimes|0\rangle\langle 0|+|1\rangle\langle 1|\otimes|+\rangle\langle +|)$ with $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$. There is no entanglement in the above state. On the one hand, one can easily see that $\overrightarrow{D}_{AB} = 0$ since $\Lambda_A(\rho_{AB}) = \rho_{AB}$. On the other hand, any projective measurement performed on the second qubit disturbs the first qubit and quantum discord $\overleftarrow{D}_{AB}$ is nonzero. Note that for some states, $\overleftarrow{D}^{ori}_{AB} = \overrightarrow{D}^{ori}_{AB}$ and this does not conflict with property (3). Property (4) follows from the fact that the von Neumann entropy
$S(\Lambda_B(\rho_{AB}))$ is equal to $S((U_A \otimes U_B)\Lambda_B(\rho_{AB})(U_A \otimes U_B)^\dagger)$. Property (5) can be directly checked by using the Schmidt decomposition of a pure state [37]. However, discord is not equal to von Neumann entropy of entanglement for mixed states.

Separable states can still have nonzero discord. For example, a Werner state $p|\psi\rangle\langle\psi| + (1-p)/4$ is disentangled if $p < 1/3$ while its discord is larger than zero for $0 < p < 1/3$. For more details, see Ref.[37]. Finally, we would like to point out that one can define a symmetric version of discord as $D^{sym}_{AB} = \frac{1}{2}(\overleftarrow{D}_{AB} + \overrightarrow{D}_{AB})$ [34]. One can also define a two-way quantum discord as $D^{max}_{AB} = \max\{\overleftarrow{D}_{AB}, \overrightarrow{D}_{AB}\}$. Then, a quantum state ρ_{AB} contains no quantum correlations completely if $D^{max}_{AB} = 0$.

2.2. Gaussian discord

2.2.1. Definition of Gaussian discord

Continuous variable systems are very important for quantum information processing [42]. One important class of continuous variable states are the Gaussian states whose Wigner functions are Gaussian. The main obstacle in calculating discord for continuous states is still the minimization over all possible measurements in one subsystem. For a Gaussian state, analytical expressions of quantum discord are derived in [43,44]. Note that measurements are Gaussian too, i.e., states after measurements are still Gaussian. It is well-known that a two-mode Gaussian state ρ_{AB} is completely described, up to local displacements, by its covariance matrix σ, with $\sigma_{ij} = Tr[\rho_{AB}(R_iR_j + R_jR_i)]$ and $\overrightarrow{R} = (x_A, p_A, x_B, p_B)$ being the vector of phase-space operators [43]. Using local unitary operations, covariance matrix of a two-mode Gaussian state can be transformed in a standard form with diagonal sub-blocks $\sigma = \begin{pmatrix} \alpha & \gamma \\ \gamma^t & \beta \end{pmatrix}$. The Gaussian discord is [43,44]

$$\overleftarrow{D}^{gau}(\rho_{AB}) = f(\sqrt{B}) - f(\lambda_+) - f(\lambda_-) + f(\sqrt{E_m}), \tag{8}$$

with $f(x) = \frac{x+1}{2}\log\left(\frac{x+1}{2}\right) - \frac{x-1}{2}\log\left(\frac{x-1}{2}\right)$, $A = \det\alpha$, $B = \det\beta$, $C = \det\gamma$, $D = \det\sigma$, $2\lambda_\pm^2 = \Delta \pm \sqrt{\Delta^2 - 4D}$, and $\Delta = A + B + 2C$. Here E_m is as follows

$$E_m = \begin{cases} \frac{2C^2+(B-1)(D-A)+2|C|\sqrt{C^2+(B-1)(D-A)}}{(B-1)^2} & \text{if } g \le 0, \\ \frac{AB-C^2+D-\sqrt{C^4+(-AB+D)^2-2C^2(AB+D)}}{2B} & \text{if } g > 0, \end{cases} \tag{9}$$

where $g = (D - AB)^2 - C^2(B+1)(D+A)$. In the case of $g \le 0$, Gaussian discord is optimized by a set of heterodyne measurements and states after measurements are squeezed thermal states. In the case of $g > 0$, Gaussian discord is optimized by a set of homodyne measurements and states after measurements are pure states of infinite squeezing [43].

2.2.2. Basic properties of Gaussian discord

Basic properties of Gaussian discord are as follows:
(1) Almost all two-mode Gaussian states have nonzero Gaussian discord [43,44].
(2) Gaussian discord of two-mode separable Gaussian states is less than or equal to 1 [43,44].
(3) Gaussian states are the most robust bipartite continuous-variable states with fixed energy against disentanglement due to noisy evolutions in Markovian Gaussian channels [45].

The authors of [46] investigated the evolution of the Gaussian discord between two resonant harmonic oscillators interacting with a common environment. They argued that Gaussian discord may be not always a good approximation of true discord since the asymptotic value of it is shown to be a nondecreasing function of temperature [46]. In [47], geometric discord for Gaussian states was discussed. The quantum correlations of non-Gaussian Werner states was also studied [48].

2.3. Global discord

2.3.1. Definition of global discord

The original definition of discord is asymmetric and is not suitable for multipartite systems. In [49], the authors proposed a symmetric measure for multipartite quantum correlations – global discord. This is a generalization of the original bipartite discord to multipartite systems. Suppose we have N particles system consisting of A_1, A_2, ... , A_N whose density matrix is $\rho_{A_1A_2\ldots A_N}$. Now, we perform a set of local measurements $\{\Pi_{A_1}^{(j_1)} \otimes \Pi_{A_2}^{(j_2)} \otimes \ldots \otimes \Pi_{A_N}^{(j_N)}\}$. The global discord is defined as [49]

$$D^{glo}(\rho_{A_1A_2\ldots A_N}) = \min_{\Pi}\{S(\rho_{A_1\ldots A_N}||\Lambda(\rho_{A_1\ldots A_N})) - \sum_{j=1}^{N} S(\rho_{A_j}||\Lambda_j(\rho_{A_j}))\}, \tag{10}$$

with $\Lambda(\rho_{A_1...A_N}) = \sum_{j_1,...,j_N} (\Pi_A^{(j)} \rho_{A_1...A_N} \Pi_A^{(j)})$ and $\Pi_A^{(j)} = (\Pi_{A_1}^{(j_1)} \otimes ... \otimes \Pi_{A_N}^{(j_N)})$. Note that $\Lambda_j(\rho_{A_j}) = \sum_k (\Pi_{A_j}^{(k)} \rho_{A_j} \Pi_{A_j}^{(k)})$. Here, $S(\rho||\sigma) = Tr(\rho \log \rho - \rho \log \sigma)$ is the quantum relative entropy [1]. In [49], an analytical expression of global discord of three-qubit GHZ-Werner states was obtained. Besides, the multipartite correlations in the Ashkin-Teller chain was investigated and discussed. They found that this model can show an infinite-order quantum critical point. This phenomenon is detected by an extremum of global discord but can not be detected by bipartite discord of Eq. (6). This example shows clearly the power of global discord in detecting multipartite correlations.

Another equivalent expression of global discord is also proposed in [50]

$$D^{glo}(\rho_{A_1 A_2 ... A_N}) = \min_{\Pi} \{I(\rho_{A_1...A_N}) - I(\Lambda(\rho_{A_1...A_N}))\}, \quad (11)$$

where the quantum mutual information is defined as follows $I(\rho_{A_1...A_N}) = \sum_{j=1}^{N} S(\rho_{A_j}) - S(\rho_{A_1...A_N})$. This kind of definition allows us to interpret global discord of multipartite systems as the minimal loss of mutual information due to all possible locally projective measurements on all subsystems which is similar to the original definition of bipartite systems in Eq. (7). Analytical expressions of global discord of N-qubit GHZ-Werner states and a specific class of N-qubit states are derived in [50].

2.3.2. Basic properties of global discord

Basic properties of global discord are as follows:
(1) Global discord is nonnegative.
(2) Global discord is symmetric.

Property (1) is similar to discord of bipartite systems in [49]. This property follows from the monotonicity of the quantum relative entropy under partial trace and the properties of projective measurements [49]. Property (2) is true since local measurements are performed on all subsystems which is different with the original one in Eq. (6).

3. Distance-based discord

In the previous section, we review some measures of discord based on measurements. Now, we introduce two kinds of distance-based discord: relative entropy-based discord [17] and square norm-based discord (geometric discord) [18]. Distance-based discord is defined as the minimal distance of a quantum state and all states with zero discord. In [17], the authors employed relative entropy as a measure of distance of two states. Another distance-based discord is based on the square norm in the Hilbert-Schmidt space which is also a measure of distance between two states [18].

3.1. Relative entropy-based discord

A unified view of correlations is established with the help of the relative entropy. According to [17], relative entropy-based discord is defined as the relative entropy between a quantum state and its closest classical states [17]

$$D^{rel}(\rho) = \min_{\chi \in C} S(\rho||\chi), \quad (12)$$

where $S(\rho||\chi) = Tr(\rho \log_2 \rho) - Tr(\rho \log_2 \chi)$. It is the relative entropy between ρ and χ. Here, C denotes all classical states. Intuitively, D^{rel} is the minimal distance between a given state ρ and all classical states χ.

The advantages of this measure is as follows. First of all, it allows us to put all correlations such as quantum discord, classical correlations, and entanglement on an equal footing [17]. This is an elegant unified view of both quantum and classical correlations. It is usually meaningless to compare entanglement, measured by concurrence or negativity, and discord directly because they are totally different measures of quantum correlations [17]. However, one can compare the relative entropy of entanglement and discord directly. This allow us to discuss and compare the behavior of entanglement and discord in a unified scheme. This approach is also applicable for multipartite systems. For Bell-diagonal states, the relative entropy of entanglement and discord can be calculated analytically [17]. Note that relative entropy-based discord is nonnegative which is a direct result from the Klein's inequality (see Eq. (11.51) of [1]). It is zero if and only if the state considered is classical itself.

3.2. Square norm-based discord or geometric discord

Geometric discord or square norm-based discord [18] is similar to the geometric measure of entanglement [19]. According to [18], Geometric discord is defined as

$$D^{geo}(\rho) = \min_{\chi} ||\rho - \chi||^2, \tag{13}$$

where the minimum is over all possible classical states χ of the form $p_1|\psi_1\rangle\langle\psi_1|\otimes\rho_1+p_2|\psi_2\rangle\langle\psi_2|\otimes\rho_2$ with $p_1+p_2 = 1$. $|\psi_1\rangle$ and $|\psi_2\rangle$ are two orthonormal basis of subsystem A, ρ_1 and ρ_2 are two density matrices of subsystem B. Here, $||\rho - \chi||^2 = Tr(\rho - \chi)^2$ is the square norm of Hilbert-Schimidt space.

Particularly, for arbitrary two-qubit systems, an analytical expression is obtained in [18]. A two-qubit state can be written as

$$\rho = \frac{1}{4}[I_A \otimes I_B + \sum_{i=1}^{3} x_i\sigma_i \otimes I_B + \sum_{i=1}^{3} y_i I_A \otimes \sigma_i$$
$$+ \sum_{i,j=1}^{3} t_{ij}\sigma_i \otimes \sigma_j], \tag{14}$$

with $x_i = Tr[\rho(\sigma_i \otimes I_B)]$, $y_i = Tr[\rho(I_B \otimes \sigma_i)]$, $t_{ij} = Tr[\rho(\sigma_i \otimes \sigma_j)]$, σ_i being the three Pauli matrices. The geometric discord of a two-qubit state of the above equation is

$$D^{geo} = \frac{1}{4}(||\overrightarrow{x}||^2 + ||T||^2 - \lambda_{\max}), \tag{15}$$

where $\lambda_{\max}$ is the maximal eigenvalues of matrix $\Omega = \overrightarrow{x}\overrightarrow{x}^t + TT^t$. Here, the superscript t stands for transpose of a vector or matrix, $\overrightarrow{x}$ is a column vector with $||\overrightarrow{x}||^2 = x_1^2 + x_2^2 + x_3^2$, and $T = (t_{ij})$ is a 3×3 matrix. This analytical expression of geometric discord allows us to calculate the quantum discord of arbitrary two-qubit states. Later, geometric discord for arbitrary states has been calculated by Luo and Fu [51]. The lower bound to geometric discord in a general bipartite states bas been investigated in [54,55]. For $2 \times N$- dimensional states, the minimization of Eq. (15) can be performed analytically. But, these expressions are in general difficult to measure experimentally. Very recently, Girolami and Adesso [52] proposed a scheme to measure a tight lower bound of geometric discord by measuring only a small number of observables on at most four copies of the state without the need for a full tomography. In [53], a scheme based on deterministic quantum computation with one quantum qubit has been proposed to measure the amount of geometric discord.

4. Other measures of quantum correlations

There are other measures of quantum correlations such as measurement-induced disturbance (MID) [56], quantum deficit [60,4], and locally inaccessible information [34].

MID introduced by Luo in [56] is based on the fact that one can, in principle, obtain all information of a classical system by performing measurements without disturbing it. If a system has quantum correlations, the situation is changed since a measurement disturbs the system. MID is defined as

$$M(\rho_{AB}) = I(\rho_{AB}) - I(\rho'_{AB}), \tag{16}$$

with $\rho'_{AB} = \sum_{i,j}[(\Pi_A^{(i)} \otimes \Pi_B^{(j)})\rho_{AB}(\Pi_A^{(i)} \otimes \Pi_B^{(j)})]$. In fact, MID is the cost of entropy of measurements [56]. From the above equation, we see that MID is easy to calculate since it does not involve optimization as in the original definition of discord in Eq. (6). Note that for some classical states MID is nonzero which is unreasonable [57]. Thus, another version of MID is introduced [58,59]

$$M^{opt} = I(\rho_{AB}) - \max_{\Pi_A \otimes \Pi_B} I(\rho'_{AB}), \tag{17}$$

where optimization is over general local measurements. This measure is studied in [57–59].

Quantum deficit is based on work extraction from a quantum system interacting with a heat bath. For a classical without any quantum correlations, the work that could be extracted from the whole system is denoted by W_{total}. This work (also denoted by W_{LOCC}) can also be extracted from subsystems using appropriate local operations and classical communication (LOCC). For classical systems $W_{total} = W_{LOCC}$. However, for systems with quantum correlations, $W_{total} \neq W_{LOCC}$. Quantum deficit is defined as the difference of these two quantities $W_{total} - W_{LOCC}$ [60,4]. There are various forms of quantum deficit such as zero-way, one-way, and two-way quantum deficit [60,4].

Locally inaccessible information (LII) is based on the fact that a fraction of quantum mutual information can not be accessed locally [34]. The distinction between quantum discord and entanglement of formation for describing quantum correlations is discussed. Several relations between entanglement of formation and LII are also derived in [34].

5. Dynamics of discord

In this section, we briefly review some dynamical properties of quantum discord under the Markovian or non-Markovian environments [22–35].

5.1. Discord in Cavity QED

In recent years, many efforts have been invested in the study of the evolution of entanglement in joint systems formed by two subsystems (each system locally interacts with its environment) [61–64]. In particular, the entanglement of a two-qubit system may disappear for a finite time during the dynamics evolution. The nonsmooth finite-time disappearance of entanglement is referred to as entanglement sudden death. This phenomenon has been observed in the laboratory by several groups for optical setups [65,66] and atomic ensembles [67].

However, it has been pointed out that almost all quantum states are discordant states (states with nonzero discord) and quantum discord is more robust than entanglement under the Markovian environments [41]. It has been shown that, for several quantum systems, there is no quantum discord sudden death [35,68–70]. Besides, it has been shown that [22] quantum discord of two noninteracting atoms within a dissipative cavity reaches an asymptotic nonzero value even when the mean photon number of the cavity field is very large. The cavity field always has some quantum correlations (measured by quantum discord) in the macroscopic limit. The authors of [25] considered the dynamics of quantum discord and entanglement of a quantum system formed by two two-level atoms within two spatially separated and dissipative cavities in the weak- or strong-coupling limit. In the weak-coupling limit, quantum discord can stay almost zero for a finite time [25]. In contrast to the weak-coupling limit, in the strong-coupling limit, sudden death of quantum discord disappears though the entanglement sudden death phenomenon occurs.

Sudden transition between quantum and classical decoherence has been studied by Mazzola, Piilo, and Maniscalco [71]. They find that there is a sharp transition between the loss of quantum and classical correlations in a composite open system. If the evolution time is smaller than a particular time interval, then quantum discord do not decay completely. This phenomenon is confirmed experimentally later [72]. Recently, this phenomenon has been investigated in two dispersive Jaynes-Cummings models [23]. Quantum discord of two atoms is completely unaffected by the dissipations of cavities if the decay rate of two cavities and the atom-field coupling strength are chosen appropriately [23].

All the previous studies are restricted to Markovian environments while non-Markovian effects has not been taken into consideration. It has been shown that quantum discord and entanglement of two qubits in independent and common environments can behave very differently. In the case of non-Markovian environments, a fraction of quantum discord can be stored in non-Markovian environments which can return to systems. As a result, quantum discord of the systems can be gained again [34]. This phenomenon is referred to as *"sudden birth of discord"*. However, there is no *"sudden birth of entanglement"*. The non-Markovian effects upon the evolution of discord of two independent qubits within two zero-temperature non-Markovian reservoirs was discussed in [35]. There is no occurrence of the sudden death, but only instantaneous disappearance of discord at some time points [35].

5.2. Discord in spin and quantum dot systems

Quantum discord in a cluster-like one-dimensional system with triplet interactions has been studied by Chen, Li, and Yin [27]. This is a topologically ordered system and the global difference of topology induced by dimension is reflected in local quantum correlations [27]. Quantum discord in general decays exponentially in both topological and magnetic phases. In [28], quantum discord of a two-dimensional systems with topological quantum phase transition has been discussed. In this system, the whole system has quantum correlations while two local spins are classically correlated [28].

Recently, the propagation of quantum correlations via a spin-1/2 chain was investigated in [29]. The role of magnetic field in the dynamics of quantum discord was also discussed. Quantum discord can be transported more efficiently than entanglement for many initial conditions and working points [29]. The effects of Dzialoshinski-Moriya interaction upon quantum discord of a spin-star system has been investigated in [30]. The results of [30] shows that strong Dzialoshinski-Moriya interaction can increase quantum discord and thermal entanglement. However, a strong magnetic field and high temperature can decrease quantum discord and thermal entanglement.

The dynamics of quantum entanglement and discord of two coupled double quantum dots interacting with an oscillator bath has been investigated in [31]. The results of this paper shows that quantum discord is more robust to dissipation than entanglement of formation of this system. Particularly, they pointed out that, even in the case of high temperatures, quantum discord could be finite in the asymptotic limit [31].

6. Conclusions

In this paper, we have reviewed different measures of quantum discord and their basic properties briefly. Quantum discord introduced about ten years ago has been extensively studied in the past few years. Several measures of discord including the original one, global discord, Gaussian discord, relative entropy-based discord, square norm-based or geometric discord have been reviewed here. We also discussed other related measures of quantum correlations such as measurement-induced disturbance, quantum deficit, and locally inaccessible information. The dynamics of quantum discord in cavity QED, spin chains, and quantum dots were summarized and compared.

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