Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/020404

Multiple Homoclinic Solutions for a Class of Fractional Hamiltonian Systems

Kaimin Teng[*]

Department of Mathematics, Taiyuan University of Technology, Taiyuan, Shanxi 030024, People's Republic of China

Received: 11 Jan. 2016, Revised: 11 Jul. 2016, Accepted: 15 Jul. 2016
Published online: 1 Oct. 2016

Abstract: In this paper, we establish the existence of infinitely many homoclinic solutions for a class of fractional Hamiltonian systems. Our main techniques are based on Fountain Theorem and dual Fountain Theorem.

Keywords: Fractional Hamiltonian systems, Fountain theorem, homoclinic solutions.

1 Introduction

In this paper, we are concerned with the existence of infinitely many homoclinic solutions for a class of fractional Hamiltonian systems of the following form

$$\begin{cases} {}_tD_\infty^\alpha[{}_{-\infty}D_t^\alpha u(t)]+L(t)u(t)=\nabla F(t,u(t)),\ t\in\mathbb{R}, \\ u\in H^\alpha(\mathbb{R},\mathbb{R}^N), \end{cases} \tag{1}$$

where ${}_tD_\infty^\alpha$ and ${}_{-\infty}D_t^\alpha$ are the Liouville fractional derivatives of order $\frac{1}{2}<\alpha<1$ respectively, $F\in C^1(\mathbb{R}\times\mathbb{R}^N,\mathbb{R})$ is a given function satisfying some assumptions and $\nabla F(t,\cdot)$ is the gradient in the second variable, and $L\in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric matrix valued function.

As we know that homoclinic orbits of dynamical systems are important in applications, such as they may be "organizing center" for the dynamics in their neighborhood, if they exist, under certain conditions, we can infer the existence of chaos nearby or the bifurcation behavior of periodic orbits. Therefore, establishing the existence of homoclinic orbits of Hamiltonian systems is one of the most important issue in the theory of Hamiltonian systems. In particular, if $\alpha=1$, problem (1) reduces to the classical second order Hamiltonian systems

$$-u''(t)+L(t)u(t)=\nabla F(t,u(t)),\quad \forall\ t\in\mathbb{R}. \tag{2}$$

During the last decades, the existence and multiplicity of homoclinic solutions for Hamiltonian systems (2) have been extensively investigated by many authors with the aid of the variational methods. For example, see [1,2,3,4,5,6,7,8, 9,10,11] and references therein. Usually many people suppose that $L(t)$ is a symmetric matrix valued function and F satisfies the global Ambrosetti-Rabinowtiz condition, that is, there exists $\mu>2$ such that

$$0<\mu F(t,x)\le(\nabla F(t,x),x),\quad \forall(t,x)\in\mathbb{R}\times\mathbb{R}^N\backslash\{0\}.$$

Especially, Rabinowtiz in [2] established the existence of homoclinic orbits for the Hamiltonian systems under the above condition. Since the domain is unbounded, there is lack of compactness of the Sobolev embedding. In order to return the compactness, there are some assumption with respect to the matrix $L(t)$, among them, Rabinowtiz and Tanaka in [3] proposed the following assumption which can guarantee the compactness of Sobolev embedding:
(L) There exists a continuous function $l:\mathbb{R}\to\mathbb{R}$ such that $l(t)>0$ for all $t\in\mathbb{R}$ with $l(t)\to+\infty$ as $|t|\to+\infty$ and

$$(L(t)x,x)\ge l(t)|x|^2,\quad \text{for all}\quad (t,x)\in\mathbb{R}\times\mathbb{R}^N.$$

[*] Corresponding author e-mail: tengkaimin2013@163.com

In this case, they proved that the system (2) has a nontrivial homoclinic solution. Omana and Willem [6] obtained an improvement result by employing a new compact embedding theorem. After [3] and [6], with the coercivity assumption (L), many results were obtained, in recent papers, Zhang and Yuan [10], Sun at el. [12] established some existence and infinitely many homoclinic solutions for problem (2), respectively.

Recently, fractional differential equations have attracted extensive attentions because of its applications in viscoelasticity, electrochemistry, control, porous media etc, please see [13,14,15]. The existence and multiplicity of solutions for BVP of fractional differential equations have been established by the tools of nonlinear analysis, such as, fixed point theorems [16,17], topological degree [18], comparison methods [19], variational methods and critical point theory [20,21,22,23], the very recent related papers [24,25,26].

This paper is motivated by some recent papers [27] and [28] where some existence and multiplicity of results concerning problem (1) are obtained by using some critical point theorems, respectively. The purpose of this paper is devoted to proving the existence of infinitely many homoclinic solutions for problem (1) with the aid of Fountain Theorem and Dual Fountain Theorem. To the best of our knowledge, it seems that no similar results are obtained in the literature for fractional Hamiltonian systems.

Next, we will state our main results. We assume that $F(t,x)$ satisfies the following conditions.

(F_1) $F(t,0)=0$ for all $t\in\mathbb{R}$, and there exist $c>0$ and $\mu>1$ such that $|\nabla F(t,x)|\le c(1+|x|^{\mu})$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$;

(F_2) $\lim\limits_{|x|\to 0}\frac{\nabla F(t,x)}{|x|}=0$ uniformly for $t\in\mathbb{R}$;

(F_3) $\lim\limits_{|x|\to\infty}\frac{F(t,x)}{|x|^2}=+\infty$ uniformly in $t\in\mathbb{R}$;

(F_4) There exists $\theta\ge 1$ such that $\theta H(t,x)\ge H(t,sx)$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$ and $s\in[0,1]$, where $H(t,x)=(\nabla F(t,x),x)-2F(t,x)$;

(F_4') There exist $\sigma>2$ and a constant $a>0$ such that $\sigma F(t,x)-(\nabla F(t,x),x)\le a(|x|^2+1)$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$, moreover, $F(t,x)\ge 0$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$;

(F_5) $F(t,x)=F(t,-x)$, for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$;

(F_6) $F(t,x)=\lambda f(t)|x|^q+\mu|x|^p$ where $f:\mathbb{R}\to\mathbb{R}^+$ is a positive continuous function $f\in L^{\frac{2}{2-q}}(\mathbb{R},\mathbb{R}^+)$ and $1<q<2<p<+\infty$, $\lambda,\mu\in\mathbb{R}$.

The first result reads as follows.

Theorem 1. *Assume that L satisfies (L) and F satisfies $(F_1)-(F_4)$ and (F_5). Then problem (1) has infinitely many homoclinic solutions $\{u_k\}$ satisfying*

$$\frac{1}{2}\int_{\mathbb{R}}\left[|{}_{-\infty}D_t^{\alpha}u_k(t)|^2+(L(t)u_k(t),u_k(t))\right]dt-\int_{\mathbb{R}}F(t,u_k(t))dt\to+\infty$$

as $k\to\infty$.

Remark. The hypothesis (F_4) implies that $F(t,x)\ge 0$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$ (see [5]).

If we replace the condition (F_4) by (F_4'), then we will get the result as follows.

Theorem 2. *Assume that L satisfies (L) and F satisfies $(F_1)-(F_3)$, (F_4') and (F_5). Then problem (1) has infinitely many homoclinic solutions $\{u_k\}$ satisfying*

$$\frac{1}{2}\int_{\mathbb{R}}\left[|{}_{-\infty}D_t^{\alpha}u_k(t)|^2+(L(t)u_k(t),u_k(t))\right]dt-\int_{\mathbb{R}}F(t,u_k(t))dt\to+\infty$$

as $k\to\infty$.

Theorem 3. *Suppose that (L) and (F_6) hold. Then, the following two statements are true:*
(a) for every $\mu>0$, $\lambda\in\mathbb{R}$, problem (1) has infinitely many homoclinic solutions $\{u_k\}$ satisfying

$$\frac{1}{2}\int_{\mathbb{R}}\left[|{}_{-\infty}D_t^{\alpha}u_k(t)|^2+(L(t)u_k(t),u_k(t))\right]dt-\int_{\mathbb{R}}F(t,u_k(t))dt\to+\infty$$

as $k\to\infty$;
(b) For every $\lambda>0$, $\mu\in\mathbb{R}$, problem (1) has infinitely many homoclinic solutions $\{u_k\}$ satisfying

$$\frac{1}{2}\int_{\mathbb{R}}\left[|{}_{-\infty}D_t^{\alpha}u_k(t)|^2+(L(t)u_k(t),u_k(t))\right]dt-\int_{\mathbb{R}}F(t,u_k(t))dt\to 0-$$

as $k\to\infty$.

The present paper is organized as follows. In section 2 we present some basic definitions and facts about the fractional calculus and give some fundamental tools for the sequel use. Section 3 is devoted to proving Theorem 1, Theorem 2 and Theorem 3.

2 Preliminaries

In this section, we first will recall some facts about the fractional calculus on the whole real axis for the readers' convenience. On the other hand, we will give some preliminaries Lemmas for using in the sequel.

2.1 Liouville Fractional Calculus

The Liouville fractional integrals of order $0<\alpha<1$ on the whole axis $\mathbb{R}$ (see [13,14,15]) are defined by

$$_{-\infty}I_t^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}(t-s)^{\alpha-1}u(s)ds \tag{3}$$

$$_{t}I_{\infty}^{\alpha}u(t)=\frac{1}{\Gamma(\alpha)}\int_{t}^{\infty}(s-t)^{\alpha-1}u(s)ds. \tag{4}$$

The Liouville fractional derivatives of order $0<\alpha<1$ (see [13,14,15]) on the whole axis $\mathbb{R}$ are defined by

$$_{-\infty}D_t^{\alpha}u(t)=\frac{d}{dt}{}_{-\infty}I_t^{1-\alpha}u(t),\quad {}_tD_{\infty}^{\alpha}u(t)=-\frac{d}{dt}{}_tI_{\infty}^{1-\alpha}u(t). \tag{5}$$

The Caputo derivatives of order $0<\alpha<1$ on the whole axis $\mathbb{R}$ (see [13,14,15]) are defined as follows

$$_{-\infty}^{\;\;C}D_t^{\alpha}u(t)={}_{-\infty}I_t^{1-\alpha}u'(t),\quad {}_t^CD_{\infty}^{\alpha}u(t)=-{}_tI_{\infty}^{1-\alpha}u'(t). \tag{6}$$

Let $u(x)$ be defined on $\mathbb{R}$, the Fourier transform of the Liouville fractional integrals and derivatives satisfies (see [13, 15])

$$\widehat{_{-\infty}I_t^{\alpha}u}(\xi)=(i\xi)^{-\alpha}\widehat{u}(\xi),\quad \widehat{_tI_{\infty}^{\alpha}u}(\xi)=(-i\xi)^{-\alpha}\widehat{u}(\xi), \tag{7}$$

$$\widehat{_{-\infty}D_t^{\alpha}u}(\xi)=(i\xi)^{\alpha}\widehat{u}(\xi),\quad \widehat{_tD_{\infty}^{\alpha}u}(\xi)=(-i\xi)^{\alpha}\widehat{u}(\xi). \tag{8}$$

Next, we present some properties for Liouville fractional integral and derivatives on the real axis, which were proved in [15].

Proposition 1.*(1) Let $1\le p\le+\infty$, $1\le q\le+\infty$, $\alpha>0$, the operator ${}_{-\infty}I_t^{\alpha}$ and ${}_tI_{\infty}^{\alpha}$ are bounded from $L^p(\mathbb{R},\mathbb{R}^N)$ to $L^q(\mathbb{R},\mathbb{R}^N)$ if and only if $0<\alpha<1$, $1<p<\frac{1}{\alpha}$, $q=\frac{p}{1-\alpha p}$.*
(2) If $\alpha>0$, for "sufficiently good" function $f(t)$, the relations

$$({}_{-\infty}D_t^{\alpha}({}_{-\infty}I_t^{\alpha}f))(t)=f(t),\quad ({}_tD_{\infty}^{\alpha}({}_tI_{\infty}^{\alpha}f))(t)=f(t)$$

are true. In particularly, these functions hold for $f\in L^1(\mathbb{R},\mathbb{R}^N)$.
(3) Let $\alpha>0$, $\beta>0$ and $p\ge1$ be such that $\alpha+\beta<\frac{1}{p}$. If $f\in L^p(\mathbb{R},\mathbb{R}^N)$, then

$$[{}_{-\infty}I_t^{\alpha}({}_{-\infty}I_t^{\beta}f)]={}_{-\infty}I_t^{\alpha+\beta}f,\quad [{}_tI_{\infty}^{\alpha}({}_tI_{\infty}^{\beta}f)]={}_tI_{\infty}^{\alpha+\beta}f.$$

(4) If $\alpha>\beta>0$, then the formulas

$$[{}_{-\infty}D_t^{\beta}({}_{-\infty}I_t^{\alpha}f)]={}_{-\infty}I_t^{\alpha-\beta}f,\quad [{}_tD_{\infty}^{\beta}({}_tI_{\infty}^{\alpha}f)]={}_tI_{\infty}^{\alpha-\beta}f$$

hold for "sufficiently good" functions f. In particularly, these functions hold for $f\in L^1(\mathbb{R},\mathbb{R}^N)$.

Proposition 2.*If $\alpha>0$, the relations*

$$\int_{-\infty}^{+\infty}\varphi(t)({}_{-\infty}I_t^{\alpha}\psi)(t)dt=\int_{-\infty}^{+\infty}\psi(t)({}_tI_{\infty}^{\alpha}\varphi)(t)dt \tag{9}$$

and

$$\int_{-\infty}^{+\infty}f(t)({}_{-\infty}D_t^{\alpha}g)(t)dt=\int_{-\infty}^{+\infty}g(t)({}_tD_{\infty}^{\alpha}f)(t)dt \tag{10}$$

are valid for "sufficiently good" functions φ,ψ,f,g. In particular, (9) holds for functions $\varphi\in L^p(\mathbb{R},\mathbb{R}^N)$ and $\psi\in L^q(\mathbb{R},\mathbb{R}^N)$, while (10) holds for $f\in{}_tI_{\infty}^{\alpha}(L^p(\mathbb{R}))$ and $g\in{}_{-\infty}I_t^{\alpha}(L^q(\mathbb{R}))$ provided that $p>1,q>1$, and $\frac{1}{p}+\frac{1}{q}=1+\alpha$, where ${}_tI_{\infty}^{\alpha}(L^p(\mathbb{R}))=\{f:f(x)={}_tI_{\infty}^{\alpha}\varphi(x),\varphi\in L^p(\mathbb{R},\mathbb{R}^N)\}$, similarly, ${}_{-\infty}I_t^{\alpha}(L^q(\mathbb{R}))$ can be defined.

*Remark.*The function of $C_0^{\infty}(\mathbb{R},\mathbb{R}^N)$ can be chosen to the "sufficiently good" function.

Proposition 3.*Let $f\in L^q(\mathbb{R},\mathbb{R}^N)$, $q=\frac{p}{1-\alpha p}$ and $\int_0^{\infty}t^{-1-\alpha}\omega_p(f,t)dt<\infty$, then $f\in{}_tI_{\infty}^{\alpha}(L^p(\mathbb{R}))$, $1<p<\frac{1}{\alpha}$, where $\omega_p(f,t)=\sup_{0<\tau<t}\|f(x+\tau)-f(x)\|_p$.*

2.2 Fractional Derivative Space

Throughout this paper, we denote by the norm of the space $L^p(\mathbb{R},\mathbb{R}^N)$ for $1 \leq p \leq +\infty$ as $\|u\|_p = \big(\int_{\mathbb{R}} |u(t)|^p dt\big)^{\frac{1}{p}}$ and $\|u\|_\infty = \sup_{t\in\mathbb{R}} |u(t)|$.

Definition 1. *Let $\alpha > 0$. The fractional derivative space E_-^α is defined by the closure of $C_0^\infty(\mathbb{R},\mathbb{R}^N)$ with respect to the norm*

$$\|u\|_{E_-^\alpha} = \Big(\int_{\mathbb{R}} |u(t)|^2 dt + \int_{\mathbb{R}} |{}_{-\infty}D_t^\alpha u(t)|^2 dt\Big)^{\frac{1}{2}}, \quad \forall u \in E_-^\alpha. \tag{11}$$

Definition 2. *Let $\alpha > 0$. The fractional derivative space E_+^α is defined by the closure of $C_0^\infty(\mathbb{R},\mathbb{R}^N)$ with respect to the norm*

$$\|u\|_{E_+^\alpha} = \Big(\int_{\mathbb{R}} |u(t)|^2 dt + \int_{\mathbb{R}} |{}_tD_\infty^\alpha u(t)|^2 dt\Big)^{\frac{1}{2}}, \quad \forall u \in E_+^\alpha. \tag{12}$$

We recall that the fractional Sobolev space $H^\alpha(\mathbb{R},\mathbb{R}^N)$ endowed with the norm

$$\|u\|_{H^\alpha} = \Big(\int_{\mathbb{R}} |u(t)|^2 dt + \int_{\mathbb{R}} ||\xi|^\alpha \widehat{u}(\xi)|^2 d\xi\Big)^{\frac{1}{2}}.$$

Note that $\int_{\mathbb{R}} |{}_{-\infty}D_t^\alpha u(t)|^2 dt = \int_{\mathbb{R}} ||\xi|^\alpha \widehat{u}(\xi)|^2 d\xi$, hence E_-^α and $H^\alpha(\mathbb{R},\mathbb{R}^N)$ are equivalent with equivalent norm. Analogous to E_-^α, by (8), E_+^α and $H^\alpha(\mathbb{R},\mathbb{R}^N)$ are equivalent with equivalent norm.

Next, we recall some Sobolev embedding results about the fractional Sobolev space $H^\alpha(\mathbb{R},\mathbb{R}^N)$.

Lemma 1. *(i) If $\alpha > \frac{1}{2}$, then $H^\alpha(\mathbb{R},\mathbb{R}^N) \subset C(\mathbb{R},\mathbb{R}^N)$ and there exists a constant C such that*

$$\|u\|_\infty \leq C\|u\|_{H^\alpha}.$$

(ii) If $u \in H^\alpha(\mathbb{R},\mathbb{R}^N)$, then $u \in L^s(\mathbb{R})$ for all $s \in [2,\infty)$ and

$$\int_{\mathbb{R}} |u(t)|^s dt \leq \|u\|_\infty^{s-2} \int_{\mathbb{R}} |u(t)|^2 dt. \tag{13}$$

In order to study problem (1) by variational methods, we introduce a new fractional Sobolev space which introduced in [29]. Let

$$E^\alpha = \{u \in H^\alpha(\mathbb{R},\mathbb{R}^N) : \int_{\mathbb{R}} |{}_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t),u(t))dt < \infty\}.$$

The space E^α is a separable Hilbert space with the inner product (since $H^\alpha(\mathbb{R},\mathbb{R}^N)$ is a separable space)

$$\langle u,v\rangle = \int_{\mathbb{R}} ({}_{-\infty}D_t^\alpha u(t), {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t),v(t))dt$$

and the corresponding norm $\|u\|^2 = \langle u,u\rangle$. Under the condition of (L), it is easy to check that E^α is continuous embedded in $H^\alpha(\mathbb{R},\mathbb{R}^N)$. By Lemma 2.2 in [27] and Lemma 1, we see that E^α is compactly embedded in $L^s(\mathbb{R},\mathbb{R}^N)$ for all $s \in [2,\infty)$.

Definition 3. *We say that $u \in E^\alpha$ is a solution of problem (1) if*

$$\int_{\mathbb{R}} ({}_{-\infty}D_t^\alpha u(t), {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t),v(t))dt - \int_{\mathbb{R}} (\nabla F(t,u(t)),v(t))dt = 0$$

for every $v \in C_0^\infty(\mathbb{R},\mathbb{R}^N)$.

Definition 4. *We call that a solution u of problem (1) is homoclinic (to 0) if $u(t) \to 0$ as $t \to \pm\infty$. In addition, if $u \not\equiv 0$ then u is called a nontrivial homoclinic solution.*

We define the functional $\varphi : E^\alpha \to \mathbb{R}$ by

$$\varphi(u) = \frac{1}{2}\int_{\mathbb{R}} [|{}_{-\infty}D_t^\alpha u(t)|^2 + (L(t)u(t),u(t))]dt - \int_{\mathbb{R}} F(t,u(t))dt.$$

Lemma 2. *Assume F satisfies $(F_1)-(F_2)$ or (F_6), then $\varphi \in C^1(E^\alpha, \mathbb{R})$ and*

$$\langle \varphi'(u), v\rangle = \int_{\mathbb{R}} [({}_{-\infty}D_t^\alpha u(t), {}_{-\infty}D_t^\alpha v(t)) + (L(t)u(t), v(t))]dt - \int_{\mathbb{R}} (\nabla F(t, u(t)), v(t))dt$$

for all $u, v \in E^\alpha$. Moreover, the critical point of φ corresponds to the solution of problem (1).

Proof. Similar to the proof of Lemma 3.1 in [27] or Lemma 2.5 in [9], we omit it.

Lemma 3. *Suppose that (L), (F_0)-(F_2) or $(F_6)$are satisfied. If $u_n \rightharpoonup u$ in E^α, then $\nabla F(t, u_n) \to \nabla F(t, u)$ in $L^2(\mathbb{R}, \mathbb{R}^N)$.*

Proof. Similar to the proof Lemma 2.4 in [27] or Lemma 2.3 in [9], we omit it.

Now, we prove a concentration-compactness principle owing to Lemma I.1 of P. L. Lions [30].

Lemma 4. *Let $l > 0$ and $2 \le q < +\infty$. If $\{u_n\}$ is bounded in $H^\alpha(\mathbb{R}, \mathbb{R}^N)$ and if*

$$\sup_{y\in\mathbb{R}} \int_{y-l}^{y+l} |u_n(t)|^q dt \to 0 \tag{14}$$

as $n \to \infty$, then $u_n \to 0$ in $L^p(\mathbb{R}, \mathbb{R}^N)$ for all $p \in (q, +\infty)$.

Proof. Fixing $\bar{q}$ ($q < \bar{q} \le +\infty$) such that $q < p < \bar{q}$, by Hölder's inequality, we have

$$\Big(\int_{y-l}^{y+l} |u_n(t)|^p dt\Big)^{\frac{1}{p}} \le \Big(\int_{y-l}^{y+l} |u_n(t)|^{\bar{q}} dt\Big)^{\frac{\lambda}{\bar{q}}} \Big(\int_{y-l}^{y+l} |u_n(t)|^q dt\Big)^{\frac{1-\lambda}{q}},$$

where $\frac{1}{p} = \frac{\lambda}{\bar{q}} + \frac{1-\lambda}{q}$. Now, covering $\mathbb{R}$ by the open set $(y-l, y+l)$, in such a way that each point of $\mathbb{R}$ is contained in at most m open sets (m is a prescribed number), we deduce

$$\int_{\mathbb{R}} |u_n(t)|^p dt \le m\Big(\int_{\mathbb{R}} |u_n(t)|^{\bar{q}} dt\Big)^{\frac{\lambda p}{\bar{q}}} \Big(\int_{y-l}^{y+l} |u_n(t)|^q dt\Big)^{\frac{(1-\lambda)p}{q}} \le m\|u_n\|_{H^\alpha}^{\bar{q}} \sup_{y\in\mathbb{R}} \Big(\int_{y-l}^{y+l} |u_n(t)|^q dt\Big)^{\frac{(1-\lambda)s}{q}}$$

which implies the conclusion.

Remark. Since $\{u_n\}$ is bounded in $H^\alpha(\mathbb{R}, \mathbb{R}^N)$, it is clear that $\{u_n\}$ is bounded in $L^\infty(\mathbb{R}, \mathbb{R}^N)$. From (14) and Lemma 1, we have for all $q < s < +\infty$,

$$\sup_{y\in\mathbb{R}} \int_{y-l}^{y+l} |u_n(t)|^s dt \to 0 \tag{15}$$

as $n \to \infty$.

Remark. Since E^α is continuously embedding to $H^\alpha(\mathbb{R}, \mathbb{R}^N)$, it is clear that if $\{u_n\}$ is bounded in E^α, the conclusion of Lemma 4 holds true.

For proving our main theorem, we present the Fountain Theorem and its dual form which were established in [31,32]. Since E^α is a separable and reflexive Banach space, there exist $\{e_n\} \subset E^\alpha$ and $\{f_n\} \subset (E^\alpha)^*$ such that

$$\langle f_n, e_m\rangle = \delta_{n,m} = \begin{cases} 1, \text{ if } n = m, \\ 0, \text{ if } n \neq m, \end{cases}$$

$E^\alpha = \text{span}\,\overline{\{e_n : n = 1, 2, \cdots\}}$, and $(E^\alpha)^* = \text{span}\,\overline{\{f_n : n = 1, 2, \cdots\}}^{w^*}$. For $k \in \mathbb{N}$, we define

$$X_k = \text{span}\{e_k\}, \quad Y_k = \oplus_{j=1}^k X_j, \quad Z_k = \overline{\oplus_{j=k}^\infty X_k}.$$

The functional φ is said to satisfy the $(C)_c$ condition if for each sequence $\{u_j\}$ satisfying that $\varphi(u_j) \to c$ and $(1 + \|u_j\|)\varphi'(u_j) \to 0$ as $j \to \infty$ has a convergent subsequence. We say that the function φ satisfies that $(PS)_c^*$ condition (with respect to Y_n) if for any sequence $\{u_{n_j}\} \subset E^\alpha$ such that $u_{n_j} \in Y_{n_j}$, $\varphi(u_{n_j}) \to c$ and $\varphi'|_{Y_{n_j}} \to 0$, as $n_j \to \infty$ has a convergent subsequence which converges to a critical point of φ.

Theorem 4. *Suppose X is a Banach space, $\varphi \in C^1(X,\mathbb{R})$ is an even functional satisfying the $(C)_c$ condition for every $c \in \mathbb{R}$. If for every $k \in \mathbb{N}$, there exists $\rho_k > r_k > 0$ such that*

$$a_k = \max_{u\in Y_k,\|u\|=\rho_k} \varphi(u) \le 0 \quad \text{and } b_k = \inf_{u\in Z_k,\|u\|=r_k} \varphi(u) \to \infty \text{ as } k \to \infty. \tag{16}$$

Then φ has an unbounded sequence of critical values.

Theorem 5. *Suppose X is a Banach space, $\varphi \in C^1(X,\mathbb{R})$ is an even functional satisfying the $(PS)^*$ condition. If there is a $k_0 > 0$ such that for every $k \ge k_0$, there exists $\rho_k > r_k > 0$ such that*

$$c_k = \max_{u\in Y_k,\|u\|=r_k} \varphi(u) < 0 \text{ and } \inf_{u\in Z_k,\|u\|=\rho_k} \varphi(u) \ge 0 \quad d_k = \inf_{u\in Z_k,\|u\|\le\rho_k} \varphi(u) \to 0 \tag{17}$$

as $k \to \infty$. Then φ has a sequence of negative critical values converging to zero.

3 Proof of the Main Results

Lemma 5. *Suppose that the conditions (L), (F_0)-(F_2) hold. $\varphi : E^\alpha \to \mathbb{R}$ satisfies the $(C)_c$ condition for every $c \in \mathbb{R}$.*

Proof. Let $\{u_n\} \subset E^\alpha$ be a sequence such that

$$\varphi(u_n) \to c \text{ and } (1+\|u_n\|)\varphi'(u_n) \to 0 \tag{18}$$

as $n \to \infty$. We claim that $\{u_n\}$ is bounded in E^α. If not, then $\|u_n\| \to +\infty$ as $n \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$.

Claim 1.

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-l}^{y+l} |v_n(t)|^2 dt = 0. \tag{19}$$

Otherwise, for some $\alpha > 0$, up to a subsequence, we get

$$\sup_{y\in\mathbb{R}} \int_{y-l}^{y+l} |v_n(t)|^2 dt \ge \alpha > 0. \tag{20}$$

We can choose $\{y_n\} \subset \mathbb{R}$ such that

$$\int_{y_n-l}^{y_n+l} |v_n(t)|^2 dt \ge \frac{\alpha}{2}. \tag{21}$$

In view of $v_n \to v$ in $L^2(\mathbb{R},\mathbb{R}^N)$ and (21), we have

$$\|v\|_2^2 + \frac{\alpha}{4} \ge \int_{\mathbb{R}} |v(t)|^2 dt + \int_{\mathbb{R}} |v_n(t)-v(t)|^2 dt \ge \int_{y_n-l}^{y_n+l} |v(t)|^2 dt + \int_{y_n-l}^{y_n+l} |v_n(t)-v(t)|^2 dt \ge \int_{y_n-l}^{y_n+l} |v_n(t)|^2 dt \ge \frac{\alpha}{2} \tag{22}$$

for n large enough. From (22), there exists $\varepsilon_0 > 0$ such that the set $\Omega = \{t \in \mathbb{R} : |v(t)| \ge \varepsilon_0\}$ has a positive Lebsegue measure. Hence, for all $t \in \Omega$, one has $|u_n(t)| \to +\infty$ as $n \to \infty$, which together with (F_3) shows

$$\frac{F(t,u_n(t))}{\|u_n\|^2} = \frac{W(t,u_n(t))}{|u_n(t)|^2}|v_n(t)|^2 \to +\infty \tag{23}$$

as $n \to \infty$ uniformly for all $t \in \Omega$. Hence by (18) and the fact that $F(t,x) \ge 0$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$, we have

$$\frac{1}{2} - \frac{c+o(1)}{\|u_n\|^2} = \frac{\frac{1}{2}\|u_n\|^2 - \varphi(u_n)}{\|u_n\|^2} = \int_{\mathbb{R}} \frac{F(t,u_n(t))dt}{\|u_n\|^2} dt \ge \int_{\Omega} \frac{F(t,u_n(t))dt}{|u_n(t)|^2} |v_n(t)|^2 dt \to +\infty \tag{24}$$

as $n \to \infty$, we get a contradiction. Therefore, the claim 1 holds. Since $\|v_n\|$ is bounded, by Lemma 4, we have

$$v_n \to 0 \text{ in } L^q(\mathbb{R},\mathbb{R}^N) \text{ for all } q > 2. \tag{25}$$

Claim 2. For any given $r \in \mathbb{R}$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} F(t, rv_n(t))dt = 0. \tag{26}$$

By conditions (F_1) and (F_2), for any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that for all $(t,x) \in \mathbb{R}\times\mathbb{R}^N$,

$$|F(t,x)| \le \varepsilon |x|^2 + c_\varepsilon |x|^{1+\mu}. \tag{27}$$

By (27), we have

$$\int_{\mathbb{R}} F(t, rv_n(t))dt \le \varepsilon |r|^2 \int_{\mathbb{R}} |v_n(t)|^2 dt + c_\varepsilon |r|^{\mu+1} \int_{\mathbb{R}} |v_n(t)|^{\mu+1} dt. \tag{28}$$

From the boundedness of $\|v_n\|_2$, (25) and the arbitrariness of ε, so (28) implies that the claim 2 is true.

We define $t_n \in [0,1]$ by $\varphi(t_n u_n) = \max_{t\in[0,1]} \varphi(tu_n)$. For any $M > 0$, let $\bar{v}_n = 2\sqrt{M} v_n$, by claim 2, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}} F(t, \bar{v}_n(t))dt = 0. \tag{29}$$

Hence, for n large enough, we have

$$\varphi(t_n u_n) \ge \varphi(\bar{v}_n) = 2M - \int_{\mathbb{R}} F(t, \bar{v}_n(t))dt \ge M.$$

This implies that $\lim_{n\to\infty} \varphi(t_n u_n) = +\infty$. Noting that $\varphi(0) = 0$ and $\varphi(u_n)$ satisfying (18), we know that $t_n \in (0,1)$ for large n, and so $\langle \varphi'(t_n u_n), t_n u_n \rangle = 0$. Hence,

$$\int_{\mathbb{R}} [(\nabla F(t, t_n u_n), t_n u_n) - 2F(t, t_n u_n)]dt = 2\varphi(t_n u_n) - \langle \varphi'(t_n u_n), t_n u_n \rangle \to +\infty \tag{30}$$

as $n \to \infty$. By (F_4), we get

$$\int_{\mathbb{R}} [(\nabla F(t, u_n), u_n) - 2F(t, u_n)]dt \ge \frac{1}{\theta} \int_{\mathbb{R}} [(\nabla F(t, t_n u_n), t_n u_n) - 2F(t, t_n u_n)]dt \tag{31}$$

for every $n \in \mathbb{N}$. By (18), we have

$$\lim_{n\to\infty} \int_{\mathbb{R}} [(\nabla F(t, u_n), u_n) - 2F(t, u_n)]dt = \lim_{n\to\infty} (2\varphi(u_n) - \langle \varphi'(u_n), u_n \rangle) = 2c. \tag{32}$$

Combining with (30), (31) and (32), we get a contradiction. Therefore, we have proved that $\{u_n\}$ is bounded.

By the compactness of embedding $E^\alpha \hookrightarrow L^p(\mathbb{R},\mathbb{R}^N)$ with $2 \le p < +\infty$, and the fact that $\{u_n\}$ is bounded in E^α, there exist $u \in E^\alpha$, and a subsequence of $\{u_n\}$ again denoted by $\{u_n\}$ such that

$$u_n \rightharpoonup u \text{ in } E^\alpha, \;\; u_n \to u \text{ in } L^p(\mathbb{R},\mathbb{R}^N). \tag{33}$$

Noting that

$$\langle \varphi'(u_n) - \varphi'(u), u_n - u \rangle = \int_{\mathbb{R}} (\nabla F(t, u_n(t)) - \nabla F(t, u(t)), u(t) - u_n(t))dt + \|u_n - u\|^2$$

By (32) and (33), we only need to prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}} (\nabla F(t, u_n(t)) - \nabla F(t, u(t)), u(t) - u_n(t))dt = 0. \tag{34}$$

For this matter, by Lemma 3 and (33), we get

$$\left| \int_{\mathbb{R}} (\nabla F(t, u_n(t)) - \nabla F(t, u(t)), u(t) - u_n(t))dt \right| \le \|\nabla F(t, u_n(t)) - \nabla F(t, u(t))\|_2 \times \|u_n - u\|_2 \to 0.$$

Consequently, $\|u_n - u\| \to 0$ as $n \to \infty$. The proof is completed.

Proof of Theorem 1. Since $\varphi \in C^1(E^\alpha, \mathbb{R})$ satisfies the $(C)_c$ condition for every $c \in \mathbb{R}$ and $\varphi(u) = \varphi(-u)$. Hence, to prove Theorem 1, we should just verify that φ satisfies (16) of Theorem 4.

(i) Let $\beta_k(p) = \sup_{u\in Z_k, \|u\|=1} \|u\|_p$, then one has $\beta_k(p) \to 0$ as $k \to \infty$. Indeed, clearly $0 < \beta_{k+1}(p) \le \beta_k(p)$, so there exits $\beta(p)$ such that $\beta_k(p) \to \beta(p)$ as $k \to \infty$ for every $2 \le p < +\infty$. From the definition of $\beta_k(p)$, there exists $\{u_k(p)\} \in Z_k$ with $\|u_k(p)\| = 1$ such that $\|u_k(p)\| > \frac{\beta_k(p)}{2}$ for every $2 \le p < +\infty$ and $k \in \mathbb{N}$. By the boundedness of $\{u_k(p)\}$, then there exists $u(p) \in E^\alpha$ such that $u_k(p) \rightharpoonup u(p)$ as $k \to \infty$. Now since $\{e_n\}$ is a basis of E^α, then for all $n \in \mathbb{N}$, $\forall k > n$, we have $0 = \langle u_k(p), e_n\rangle \to \langle u(p), e_n\rangle$ as $k \to \infty$. This implies that $u(p) = 0$. By the compactness of the embedding $E^\alpha \hookrightarrow L^p(\mathbb{R}, \mathbb{R}^N)$ with $2 \le p < +\infty$, we have $u_k(p) \to 0$ in $L^p(\mathbb{R}, \mathbb{R}^N)$ for all $2 \le p < +\infty$. Hence, $\beta(p) = 0$.

By the definition of $\beta_k(p)$, and (27), for all $u \in Z_k$, we have

$$\varphi(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} F(t,u(t))dt \ge \frac{1}{2}\|u\|^2 - \varepsilon\int_{\mathbb{R}} |u(t)|^2 dt - c_\varepsilon \int_{\mathbb{R}} |u(t)|^{\mu+1} dt \ge \frac{1}{4}\|u\|^2 - C\beta_k^{\mu+1}(\mu+1)\|u\|^{\mu+1} \quad (35)$$

for ε small enough. Therefore, taking $r_k = (8C)^{\frac{1}{1-\mu}}(\beta_k(\mu+1))^{\frac{\mu+1}{1-\mu}}$, then $r_k \to +\infty$ as $k \to \infty$ and for every $u \in Z_k$ with $\|u\| = r_k$, by (35), one has

$$b_k = \inf_{u\in Z_k, \|u\|=1} \varphi(u) \ge \frac{1}{8} r_k^2 \to +\infty, \quad (36)$$

as $k \to \infty$.

(ii) Similarly as in the proof of Lemma 3.1 of [12], we see that there exits a constant $\delta > 0$ such that

$$\mathrm{meas}\{t \in \mathbb{R} : |u(t)| \ge \delta\|u\|\} \ge \delta \quad (37)$$

for all $u \in Y_k\backslash\{0\}$. We denote by $\Omega_u = \{t \in \mathbb{R} : |u(t)| \ge \delta\|u\|\}$. By the hypothesis (F_3), there exits $R > 0$ such that

$$F(t,x) \ge \frac{|x|^2}{\delta^3} \quad (38)$$

for all $x \in \mathbb{R}^N$ with $|x| \ge R$ and $t \in \mathbb{R}$. Observing that for any $u \in Y_k$ with $\|u\| \ge \frac{R}{\delta}$, there holds

$$|u(t)| \ge R, \ \text{ for all } \ t \in \Omega_u. \quad (39)$$

By (37), (38), (39)and $F(t,x) \ge 0$, we have

$$\varphi(u) \le \frac{1}{2}\|u\|^2 - \int_{\Omega_u} F(t,u(t))dt \le \frac{1}{2}\|u\|^2 - \frac{1}{\delta^3}\int_{\Omega_u} |u(t)|^2 dt \le \frac{1}{2}\|u\|^2 - \frac{1}{\delta}\|u\|^2 \mathrm{meas}(\Omega_u) \le -\frac{1}{2}\|u\|^2$$

for all $u \in Y_k$ with $\|u\| \ge \frac{R}{\delta}$. Therefore, we can choose $\rho_k > \max\{r_k, \frac{R}{\delta}\}$, then

$$a_k = \max_{u\in Y_k, \|u\|=\rho_k} \varphi(u) < 0.$$

Hence, combining with Lemma 5, by Theorem 4, we obtain that problem (1) has infinitely many homoclinic solutions $\{u_k\}$ satisfying

$$\frac{1}{2}\int_{\mathbb{R}} |{}_{-\infty}D_t^\alpha u_k(t)|^2 + (L(t)u_k(t), u_k(t))dt - \int_{\mathbb{R}} F(t,u_k(t))dt \to +\infty$$

as $k \to \infty$.

Proof of Theorem 2. We only need to verify that the functional φ satisfies the $(C)_c$ condition. Let $\{u_n\} \subset E^\alpha$ be a sequence such that

$$\varphi(u_n) \to c \ \text{ and } \ (1 + \|u_n\|)\varphi'(u_n) \to 0 \quad (40)$$

as $n \to \infty$. We claim that $\{u_n\}$ is bounded in E^α. If not, then $\|u_n\| \to +\infty$ as $n \to \infty$. Let $v_n = \frac{u_n}{\|u_n\|}$, then $\|v_n\| = 1$. Hence, up to a subsequence, there exists $v \in E^\alpha$ such that $v_n \rightharpoonup v$ in E^α, $v_n \to v$ in $L^p(\mathbb{R}, \mathbb{R}^N)$ and $v_n(t) \to v(t)$ a.e. $t \in \mathbb{R}$.

By (40), we have

$$\int_{\mathbb{R}} \frac{2F(t,u_n(t))}{|u_n(t)|^2}|v_n(t)|^2 dt \to 1 \quad (41)$$

as $n \to \infty$. Let $\Sigma_1 = \{t \in \mathbb{R} : v(t) \ne 0\}$ and $\Sigma_2 = \mathbb{R}\backslash\Sigma_1$. Obviously,

$$\frac{2F(t,u_n(t))}{|u_n(t)|^2}|v_n(t)|^2 \to +\infty, \quad t \in \Sigma_1 \quad (42)$$

as $n\to\infty$. By the hypothesis (F_3), for a constant $a_0>0$, there exists $R_0>0$ such that $F(t,x)\geq a_0|x|^2$, for all $x\in\mathbb{R}^n$ with $|x|\geq R_0$ and $t\in\mathbb{R}$. It follows that

$$\begin{aligned}\int_{\mathbb{R}}\frac{2F(t,u_n(t))}{|u_n(t)|^2}|v_n(t)|^2dt &\geq \int_{\Sigma_1}\frac{2F(t,u_n(t))}{|u_n(t)|^2}|v_n(t)|^2dt+\int_{\Sigma_2\cap\{t\in\mathbb{R}:|u_n(t)|\geq R_0\}}\frac{2F(t,u_n(t))}{|u_n(t)|^2}|v_n(t)|^2dt+\int_{\Sigma_2\cap\{t\in\mathbb{R}:|u_n(t)|\leq R_0\}}\frac{2F(t,u_n(t))}{\|u_n(t)\|^2}dt\\ &\geq \int_{\Sigma_1}\frac{2F(t,u_n(t))}{|u_n(t)|^2}|v_n(t)|^2dt+2a_0\int_{\Sigma_2\cap\{t\in\mathbb{R}:|u_n(t)|\geq R_0\}}|v_n(t)|^2dt+\int_{\Sigma_2\cap\{t\in\mathbb{R}:|u_n(t)|\leq R_0\}}\frac{2F(t,u_n(t))}{\|u_n(t)\|^2}dt\\ &\geq \int_{\Sigma_1}\frac{2F(t,u_n(t))}{|u_n(t)|^2}|v_n(t)|^2dt+\int_{\Sigma_2\cap\{t\in\mathbb{R}:|u_n(t)|\leq R_0\}}\frac{2F(t,u_n(t))}{\|u_n(t)\|^2}dt. \end{aligned}\tag{43}$$

By (27) and Lebesgue dominated convergent theorem, we have

$$\Big|\int_{\Sigma_2\cap\{t\in\mathbb{R}:|u_n(t)|\leq R_0\}}\frac{2F(t,u_n(t))}{\|u_n(t)\|^2}dt\Big|\leq c\int_{\Sigma_2}|v_n(t)|^2dt\to 0.\tag{44}$$

If Σ_1 has positive measure, (41)-(44) implies a contradiction. Hence, the measure of Σ_1 must be 0, i.e., we must have $v(t)\equiv 0$, a.e. $t\in\mathbb{R}$. Moreover, from (40), we get

$$\int_{\mathbb{R}}\frac{(\nabla F(t,u_n(t)),u_n(t))-\sigma F(t,u_n(t))}{|u_n(t)|^2}|v_n(t)|^2dt\to 1-\frac{\sigma}{2}.\tag{45}$$

But by the hypothesis (F_4'), we have

$$\liminf_{n\to\infty}\frac{(\nabla F(t,u_n(t)),u_n(t))-\sigma F(t,u_n(t))}{|u_n(t)|^2}|v_n(t)|^2\geq\liminf_{n\to\infty}\Big(-a\frac{|u_n|^2+1}{|u_n|^2}|v_n(t)|^2\Big)=0.\tag{46}$$

Hence, (45) and (46) implies that $1-\frac{\sigma}{2}\geq 0$, this contradicts with the assumption $\sigma>2$. Consequently, $\{u_n\}$ is bounded in E^α. As in the proof of Lemma 5, we conclude that φ satisfies the $(C)_c$ condition.

*Remark.*In the proof of (43), if we use the hypothesis $F(t,x)\geq 0$ for all $(t,x)\in\mathbb{R}\times\mathbb{R}^N$, it is easy to check that it holds.

Proof of Theorem 3. Case (a). Note that $p>2$, we may choose $\|u\|\geq R_1$ large enough such that $\frac{1}{4}\|u\|^2-c_1\frac{|\lambda|}{q}\|f\|_{\frac{2}{2-q}}\|u\|^q\geq 0$ for all $u\in Z_k$ with $\|u\|\geq R_1$. By the definition of $\beta_k(p)$ in the proof of Theorem 1, we have

$$\varphi(u)=\frac{1}{2}\|u\|^2-\frac{\lambda}{q}\|f^{\frac{1}{q}}u\|_q^q-\frac{\mu}{p}\|u\|_p^p\geq\frac{1}{2}\|u\|^2-\frac{|\lambda|}{q}c_1\|f\|_{\frac{2}{2-q}}\|u\|^q-\frac{\mu}{p}\beta_k^p(p)\|u\|^p\geq\frac{1}{4}\|u\|^2-\frac{\mu}{p}\beta_k^p(p)\|u\|^p$$

for any $u\in Z_k$ with $\|u\|\geq R_1$. Taking $r_k=\Big(\frac{p}{8\mu\beta_k^p(p)}+R_1^{p-2}\Big)^{\frac{1}{p-2}}$, since $p>2$, by the fact that $\beta_k(p)\to 0$, then $r_k\to+\infty$ as $k\to\infty$. Thus for $u\in Z_k$ with $\|u\|=r_k$, we get that

$$\inf_{u\in Z_k,\|u\|=r_k}\varphi(u)\geq\frac{1}{8}r_k^2\to+\infty,\quad \text{as }\ k\to\infty.$$

Clearly, $F(t,x)$ satisfies (F_3), so, similarly as the proof of Theorem 1, we can obtain that

$$a_k=\max_{u\in Y_k,\|u\|=\rho_k}\varphi(u)<0.$$

Next, we show that φ satisfies the $(C)_c$ condition. In fact, we assume that $\{u_n\}\subset E^\alpha$ is a sequence such that

$$\varphi(u_n)\to c\ \text{ and }\ (1+\|u_n\|)\varphi'(u_n)\to 0$$

as $n\to\infty$. It follows that

$$\varphi(u_n)-\frac{1}{p}\langle\varphi'(u_n),u_n\rangle=(\frac{1}{2}-\frac{1}{p})\|u_n\|^2-\lambda(\frac{1}{q}-\frac{1}{p})\|f^{\frac{1}{q}}u_n\|_q^q\geq(\frac{1}{2}-\frac{1}{p})\|u_n\|^2-|\lambda|(\frac{1}{q}-\frac{1}{p})\|f\|_{\frac{2}{2-q}}\|u_n\|_2^q,$$

which implies that $\{u_n\}$ is bounded in E^α from the fact that $f \in L^{\frac{2}{2-q}}(\mathbb{R},\mathbb{R}^+)$ and $1 < q < 2 < p < +\infty$. Thus, the remainder proof is similar to the proof of Theorem 1.

Case (b). Firstly we verify that (17) of Theorem 5 holds. Note that $q < 2$, we may choose $\|u\| \le R_2$ small enough such that $\frac{1}{4}\|u\|^2 - c_1\frac{|\mu|}{p}c_2\|u\|^p \ge 0$ for all $u \in Z_k$ with $\|u\| \le R_2$. By the definition of $\beta_k(2)$ in the proof of Theorem 1, we have

$$\varphi(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\|f^{\frac{1}{q}}u\|_q^q - \frac{\mu}{p}\|u\|_p^p \ge \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\beta_k^q(2)\|f\|_{\frac{2}{2-q}}\|u\|^q - \frac{|\mu|}{p}c_2\|u\|^p \ge \frac{1}{4}\|u\|^2 - \frac{\lambda}{q}\beta_k^q(2)\|f\|_{\frac{2}{2-q}}\|u\|^q. \tag{47}$$

We choose $\rho_k = \left(\frac{8\lambda}{q}\beta_k^q(2)\|f\|_{\frac{2}{2-q}}\right)^{\frac{1}{2-q}}$. Obviously, $\rho_k \to 0$ as $k \to \infty$. Hence, there exits $k_0 > 0$ such that $\rho_k \le R_2$ for all $k \ge k_0$. Combining with (47), straightforward computation gives that

$$\inf_{u\in Z_k,\|u\|=\rho_k} \varphi(u) \ge \frac{1}{8}\rho_k^2 > 0.$$

Furthermore, by (47), for any $u \in Z_k$ with $\|u\| \le \rho_k$, we have

$$0 \ge d_k = \inf_{u\in Z_k,\|u\|\le\rho_k} \varphi(u) \ge -\frac{\lambda}{q}\beta_k^q(2)\|f\|_{\frac{2}{2-q}}\|u\|^q.$$

Hence, $d_k \to 0-$ as $k \to \infty$.

Next, we show that for the sequence $\{\rho_k\}$ obtained above, there exist $0 < r_k < \rho_k$ for all $k \ge k_0$ such that

$$c_k = \inf_{u\in Y_k,\|u\|=r_k} \varphi(u) < 0. \tag{48}$$

As the proof of Lemma 3.1 in [12], we get that for any finite dimensional subspace $F \subset E^\alpha$, there exists $\delta_1 > 0$ such that

$$\text{meas}\{t \in \mathbb{R} : f(t)|u(t)|^q \ge \delta_1\|u\|^q\} \ge \delta_1, \quad \forall u \in F\backslash\{0\}.$$

We denote by $\Omega'_u = \{t \in \mathbb{R} : f(t)|u(t)|^q \ge \delta_1\|u\|^q\}$. Then for all $u \in Y_k$, if $\mu \ge 0$, we have

$$\varphi(u) \le \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\|f^{\frac{1}{q}}u\|_q^q - \frac{\mu}{p}\|u\|_p^p \le \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\int_{\Omega'_u} f(t)|u(t)|^q dt \le \frac{1}{2}\|u\|^2 - \frac{\lambda\delta_1}{q}\|u\|^q \text{meas}(\Omega'_u) \le \frac{1}{2}\|u\|^2 - \frac{\lambda\delta_1^2}{q}\|u\|^q.$$

Choosing $0 < r_k < \min\{\rho_k, (\frac{\lambda\delta_1^2}{q})^{\frac{1}{2-q}}\}$, directly computation shows that

$$c_k \le -\frac{r_k^2}{2} < 0.$$

On the other hand, if $\mu < 0$, we have

$$\begin{aligned}\varphi(u) &\le \frac{1}{2}\|u\|^2 - \frac{\mu}{p}c_2\|u\|^p - \frac{\lambda}{q}\int_{\Omega'_u} f(t)|u(t)|^q dt \le \frac{1}{2}\|u\|^2 - \frac{\mu}{p}c_2\|u\|^p - \frac{\lambda\delta_1^2}{q}\|u\|^q \\ &= (\frac{1}{2}\|u\|^2 - \frac{\lambda\delta_1^2}{2q}\|u\|^q) + (-\frac{\mu}{p}c_2\|u\|^p - \frac{\lambda\delta_1^2}{2q}\|u\|^q).\end{aligned}$$

Choosing $0 < r_k < \min\{\rho_k, (\frac{\lambda\delta_1^2}{2q})^{\frac{1}{2-q}}, (\frac{p\lambda\delta_1^2}{4q(-\mu)c_2})^{\frac{1}{p-q}}\}$, then $r_k^{2-q} \le \frac{\lambda\delta_1^2}{2q}$ and $r_k^{p-q} \le \frac{p\lambda\delta_1^2}{4q(-\mu)c_2}$. It follows that

$$\begin{aligned}c_k &\le \frac{1}{2}r_k^2 - \frac{\lambda\delta_1^2}{2q}r_k^q + (-\frac{\mu}{p}c_3 r_k^{p-q}r_k^q - \frac{\lambda\delta_1^2}{2q}r_k^q \le \frac{1}{2}r_k^2 - r_k^2 + \frac{(-\mu)c_2}{p}\frac{p\lambda\delta_1^2}{4q(-\mu)c_2}r_k^q - \frac{\lambda\delta_1^2}{2q}r_k^q \\ &= -\frac{1}{2}r_k^2 - \frac{\lambda\delta_1^2}{4q}r_k^q \le -\frac{1}{2}r_k^2 - \frac{1}{2}r_k^2 = -r_k^2 < 0.\end{aligned}$$

Consequently, (48) holds.

Finally, we show that φ satisfies the $(PS)_c^*$ condition. Indeed, let $\{u_{n_j}\} \subset E^\alpha$ be a sequence such that

$$n_j \to \infty, \quad u_{n_j} \in Y_{n_j}, \quad \varphi(u_{n_j}) \to c, \quad \varphi'|_{Y_{n_j}}(u_{n_j}) \to 0. \tag{49}$$

For n_j large enough, we have

$$c+1+\|u_{n_j}\| = \varphi(u_{n_j}) - \frac{1}{p}\langle\varphi'(u_{n_j}),u_{n_j}\rangle = (\frac{1}{2}-\frac{1}{p})\|u_{n_j}\|^2 - \lambda(\frac{1}{q}-\frac{1}{p})\|f^{\frac{1}{q}}u_{n_j}\|_q^q$$
$$\geq (\frac{1}{2}-\frac{1}{p})\|u_{n_j}\|^2 - \lambda(\frac{1}{q}-\frac{1}{p})c_3\|f\|_{\frac{2}{2-q}}\|\|u_{n_j}\|^q.$$

Since $1<q<2<p$, it yields that $\{u_{n_j}\}$ is bounded in E^α. Hence, up to a subsequence, we may assume that $u_{n_j} \rightharpoonup u$ as $j \to \infty$ for some $u \in E^\alpha$ and $u_{n_j} \to u$ in $L^s(\mathbb{R},\mathbb{R}^N)$ for all $2 \leq s < +\infty$. Since

$$|\langle\varphi'(u_{n_j}),u_{n_j}-u\rangle| \leq \|\varphi'(u_{n_j})\|_{(E^\alpha)^*}\|u_{n_j}-u\| \to 0$$

as $j \to \infty$, and by Hölder's inequality, we have

$$|\int_{\mathbb{R}} f(t)|u_{n_j}(t)|^{q-2}u_{n_j}(t)\cdot(u_{n_j}(t)-u(t))dt| \leq \|f\|_{\frac{2}{2-q}}\|u_{n_j}\|_2\|u_{n_j}-u\|_2 \to 0$$

and

$$|\int_{\mathbb{R}} |u_{n_j}(t)|^{p-2}u_{n_j}(t)\cdot(u_{n_j}(t)-u(t))dt| \leq \|u_{n_j}\|_p^{p-1}\|u_{n_j}-u\|_p \to 0$$

as $j \to \infty$. Therefore, we get that

$$\lim_{j\to\infty}\|u_{n_j}\|^2 = \|u\|^2.$$

This implies that $u_{n_j} \to u$ in E^α. In fact

$$\lim_{j\to\infty}\|u_{n_j}-u\|^2 = \lim_{j\to\infty}[\|u_{n_j}\|^2+\|u\|^2-2\langle u_{n_j},u\rangle] = 0.$$

Hence, it is easy to check that $\varphi'(u)=0$. The proof is completed.

4 Conclusion

The existence of infinitely many high or small energy solutions for fractional Hamiltonian systems (1) was established by the well known fountain theorem and dual fountain theorem. Here we assume that the nonlinearities satisfy the superquadratic growth (but not the classical Ambrosetti-Rabinowtiz condition) and concave-convex conditions.

Acknowledgement

The authors would like to express their sincere thanks to the anonymous referee for his/her careful reading of the manuscript and valuable comments and suggestions. This work was supported by NSFC grant 11501403 and by the Shanxi Province Science Foundation for Youths under grant 2013021001-3.

References

[1] V. C. Zelati and P. H. Rabinowtiz, Homoclinic orbits for second order Hamiltonian systems possesing superquadratic potentials, *J. Amer. Math. Soc.* **4**, 693–727 (1991).
[2] P. H. Rabinowitz, Homoclinic orbits for a class of Hamiltonian systems, *Proc. Roy. Soc. Edinburgh*, **114**, 33–38 (1990).
[3] P. H. Rabinowitz and K. Tanaka, Some results on connecting orbits for a class of Hamiltonian systems, *Math. Z.* **206**, 473–499 (1991).
[4] Y. Ding, Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems, *Nonlinear Anal.* **25**, 1095–1113 (1995).
[5] Y. Lv and C. Tang, Existence and multiplicity of Homoclinic orbits for second order Hamiltonian systems with superquadratic potential, *Abstr. Appl. Anal.* **2013**, Artical ID 328630 (2013).
[6] W. Omana and M. Willem, Homoclinic orbits for a class of Hamiltonian systems, *Differ. Int. Equ.* **5**, 1115–1120 (1992).

[7] Z. Ou and C. Tang, Existence of homoclinic solution for the second order Hamiltonian systems, *J. Math. Anal. Appl.* **291**, 203–213 (2004).
[8] X. Tang and L. Xiao, Homoclinic solutions for non-autonomous second-order Hamiltonian systems with a coercive potential, *J. Math. Anal. Appl.* **351**, 586–594 (2009).
[9] L. Yang, H. Chen and J. Sun, Infinitely many homoclinic solutions for second-order Hamiltonian systems, *Nonlinear Anal.* **74**, 6459–6468 (2011).
[10] Z. Zhang and R. Yuan, Homoclinic solutions for a class of non-autonomous subquadratic second order Hamiltonian systems, *Nonlinear Anal.* **71**, 4125–4130 (2009).
[11] W. Zou and S. Li, Infinitely many homoclinic orbits for the second-order Hamiltonian systems, *Appl. Math. Lett.* **16**, 1283–1287 (2003).
[12] J. Sun, H. Chen and J. J. Nieto, Homoclinic solutions for a class of subquadratic second-order Hamiltonian systems, *J. Math. Anal. Appl.* **373**, 20–29 (2011).
[13] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematical Studies, Vol. **204**, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006.
[14] I. Podlubny, *Fractional differential equations*, Academic Press, 1999.
[15] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives: theory and applications*, Gordon and Breach, Switzerland, 1993.
[16] Z. Bai and H. Lü, Positive solutions for boundary value problem of nonlinear fractional differential equation, *J. Math. Anal. Appl.* **311**, 495–505 (2005).
[17] S. Zhang, Existence of solution for a boundary value problem of fractional order, *Acta Math. Sci.* **26**, 220–228 (2006).
[18] W. Jiang, The existence of solutions for boundary value problems of fractional differential equations at resonance, *Nonlinear Anal.* **74**, 1987–1994 (2011).
[19] S. Liang and J. Zhang, Positive solutions for boundary value problems of nonlinear fractional differential equation, *Nonlinear Anal.* **71**, 5545–5550 (2009).
[20] F. Jiao and Y. Zhou, Existence of solutions for a class of fractional boundary value problem via critical point theory, *Comput. Math. Appl.* **62**, 1181–1199 (2011).
[21] F. Jiao and Y. Zhou, Existence results for fractional boundary value problem via critical point theory, *Int. J. Bifur. Chaos*, **22**, 1250086 (2012).
[22] J. Chen and X. Tang, Existence and multiplicity of solutions for some fractional boundary value problem via critical point theory, *Abstr. Appl. Anal.* **2012**, Article ID 648635 (2012).
[23] K. Teng, H. Jia and H. Zhang, Existence and multiplicity results for fractional differential inclusions with Dirichlet boundary conditions, *Appl. Math. Comput.*, **220**, 792–801 (2013).
[24] W. Fan, X. Jiang and S. Chen, Parameter estimation for the fractional fractal diffusion model based on its numerical solution, *Compu. Math. Appl.* **71**, 642–651 (2016).
[25] X. Jiang and S. Chen, Analytical and numerical solutions of time fractional anomalous thermal diffusion equation in composite medium, *Z. Angew. Math. Mech.* **95**, 156–164 (2015).
[26] B. Yu, X. Jiang and C. Wang, Numerical algorithms to estimate relaxation parameters and Caputo fractional derivative for a fractional thermal wave model in spherical composite medium, *Appl. Math. Comput.* **274**, 106–118 (2016).
[27] C. Torres, Existence of solution for a class of fractional Hamitonian systems, *E. J. Differ. Equ.* **259**, 1–9 (2013).
[28] Z. Zhang and R. Yuan, Variational approach to solutions for a class of fractional Hamiltonian systems, *Math. Meth. Appl. Sci.* **37**, 1873–1883 (2014).
[29] V. Ervin and J. Roop, Variational formulation for the stationary fractional advection dispersion equation, *Numer. Math. Part. Diff. Eqs.*, **22**, 58–76 (2006).
[30] P. L. Lions, The concentration compactness principle in the calculus of variations. The local compactness case. II, *Annales de Institut Henri Poincare. Analyse Non Lineaire*, **1**, 223–283 (1984).
[31] T. Bartsch, Infinitely many solutions of a symmetric Dirichlet problem, *Nonlinear Anal.* **20**, 1205–1216 (1993).
[32] T. Bartsch and M. Willem, On an elliptic equation with concave and convex nonlinearities, *Proc. Amer. Math. Soc.* **123**, 3555–3561 (1995).