Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/020406

A Mean Value Theorem for the Conformable Fractional Calculus on Arbitrary Time Scales

Eze R. Nwaeze[*]

Tuskegee University, Tuskegee AL, 36088, USA

Received: 19 Mar. 2016, Revised: 19 Jun. 2016, Accepted: 22 Jun. 2016
Published online: 1 Oct. 2016

Abstract: In this paper, we continue with the development of the newly Benkhettou–Hassani–Torres fractional (noninteger order) calculus on time scales by proving Rolle's Theorem, Mean Value Theorem, generalized Mean Value Theorem and some other auxiliary results for the fractional derivative T_α. Our results coincide with well-known classical results when the operator T_α is of (integer) order $\alpha = 1$ and the time scale coincides with the set of real numbers.

Keywords: $\alpha-$differentiable functions, time scales, mean value theorem.

1 Introduction

The concept of fractional derivative is traditionally associated to non-integers where the order of derivative is considered to be non-integer. In 1988, Hilger [1] initiated the concept of time scale for which the notion of the delta derivative was defined. For this derivative, Guseinov and Kaymakçalan [2] obtained, among other things, a Rolle's and Mean Value Theorem for the delta derivative. It is now a subject of interest to combine this concept of fractional derivative with the time scale theory. For more on this see [3,4,5,6]. In 2014, Khalil et al. in [7] came up with an interesting idea of the fractional derivative that extends the familiar limit definition of the derivatives of a function called conformable fractional derivative. The simple nature of this definition allows for many extensions of some classical theorems in calculus for which the applications are indispensable in the fractional differential models that the existing definitions do not permit. Recently, Benkhettou, et al. [8] extended this definition to an arbitrary time scale $\mathbb{T}$ by introducing T_α differentiation operator and the α-fractional integral.

Motivated by the work in [2], we continue with the development of the conformable time-scale fractional calculus initiated in [8]. Precisely, we prove Rolle's Theorem (Theorem 5), Mean Value Theorem (Theorem 6), generalized Mean Value Theorem (Theorem 7) and some other auxiliary results for the fractional derivative T_α. For the case $\alpha = 1$ and $\mathbb{T} = \mathbb{R}$, see [2] and [9], respectively.

The paper is organized as follows. In Section 2 we recall the basics of the conformable fractional calculus on time scales. Our results are then stated and proved in Section 3.

2 Preliminaries

We start by presenting some basic notions in time scale theory. For more on this subject, we refer the reader to the book [10].

Definition 1. *A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ are defined by*

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}$$

[*] Corresponding author e-mail: enwaeze@mytu.tuskegee.edu

for $t \in \mathbb{T}$. In this definition, we put $\sigma(\sup \mathbb{T}) := \sup \mathbb{T}$. and $\rho(\inf \mathbb{T}) := \inf \mathbb{T}$. Clearly, we see that $\sigma(t) \geq t$ and $\rho(t) \leq t$ for all $t \in \mathbb{T}$. We say that t is right-scattered, right-dense, left-scattered, and left-dense if $\sigma(t) > t$, $\sigma(t) = t$, $\rho(t) < t$, $\rho(t) = t$, respectively. The set $\mathbb{T}^k$ is derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum t^*, then $\mathbb{T}^k = \mathbb{T} \setminus \{t^*\}$. Otherwise, $\mathbb{T}^k = \mathbb{T}$. For $a, b \in \mathbb{T}$ with $a \leq b$, we define the interval $[a,b]$ in $\mathbb{T}$ by $[a,b] = \{t \in \mathbb{T} : a \leq t \leq b\}$. Open intervals and half-open intervals are defined in the same manner.

Now, we briefly recall the necessary definitions and results from the conformable fractional calculus on time scales [8].

Definition 2(See [8]). *Let $\mathbb{T}$ be a time scale, $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^\kappa$, and $\alpha \in (0,1]$. For $t > 0$, we define $T_\alpha(f)(t)$ to be the number, provided it exists, with the property that, given any $\varepsilon > 0$, there is a δ-neighbourhood*

$$\mathscr{V}_t = (t-\delta, t+\delta) \cap \mathbb{T} \quad ([t, t+\delta) \cap \mathbb{T})$$

of t, $\delta > 0$, such that $\left|[f(\sigma(t)) - f(s)]t^{1-\alpha} - T_\alpha(f)(t)[\sigma(t) - s]\right| \leq \varepsilon |\sigma(t) - s|$ for all $s \in \mathscr{V}_t$. We call $T_\alpha(f)(t)$ the α-fractional derivative (right-sided α-differentiable) of f of order α at t, and we define the α-fractional derivative at 0 as $T_\alpha(f)(0) := \lim\limits_{t \to 0^+} T_\alpha(f)(t)$.

In what follows, we will simply say "f is α-differentiable at t" instead of "f is α-differentiable of order f at t." We say that f is α-differentiable on $[a,b)$ if it is α-differentiable at every point in (a,b), and right-sided α-differentiable at a.

If $\alpha = 1$, then we obtain from Definition 2 the Hilger delta derivative of time scales [10]. The α-fractional derivative of order zero is defined by the identity operator: $T_0(f) := f$.

Theorem 1(See [8]). *Let $\alpha \in (0,1]$, $\lambda \in \mathbb{R}$, and $\mathbb{T}$ be a time scale. Assume $f, g : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^k$. The following holds*

(i) *if f is continuous at t and t is right-scattered, then f is α-differentiable at t with*

$$T_\alpha(f)(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} t^{1-\alpha}.$$

(ii) *if t is right-dense, then f is α-differentiable at t if and only if the limit $\lim\limits_{s \to t} \frac{f(t) - f(s)}{t - s} t^{1-\alpha}$ exists as a finite number. In this case,*

$$T_\alpha(f)(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} t^{1-\alpha}.$$

(iii) *if $f(t) = t$ for all $t \in \mathbb{T}$, then $T_\alpha(f)(t) = t^{1-\alpha}$, if $0 < \alpha < 1$ and 1 if $\alpha = 1$. In addition, the α-derivative of a constant function is zero.*

(iv) *if f and g are α-differentiable, then $f + g$ and λf are both α-differentiable with $T_\alpha(f+g) = T_\alpha(f) + T_\alpha(g)$ and $T_\alpha(\lambda f) = \lambda T_\alpha(f)$.*

Let f be a real-valued function defined on an interval I. We say that f is increasing, decreasing, nondecreasing, and nonincreasing on I if $t_1, t_2 \in I$ and $t_1 < t_2$ implies $f(t_1) < f(t_2)$, $f(t_1) > f(t_2)$, $f(t_1) \leq f(t_2)$, and $f(t_1) \leq f(t_2)$, respectively.

Definition 3. *We say a function $f : \mathbb{T} \to \mathbb{R}$ is right-increasing (right-decreasing) at $t_0 \in \mathbb{T}^k$ provided that*

(i) *if t_0 is right scattered, then $f(\sigma(t_0)) > f(t_0)$, $(f(\sigma(t_0)) < f(t_0))$,*
(ii) *if t_0 is right dense, then there is a neighborhood U of t_0 such that $f(t) > f(t_0)$, $(f(t) < f(t_0))$, for all $t \in U$, $t > t_0$.*

Definition 4. *We say a function $f : \mathbb{T} \to \mathbb{R}$ is local right-maximum (local right-minimum) at $t_0 \in \mathbb{T}^k$ provided that*

(i) *if t_0 is right scattered, then $f(\sigma(t_0)) \leq f(t_0)$, $(f(\sigma(t_0)) \geq f(t_0))$,*
(ii) *if t_0 is right dense, then there is a neighborhood U of t_0 such that $f(t) \leq f(t_0)$, $(f(t) \geq f(t_0))$, for all $t \in U$, $t > t_0$.*

3 Main Results

Throughout this paper, $\alpha \in (0,1]$.

Theorem 2. *Let* $f : \mathbb{T} \to \mathbb{R}$ *be* α*-differentiable at* $t_0 \in \mathbb{T}^k$ *and* $T_\alpha(f)(t_0) > 0$, $(T_\alpha(f)(t_0) < 0)$. *Then* f *is right-increasing, (right-decreasing), at* t_0.

Proof. We prove the case when $T_\alpha(f)(t_0) > 0$, since the proof for the case $T_\alpha(f)(t_0) < 0$ is similar. Now, if t_0 is right scattered (i.e. $\sigma(t_0) > t_0$), then by item (i) of Theorem 1, we obtain

$$T_\alpha(f)(t_0) = \frac{f(\sigma(t_0)) - f(t_0)}{\sigma(t_0) - t_0} t_0^{1-\alpha}.$$

$T_\alpha(f)(t_0) > 0$ implies that $f(\sigma(t_0)) > f(t_0)$ if $\sigma(t_0) > t_0$ (since $t_0 > 0$). In the other hand, if t_0 is right dense, then by item (ii) of Theorem 1, we have

$$T_\alpha(f)(t_0) = \lim_{t \to t_0} \frac{f(t_0) - f(t)}{t_0 - t} t_0^{1-\alpha}.$$

So, for $\varepsilon = T_\alpha(f)(t_0)$ there is a neighborhood U of t_0 such that

$$\left| \frac{f(t_0) - f(t)}{t_0 - t} t_0^{1-\alpha} - T_\alpha(f)(t_0) \right| < T_\alpha(f)(t_0)$$

for all $t \in U$, $t \neq t_0$. Hence $0 < \dfrac{f(t_0) - f(t)}{t_0 - t} t_0^{1-\alpha} < 2\, T_\alpha(f)(t_0)$ for all $t \in U$. Therefore, $f(t) > f(t_0)$ for all $t \in U$, $t > t_0$, and hence the proof is complete.

Theorem 3. *Let* $f : \mathbb{T} \to \mathbb{R}$ *be* α*-differentiable at* $t_0 \in \mathbb{T}^k$ *and* $T_\alpha(t_0) > 0$ $(T_\alpha(t_0) < 0)$*. Then* f *attains its local right-minimum (local right-maximum), at* t_0.

Proof. If $T_\alpha(f)(t_0) > 0$, then by Theorem 2, f will be right-increasing at t_0 and therefore f will attain its local right-minimum at t_0.

Theorem 4. *Let* $f : \mathbb{T} \to \mathbb{R}$ *be* α*-differentiable at* $t_0 \in \mathbb{T}^k$. *If* f *attains its local right-minimum (local right-maximum), at* t_0*, then*

$$T_\alpha(f)(t_0) \geq 0,\ (T_\alpha(f)(t_0) \leq 0).$$

Proof. Let f attain its local right-minimum at t_0. We need to show that $T_\alpha(f)(t_0) \geq 0$. To do this, we assume the contrary, viz, that $T_\alpha(f)(t_0) < 0$. Then by Theorem 2, f will be right-decreasing, hence contradicting our assumption that f attains its local right-minimum at t_0. Thus we must have that $T_\alpha(f)(t_0) \geq 0$.

Theorem 5(A fractional version of the Rolle's Theorem). *Let* f *be a function satisfying the following*

(a) *continuous on* $[a,b]$
(b) α*-differentiable on* $[a,b)$
(c) $f(a) = f(b)$.

Then there exist $\eta, \eta' \in [a,b)$ *such that*

$$T_\alpha(f)(\eta) \leq 0 \leq T_\alpha(f)(\eta')$$

Proof. By the extreme value theorem, there exist $\eta, \eta' \in [a,b]$ such that f attains its minimum value at $t = \eta'$ and maximum value at $t = \eta$. Since $f(a) = f(b)$, we may assume that $\eta, \eta' \in [a,b)$. Clearly, f attains its local right-minimum at η' and its local right-maximum at η. Then by Theorem 4 we have $T_\alpha(f)(\eta) \leq 0$ and $T_\alpha(f)(\eta') \geq 0$.

Theorem 6(A fractional version of the Mean Value Theorem). *Let* f *be a function satisfying the following*

(a) *continuous on* $[a,b]$
(b) α*-differentiable on* $[a,b)$.

Then there exist $\eta, \eta' \in [a,b)$ *such that*

$$\eta^{\alpha-1} T_\alpha(f)(\eta) \leq \mathscr{R}(f;a,b) \leq (\eta')^{\alpha-1} T_\alpha(f)(\eta'),$$

where $\mathscr{R}(f;a,b) = \dfrac{f(b) - f(a)}{b - a}$.

Proof. For the case when $\alpha = 1$, see [2, Theorem 2.7]. So we only prove the case when $0 < \alpha < 1$. Consider the function F defined on $[a,b]$ by

$$F(t) = f(t) - f(a) - \frac{f(b) - f(a)}{b-a}(t-a).$$

As a difference of continuous functions, F is continuous on $[a,b]$. Also, by item (iv) of Theorem 1, F is α-differentiable on $[a,b)$ and $F(a) = 0 = F(b)$. Applying Theorem 5 to F, then there exist $\eta, \eta' \in [a,b)$ such that

$$T_\alpha(F)(\eta) \leq 0 \leq T_\alpha(F)(\eta').$$

Using items (iii) and (iv) of Theorem 1, we get

$$T_\alpha(F)(t) = T_\alpha(f)(t) - \frac{f(b) - f(a)}{b-a} t^{1-\alpha},$$

and hence our result is proven.

Corollary 1. *Let f be a continuous function on $[a,b]$ that is α-differentiable on $[a,b)$. If $T_\alpha(f)(t) = 0$ for all $t \in [a,b)$, then f is a constant function on $[a,b]$.*

Corollary 2. *Let f be a continuous function on $[a,b]$ that is α-differentiable on $[a,b)$. Then f is increasing, decreasing, nondecreasing, and nonincreasing on $[a,b]$ if $T_\alpha(f)(t) > 0,\ T_\alpha(f)(t) < 0,\ T_\alpha(f)(t) \geq 0$, and $T_\alpha(f)(t) \leq 0$ for all $t \in [a,b)$, respectively.*

We now present a generalized Mean Value Theorem which generalizes Theorem 6 by taking $g(t) = t$ for all $t \in \mathbb{T}$.

Theorem 7(A fractional version of the generalized Mean Value Theorem). *Let f and g be functions satisfying the following*

(a) *continuous on $[a,b]$*
(b) *α-differentiable on $[a,b)$.*

Suppose $T_\alpha(g)(t) > 0$ for all $t \in [a,b)$. Then there exist $\eta, \eta' \in [a,b)$ such that

$$\frac{T_\alpha(f)(\eta)}{T_\alpha(g)(\eta)} \leq \frac{f(b) - f(a)}{g(b) - g(a)} \leq \frac{T_\alpha(f)(\eta')}{T_\alpha(g)(\eta')}.$$

Proof. If $T_\alpha(g)(t) > 0$ for all $t \in [a,b)$, then it follows from Theorem 6 that $g(a) \neq g(b)$. Now we consider the following function

$$F(t) = f(t) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(t) - g(a)).$$

As a difference of continuous functions, F is continuous on $[a,b]$. Also, by item (iv) of Theorem 1, F is α-differentiable on $[a,b)$ and $F(a) = 0 = F(b)$. Applying Theorem 5 to F, then there exist $\eta, \eta' \in [a,b)$ such that

$$T_\alpha(F)(\eta) \leq 0 \leq T_\alpha(F)(\eta').$$

Using items (iii) and (iv) of Theorem 1, we get

$$T_\alpha(F)(t) = T_\alpha(f)(t) - \frac{f(b) - f(a)}{g(b) - g(a)} T_\alpha(g)(t),$$

and hence the desired result.

4 Conclusion

We continue the development of the newly proposed definition of the conformable fractional calculus (on time scale) initiated in [8]. For this, we proved the mean value theorem and some related results in this direction for the conformable fractional operator on an arbitrary time scale.

References

[1] S. Hilger, Ein Maβkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten, Ph.D. thesis, Universität Würzburg, Würzburg, Germany, 1988.
[2] G.Sh. Guseinov and B. Kaymakçalan, On a disconjugacy criterion for second order dynamic equations on time scales, *J. Comput. Appl. Math.* **141**, 187–196 (2002).
[3] N. R. O. Bastos, R. A. C. Ferreira and D. F. M. Torres, Necessary optimality conditions for fractional difference problems of the calculus of variations, *Discrete Contin. Dyn. Syst.* **29**, No. 2, 417–437 (2011).
[4] N. R. O. Bastos, D. Mozyrska and D. F. M. Torres, Fractional derivatives and integrals on time scales via the inverse generalized Laplace transform, *Int. J. Math. Comput.* **11**, J11, 1–9 (2011).
[5] N. Benkhettou, A. M. C. Brito da Cruz and D. F. M. Torres, A fractional calculus on arbitrary time scales: Fractional differentiation and fractional integration, *Signal Proc.* **107**, 230–237 (2015).
[6] N. Benkhettou, A. M. C. Brito da Cruz and D. F. M. Torres, Nonsymmetric and symmetric fractional calculi on arbitrary nonempty closed sets, *Math. Meth. Appl. Sci.* **39**, No. 2, 261–279 (2016).
[7] R. Khalil, M. A. Horani, A. Yousef and M. Sababheh, A new definition of fractional derivativel, *J. Comput. Appl. Math.* **264**, 65–70 (2014).
[8] N. Benkhettou, S. Hassani and D. F. M. Torres, A conformable fractional calculus on arbitrary time scales, *J. King Saud Univ. Sci.* **28**, No. 1, 93–98 (2016).
[9] O. S. Iyiola and E. R. Nwaeze, Some new results on the new conformable fractional calculus with application using D'Alambert approach, *Progr. Fract. Differ. Appl.* **2**, No. 1, 115–121 (2016).
[10] M. Bohner and A. Peterson, *Dynamic equations on time scales*, Birkhäuser Boston, Boston, MA, 2001.