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Some Families of Integral, Trigonometric and Other Related Inequalities

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In this lecture, we propose to survey several interesting recent developments on improvements and generalizations of what is popularly known as Steffensen's integral inequality. Moreover, as a by-product of the investigation presented here, we show how one can correct an error in a recent generalization of Steffensen's inequality. We also present a brief survey of some old and new inequalities associated with trigonometric functions. These include (among other results) a weighted and exponential generalization of what is popularly known as Wilker's inequality and a substantially improved version of the Sándor-Bencze conjectured inequality. References to many related developments in recent years are also provided for the interested reader.

Keywords: Integral inequalities; Trigonometric inequalities; Trigonometric functions; Steffensen's inequality; Mercer's inequality; Huygens's inequality; Weighted generalizations.

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1 Introduction

We begin by recalling here the well-known Steffensen's integral inequality as Theorem A below.

Theorem A. *Let f and g be integrable functions defined on $[a, b]$ with f nonincreasing. Also let*

$$\lambda = \int_a^b g(t)dt \qquad \big(0 \leqq g(t) \leqq 1;\ t \in [a, b]\big).$$

Then

$$\int_{b-\lambda}^b f(t)dt \leqq \int_a^b f(t)g(t)dt \leqq \int_a^{a+\lambda} f(t)dt. \tag{1}$$

This classical inequality (1) was established by Steffensen in 1919 (see [27]). A comprehensive survey on this inequality can be found in [12]. Steffensen's inequality (1) plays an important rôle in the study of integral inequalities. As tools, Steffensen's inequality (1) can be used for dealing with comparison between integrals over a whole set and integrals over a subset. Due to the importance of Steffensen's inequality (1), it has been given considerable attention by mathematicians and has motivated a large number of research papers on this subject (see [1], [3], [4], [5], [13], [14], [19], [20] and [23]; see also [24] and the references cited in each of these earlier works).

Mercer [11] generalized Steffensen's inequality (1) in the form asserted by Theorem B below.

Theorem B. *Let f, g and h be integrable functions defined on $[a, b]$ with f nonincreasing. Also let*

$$0 \leqq g(t) \leqq h(t) \qquad (t \in [a, b]).$$

Then

$$\int_{b-\lambda}^{b} f(t)h(t)dt \leqq \int_{a}^{b} f(t)g(t)dt \leqq \int_{a}^{a+\lambda} f(t)h(t)dt, \tag{2}$$

where λ is given by

$$\int_{a}^{a+\lambda} h(t)dt = \int_{a}^{b} g(t)dt.$$

In order to observe the fact that there is an error in Mercer's result (2), we consider

$$f(t) = 8 - 3t, \ g(t) = t \quad \text{and} \quad h(t) = 4t \qquad (t \in [0, 2]).$$

Clearly, f is nonincreasing on $[0, 2]$ and

$$0 \leqq t \leqq 4t \qquad (t \in [0, 2]) \quad \text{and} \quad \int_{0}^{0+\lambda} 4t dt = \int_{0}^{2} t dt \qquad (\lambda = 1),$$

which shows that the conditions of Theorem B are fully satisfied. However, we find in this case that

$$\begin{aligned} &\int_{b-\lambda}^{b} f(t)h(t)dt - \int_{a}^{b} f(t)g(t)dt \\ &\quad = \int_{1}^{2} (8 - 3t)(4t)dt - \int_{0}^{2} (8 - 3t)tdt \\ &\quad = 12, \end{aligned}$$

which obviously contradicts the first (left-hand) inequality in (2). Consequently, the inequality (2) is not true in general.

The main object of this lecture is to establish several new generalized and sharpened versions of Steffensen's inequality (1). We shall also reconsider and present a *duly-corrected* version of Mercer's inequality (2).

2 A Useful Lemma

We begin by stating a useful integral inequality asserted by the following Lemma.

Lemma 1. *Let f, g and h be integrable functions defined on $[a,b]$. Suppose also that λ is a real number such that*

$$\int_a^{a+\lambda} h(t)dt = \int_a^b g(t)dt = \int_{b-\lambda}^b h(t)dt.$$

Then

$$\int_a^b f(t)g(t)dt = \int_a^{a+\lambda} \big(f(t)h(t) - [f(t) - f(a+\lambda)][h(t) - g(t)]\big)dt + \int_{a+\lambda}^b [f(t) - f(a+\lambda)]g(t)dt, \tag{3}$$

and

$$\int_a^b f(t)g(t)dt = \int_a^{b-\lambda} [f(t) - f(b-\lambda)]g(t)dt + \int_{b-\lambda}^b \big(f(t)h(t) - [f(t) - f(b-\lambda)][h(t) - g(t)]\big)dt. \tag{4}$$

Proof. The assumptions of Lemma 1 imply that

$$a \leqq a+\lambda \leqq b \qquad \text{and} \qquad a \leqq b-\lambda \leqq b.$$

Firstly, we prove the validity of the integral identity (3). Indeed, by direct computation, we find that

$$\begin{aligned}
&\int_a^{a+\lambda} \big(f(t)h(t) - [f(t) - f(a+\lambda)][h(t) - g(t)]\big)dt - \int_a^b f(t)g(t)dt \\
&= \int_a^{a+\lambda} \big(f(t)h(t) - f(t)g(t) - [f(t) - f(a+\lambda)][h(t) - g(t)]\big)dt \\
&\qquad + \int_a^{a+\lambda} f(t)g(t)dt - \int_a^b f(t)g(t)dt \\
&= \int_a^{a+\lambda} f(a+\lambda)[h(t) - g(t)]dt - \int_{a+\lambda}^b f(t)g(t)dt \\
&= f(a+\lambda)\left(\int_a^{a+\lambda} h(t)dt - \int_a^{a+\lambda} g(t)dt\right) - \int_{a+\lambda}^b f(t)g(t)dt.
\end{aligned} \tag{5}$$

Now, if we apply the following assumption of Lemma 1:

$$\int_a^{a+\lambda} h(t)dt = \int_a^b g(t)dt$$

to (5), we obtain

$$\begin{aligned}
&f(a+\lambda)\left(\int_a^{a+\lambda} h(t)dt - \int_a^{a+\lambda} g(t)dt\right) - \int_{a+\lambda}^b f(t)g(t)dt \\
&= f(a+\lambda)\left(\int_a^b g(t)dt - \int_a^{a+\lambda} g(t)dt\right) - \int_{a+\lambda}^b f(t)g(t)dt \\
&= f(a+\lambda)\int_{a+\lambda}^b g(t)dt - \int_{a+\lambda}^b f(t)g(t)dt \\
&= \int_{a+\lambda}^b [f(a+\lambda) - f(t)]g(t)dt. \qquad (6)
\end{aligned}$$

By combining the integral identities (5) and (6), we are led to the desired integral identity (3) asserted by Lemma 1.

Secondly, we observe that the following assumption of Lemma 1:

$$\int_{b-\lambda}^b h(t)dt = \int_a^b g(t)dt$$

implies that

$$\int_a^{a+(b-a-\lambda)} h(t)dt = \int_a^b [h(t) - g(t)]dt.$$

By appealing to the integral identity (3) with the following substitutions:

$$\lambda \longmapsto b-a-\lambda \qquad \text{and} \qquad g(t) \longmapsto h(t) - g(t),$$

the integral identity (4) would follow immediately.

The proof of Lemma 1 is thus completed.

3 The Main Integral Inequalities

In this section, we present the main integral inequalities as Theorems 1, 2 and 3 below (see also the recent works [30] and [31]).

Theorem 1. *Let f, g and h be integrable functions defined on $[a,b]$ with f nonincreasing. Also let*

$$0 \leqq g(t) \leqq h(t) \qquad (t \in [a,b]).$$

Then the following integral inequalities hold true:

$$\begin{aligned}\int_{b-\lambda}^{b} f(t)h(t)dt &\leqq \int_{b-\lambda}^{b} \big(f(t)h(t) - [f(t) - f(b-\lambda)][h(t) - g(t)]\big)dt \\ &\leqq \int_{a}^{b} f(t)g(t)dt \\ &\leqq \int_{a}^{a+\lambda} \big(f(t)h(t) - [f(t) - f(a+\lambda)][h(t) - g(t)]\big)dt \\ &\leqq \int_{a}^{a+\lambda} f(t)h(t)dt, \qquad (7)\end{aligned}$$

where λ is given by

$$\int_{a}^{a+\lambda} h(t)dt = \int_{a}^{b} g(t)dt = \int_{b-\lambda}^{b} h(t)dt.$$

Proof. In view of the assumptions that the function f is nonincreasing on $[a, b]$ and that

$$0 \leqq g(t) \leqq h(t) \qquad (t \in [a, b]),$$

we conclude that

$$\int_{a}^{b-\lambda} [f(t) - f(b-\lambda)]g(t)dt \geqq 0 \qquad (8)$$

and

$$\int_{b-\lambda}^{b} [f(b-\lambda) - f(t)][h(t) - g(t)]dt \geqq 0. \qquad (9)$$

Using the integral identity (4) together with the integral inequalities (8) and (9), we find that

$$\begin{aligned}\int_{b-\lambda}^{b} f(t)h(t)dt &\leqq \int_{b-\lambda}^{b} \big(f(t)h(t) - [f(t) - f(b-\lambda)][h(t) - g(t)]\big)dt \\ &\leqq \int_{a}^{b} f(t)g(t)dt. \qquad (10)\end{aligned}$$

In the same way as above, we can prove that

$$\begin{aligned}\int_{a}^{b} f(t)g(t)dt &\leqq \int_{a}^{a+\lambda} \big(f(t)h(t) - [f(t) - f(a+\lambda)][h(t) - g(t)]\big)dt \\ &\leqq \int_{a}^{a+\lambda} f(t)h(t)dt. \qquad (11)\end{aligned}$$

The proof of Theorem 1 is completed by combining the integral inequalities (10) and (11).

Remark 1. The results asserted by Theorem 1 show that the Mercer's inequality (2) holds true under the following *additional* condition:

$$\int_a^b g(t)dt = \int_{b-\lambda}^b h(t)dt.$$

Furthermore, as a direct consequence of Theorem 1, a modified version of Mercer's inequality (2) can be deduced as follows.

Corollary 1. *Let f, g and h be integrable functions defined on $[a,b]$ with f nonincreasing. Also let*

$$0 \leqq g(t) \leqq h(t) \qquad (t \in [a,b]).$$

Then

$$\int_{b-\lambda}^b f(t)h(t)dt \leqq \int_a^b f(t)g(t)dt \leqq \int_a^{a+\lambda} f(t)h(t)dt, \tag{12}$$

where λ is given by

$$\int_a^{a+\lambda} h(t)dt \ = \int_a^b g(t)dt \ = \int_{b-\lambda}^b h(t)dt.$$

In particular, upon setting $h(t) \equiv 1$ in (7), we obtain the following refinement of Steffensen's inequality (1).

Corollary 2. *Let f and g be integrable functions defined on $[a,b]$ with f nonincreasing. Also let*

$$\lambda = \int_a^b g(t)dt \qquad \textit{and} \qquad 0 \leqq g(t) \leqq 1 \quad (t \in [a,b]).$$

Then

$$\begin{aligned}\int_{b-\lambda}^b f(t)dt &\leqq \int_{b-\lambda}^b \big(f(t) - [f(t) - f(b-\lambda)][1-g(t)]\big)dt \\ &\leqq \int_a^b f(t)g(t)dt \\ &\leqq \int_a^{a+\lambda} \big(f(t) - [f(t) - f(a+\lambda)][1-g(t)]\big)dt \\ &\leqq \int_a^{a+\lambda} f(t)dt. \end{aligned} \tag{13}$$

In Theorem 2 below, we present a new sharpened and generalized version of Mercer's inequality (2).

Theorem 2. *Let f, g, h and ψ be integrable functions defined on $[a,b]$ with f nonincreasing. Also let*

$$0 \leqq \psi(t) \leqq g(t) \leqq h(t) - \psi(t) \qquad (t \in [a,b]).$$

Then

$$\begin{aligned}\int_{b-\lambda}^{b} f(t)h(t)dt &+ \int_a^b \big|[f(t)-f(b-\lambda)]\psi(t)\big|dt \\ &\leqq \int_a^b f(t)g(t)dt \\ &\leqq \int_a^{a+\lambda} f(t)h(t)dt - \int_a^b \big|[f(t)-f(a+\lambda)]\psi(t)\big|dt, \end{aligned} \tag{14}$$

where λ is given by

$$\int_a^{a+\lambda} h(t)dt = \int_a^b g(t)dt = \int_{b-\lambda}^{b} h(t)dt.$$

Proof. By the assumptions that the function f is nonincreasing on $[a,b]$ and that

$$0 \leqq \psi(t) \leqq g(t) \leqq h(t) - \psi(t) \qquad (t \in [a,b]),$$

it follows that

$$\begin{aligned}&\int_a^{a+\lambda} [f(t)-f(a+\lambda)][h(t)-g(t)]dt + \int_{a+\lambda}^{b} [f(a+\lambda)-f(t)]g(t)dt \\ &= \int_a^{a+\lambda} |f(t)-f(a+\lambda)|\,[h(t)-g(t)]dt + \int_{a+\lambda}^{b} |f(a+\lambda)-f(t)|\,g(t)dt \\ &\geqq \int_a^{a+\lambda} |f(t)-f(a+\lambda)|\psi(t)dt + \int_{a+\lambda}^{b} |f(a+\lambda)-f(t)|\psi(t)dt \\ &= \int_a^b \big|[f(t)-f(a+\lambda)]\psi(t)\big|dt.\end{aligned}$$

We thus have

$$\begin{aligned}\int_a^{a+\lambda} [f(t)-f(a+\lambda)][h(t)-g(t)]dt &+ \int_{a+\lambda}^{b} [f(a+\lambda)-f(t)]g(t)dt \\ &\geqq \int_a^b \big|[f(t)-f(a+\lambda)]\psi(t)\big|dt.\end{aligned} \tag{15}$$

Similarly, we find that

$$\begin{aligned}\int_a^{b-\lambda} [f(t)-f(b-\lambda)]g(t)dt &+ \int_{b-\lambda}^{b} [f(b-\lambda)-f(t)][h(t)-g(t)]dt \\ &\geqq \int_a^b \big|[f(t)-f(b-\lambda)]\psi(t)\big|dt.\end{aligned} \tag{16}$$

By combining the integral identities (3) and (4) and the integral inequalities (15) and (16), we arrive at the inequality (14) asserted by Theorem 2. This evidently completes the proof of Theorem 2.

Remark 2. It is obvious that the Mercer's inequality (2) in the *modified* form (12) follows from Theorem 2 with $\psi(t) \equiv 0$. In addition, by putting

$$h(t) \equiv 1 \qquad \text{and} \qquad \psi(t) \equiv M \qquad (M \in \mathbb{R}^+ \cup \{0\})$$

in (14), we deduce Corollary 3 below.

Corollary 3. *Let f and g be integrable functions defined on $[a, b]$ with f nonincreasing. Also let*

$$\lambda = \int_a^b g(t)dt \qquad \textit{and} \qquad 0 \leqq M \leqq g(t) \leqq 1 - M \quad (t \in [a, b]).$$

Then

$$\begin{aligned} \int_{b-\lambda}^b f(t)dt + M \int_a^b |f(t) - f(b-\lambda)|dt \\ \leqq \int_a^b f(t)g(t)dt \\ \leqq \int_a^{a+\lambda} f(t)dt - M \int_a^b |f(t) - f(a+\lambda)|dt. \end{aligned} \tag{17}$$

Remark 3. Clearly, the integral inequality (17) is a sharpened and generalized version of Steffensen's inequality (1). Indeed, in its special case when $M = 0$, the inequality (17) would reduce to Steffensen's inequality (1).

Finally, we give a general result on a considerably improved version of Steffensen's inequality (1) by introducing the *additional* parameters λ_1 and λ_2.

Theorem 3. *Let f and g be integrable functions defined on $[a, b]$ with f nonincreasing. Also let*

$$0 \leqq \lambda_1 \leqq \int_a^b g(t)dt \leqq \lambda_2 \leqq b - a$$

and

$$0 \leqq M \leqq g(t) \leqq 1 - M \qquad (t \in [a, b]).$$

Then

$$\int_{b-\lambda_1}^{b} f(t)dt + f(b)\left(\int_a^b g(t)dt - \lambda_1\right) + M\int_a^b \left|f(t) - f\left(b - \int_a^b g(t)dt\right)\right| dt$$
$$\leqq \int_a^b f(t)g(t)dt$$
$$\leqq \int_a^{a+\lambda_2} f(t)dt - f(b)\left(\lambda_2 - \int_a^b g(t)dt\right) - M\int_a^b \left|f(t) - f\left(a + \int_a^b g(t)dt\right)\right| dt. \tag{18}$$

Proof. By direct computation, we get

$$\int_a^b f(t)g(t)dt - \int_a^{a+\lambda_2} f(t)dt + f(b)\left(\lambda_2 - \int_a^b g(t)dt\right)$$
$$= \int_a^b f(t)g(t)dt - \int_a^{a+\lambda_2} f(t)dt + \int_a^{a+\lambda_2} f(b)dt - \int_a^b f(b)g(t)dt$$
$$= \int_a^b [f(t) - f(b)]g(t)dt - \int_a^{a+\lambda_2} [f(t) - f(b)]dt$$
$$\leqq \int_a^b [f(t) - f(b)]g(t)dt - \int_a^{a+\int_a^b g(t)dt} [f(t) - f(b)]dt, \tag{19}$$

where the last inequality follows from the assumption that

$$a \leqq a + \int_a^b g(t)dt \leqq a + \lambda_2 \leqq b$$

and

$$f(t) - f(b) \geqq 0 \qquad (t \in [a, b]).$$

On the other hand, from the hypothesis of Theorem 3, we conclude that the function $f(t) - f(b)$ is integrable and nonincreasing on $[a, b]$. Thus, by using Corollary 3 with the following substitution:

$$f(t) \longmapsto f(t) - f(b)$$

in (17), we find that

$$\int_a^b [f(t) - f(b)]g(t)dt - \int_a^{a+\int_a^b g(t)dt} [f(t) - f(b)]dt$$
$$\leqq -M\int_a^b \left|f(t) - f\left(a + \int_a^b g(t)dt\right)\right| dt. \tag{20}$$

By combining the integral inequalities (19) and (20), we obtain

$$\int_a^b f(t)g(t)dt - \int_a^{a+\lambda_2} f(t)dt + f(b)\left(\lambda_2 - \int_a^b g(t)dt\right)$$
$$\leqq -M\int_a^b \left|f(t) - f\left(a + \int_a^b g(t)dt\right)\right| dt,$$

which is the second inequality in the assertion (18) of Theorem 3.

In the same way as above, we can prove that

$$\int_a^b f(t)g(t)dt - \int_{b-\lambda_1}^b f(t)dt - f(b)\left(\int_a^b g(t)dt - \lambda_1\right)$$
$$\geqq \int_a^b [f(t) - f(b)]g(t)dt \ + \int_{b-\int_a^b g(t)dt}^b [f(b) - f(t)]dt$$
$$\geqq M\int_a^b \left|f(t) - f\left(b - \int_a^b g(t)dt\right)\right| dt, \tag{21}$$

which implies the first inequality in the assertion (18) of Theorem 3. This completes the proof of Theorem 3.

Remark 4. It is clear that Steffensen's inequality (1) would follow as a special case of the inequality (18) when

$$M = 0 \qquad \text{and} \qquad \lambda_1 = \lambda_2.$$

Moreover, it is worth noticing that the integral inequality (18) is stronger than Steffensen's inequality (1) if

$$f(b) \geqq 0.$$

4 Trigonometric Inequalities Emerging from Wilker's Inequality

The following trigonometric inequality is known in the literature as Wilker's inequality [29]:

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 \qquad \left(0 < x < \frac{\pi}{2}\right). \tag{22}$$

Wilker's inequality (22) has attracted remarkable interest of many mathematicians and has motivated a large number of research papers involving different proofs, various generalizations and improvements (see [6], [7], [8], [21], [22], [28], [29] and [35]; see also the references cited therein). Recently, the following similar inequality (proved by Huygens [10]) was considered by Sándor and Bencze [25]:

$$2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} > 3 \qquad \left(0 < x < \frac{\pi}{2}\right). \tag{23}$$

Huygens's inequality (23) prompts us to ask a natural question: Does there exist an inequality which unifies (and possibly also extends) Wilker's inequality (22) and Huygens's inequality (23)? The following theorem gives an affirmative answer to this question (see also the recent investigations [32], [33] and [34]).

Theorem 4. *Let*

$$0 < x < \frac{\pi}{2},\ \lambda > 0,\ \mu > 0 \quad \textit{and} \quad p \leqq \frac{2q\mu}{\lambda}.$$

Then, for

$$q > 0 \quad \textit{or} \quad q \leqq \min\left\{-\frac{\lambda}{\mu}, -1\right\},$$

the following inequality holds true:

$$\frac{\lambda}{\mu+\lambda}\left(\frac{\sin x}{x}\right)^p + \frac{\mu}{\mu+\lambda}\left(\frac{\tan x}{x}\right)^q > 1. \tag{24}$$

5 A Further Set of Useful Lemmas

In order to prove Theorem 4, we need each of the following lemmas.

Lemma 2 (see [9, p. 17]). *If*

$$x_i > 0,\ \lambda_i > 0 \quad (i = 1, \cdots, n) \quad \textit{and} \quad \sum_{i=1}^{n} \lambda_i = 1,$$

then

$$\sum_{i=1}^{n} \lambda_i x_i \geqq \prod_{i=1}^{n} x_i^{\lambda_i}. \tag{25}$$

Lemma 3 (see [14, p. 238]). *The following two-sided trigonometric inequality holds true*:

$$\cos x < \left(\frac{\sin x}{x}\right)^3 < 1 \qquad \left(0 < x < \frac{\pi}{2}\right). \tag{26}$$

Lemma 4. *The following trigonometric inequality holds true*:

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2 \qquad \left(0 < x < \frac{\pi}{2}\right). \tag{27}$$

Proof. Define a function

$$f : \left(0, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}$$

by

$$f(x) = \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x}.$$

Then, upon differentiating $f(x)$ with respect to x, we get

$$f'(x) = \frac{1}{\sin^3 x}\left(\sin^2 x \cos x - 2x^2 \cos x + x \sin x\right).$$

Next, by applying Lemma 3 followed by a simple calculation, we find that

$$\begin{aligned}
f'(x) &= \frac{x^2}{\sin^3 x}\left[\cos x\left(\left(\frac{\sin x}{x}\right)^2 - 2\right) + \frac{\sin x}{x}\right] \\
&= \frac{x^2}{\sin^3 x}\left[\left(\cos x - \left(\frac{\sin x}{x}\right)^3\right)\left(\left(\frac{\sin x}{x}\right)^2 - 2\right)\right. \\
&\qquad\qquad \left. + \left(\frac{\sin x}{x}\right)^3\left(\left(\frac{\sin x}{x}\right)^2 - 2\right) + \frac{\sin x}{x}\right] \\
&= \frac{x^2}{\sin^3 x}\left[\left(\cos x - \left(\frac{\sin x}{x}\right)^3\right)\left(\left(\frac{\sin x}{x}\right)^2 - 2\right)\right. \\
&\qquad\qquad \left. + \left(\frac{\sin x}{x}\right)\left(\left(\frac{\sin x}{x}\right)^2 - 1\right)^2\right] \\
&> 0 \qquad\qquad \left(0 < x < \frac{\pi}{2}\right).
\end{aligned}$$

This means that $f(x)$ is *strictly increasing* on the open interval $\left(0, \frac{\pi}{2}\right)$. Consequently, we can deduce from the following observation:

$$\lim_{x \to 0+} f(x) = 2$$

that

$$f(x) > 2 \qquad\qquad \left(0 < x < \frac{\pi}{2}\right),$$

which leads us to the inequality (27) asserted by Lemma 4.

6 Demonstration of Theorem 4

In our proof of Theorem 4, we consider the following two cases.

Case I. Let

$$\lambda > 0,\ \mu > 0,\ p \leqq \frac{2q\mu}{\lambda} \quad \text{and} \quad q > 0.$$

Then, by applying Lemma 2 and Lemma 3, we obtain

$$\begin{aligned}
&\frac{\lambda}{\mu+\lambda}\left(\frac{\sin x}{x}\right)^{p}+\frac{\mu}{\mu+\lambda}\left(\frac{\tan x}{x}\right)^{q} \\
&\qquad \geqq \left(\frac{\sin x}{x}\right)^{p\lambda/(\mu+\lambda)}\left(\frac{\tan x}{x}\right)^{q\mu/(\mu+\lambda)} \\
&\qquad = \left(\frac{\sin x}{x}\right)^{p\lambda/(\mu+\lambda)}\left(\frac{\sin x}{x}\right)^{q\mu/(\mu+\lambda)}\left(\frac{1}{\cos x}\right)^{q\mu/(\mu+\lambda)} \\
&\qquad > \left(\frac{\sin x}{x}\right)^{p\lambda/(\mu+\lambda)}\left(\frac{\sin x}{x}\right)^{q\mu/(\mu+\lambda)}\left(\frac{\sin x}{x}\right)^{-3q\mu/(\mu+\lambda)} \\
&\qquad = \left(\frac{\sin x}{x}\right)^{(p\lambda-2q\mu)/(\mu+\lambda)} \\
&\qquad \geqq 1 \qquad\qquad \left(0<x<\frac{\pi}{2}\right),
\end{aligned}$$

which is the desired inequality (24).

Case II. Let

$$\lambda>0,\ \mu>0,\ p\leqq\frac{2q\mu}{\lambda}\quad\text{and}\quad q\leqq\min\left\{-\frac{\lambda}{\mu},-1\right\}.$$

Then it follows from the hypothesis of Theorem 4 that

$$\begin{aligned}
&\frac{\lambda}{\mu+\lambda}\left(\frac{\sin x}{x}\right)^{p}+\frac{\mu}{\mu+\lambda}\left(\frac{\tan x}{x}\right)^{q} \\
&\qquad \geqq \frac{\lambda}{\mu+\lambda}\left(\frac{\sin x}{x}\right)^{2q\mu/\lambda}+\frac{\mu}{\mu+\lambda}\left(\frac{\tan x}{x}\right)^{q} \\
&\qquad = \frac{\lambda}{\mu+\lambda}\left(\frac{x}{\sin x}\right)^{-2q\mu/\lambda}+\frac{\mu}{\mu+\lambda}\left(\frac{x}{\tan x}\right)^{-q} \qquad (28) \\
&\qquad\qquad \left(-\frac{q\mu}{\lambda}\geqq 1;\ -q\geqq 1\right).
\end{aligned}$$

Moreover, we find from Lemma 4 that

$$\left(\frac{x}{\sin x}\right)^{2}>2-\frac{x}{\tan x}>0 \qquad \left(0<x<\frac{\pi}{2}\right).$$

By combining the inequality (28) with the above trigonometric inequality, we obtain

$$\begin{aligned}
&\frac{\lambda}{\mu+\lambda}\left(\frac{\sin x}{x}\right)^{p}+\frac{\mu}{\mu+\lambda}\left(\frac{\tan x}{x}\right)^{q} \\
&\qquad > \frac{\lambda}{\mu+\lambda}\left(2-\frac{x}{\tan x}\right)^{-q\mu/\lambda}+\frac{\mu}{\mu+\lambda}\left(\frac{x}{\tan x}\right)^{-q} \qquad (29) \\
&\qquad\qquad \left(0<x<\frac{\pi}{2}\right).
\end{aligned}$$

We now define a function

$$f : (0,1) \longrightarrow \mathbb{R}$$

by

$$f(x) = \frac{\lambda}{\mu+\lambda}\ (2-x)^{-q\mu/\lambda} + \frac{\mu}{\mu+\lambda}\ x^{-q},$$

which, upon differentiating with respect to x, yields

$$f'(x) = \frac{q\mu}{\mu+\lambda}\left[(2-x)^{-(q\mu/\lambda)-1} - x^{-q-1}\right].$$

For

$$-\frac{q\mu}{\lambda} \geqq 1,\ -q \geqq 1 \quad \text{and} \quad 0 < x < 1,$$

it is easy to verify that

$$(2-x)^{-(q\mu/\lambda)-1} \geqq 1 \geqq x^{-q-1}.$$

We thus conclude that

$$f'(x) \leqq 0 \qquad\qquad (0 < x < 1),$$

which immediately implies that $f(x)$ is *decreasing* on the open interval $(0,1)$. Hence we have

$$f(x) \geqq f(1) = 1 \qquad\qquad (0 < x < 1).$$

Now, by making use of the inequality (29) together with the following well-known trigonometric inequality:

$$0 < \frac{x}{\tan x} < 1 \qquad\qquad \left(0 < x < \frac{\pi}{2}\right),$$

we deduce that

$$\frac{\lambda}{\mu+\lambda}\left(\frac{\sin x}{x}\right)^p + \frac{\mu}{\mu+\lambda}\left(\frac{\tan x}{x}\right)^q > f\left(\frac{x}{\tan x}\right) \geqq 1,$$

which proves the inequality (24). This completes the proof of Theorem 4.

By setting $(p,q) = (2,1)$ and $(\lambda,\mu) = (2,1)$ in Theorem 4, we obtain a weighted generalization of Wilker's inequality (22) and an exponential generalization of Huygens's inequality (23), given by Corollary 4 and Corollary 5, respectively.

Corollary 4. *Let*

$0 < x < \frac{\pi}{2}$ *and* $0 < \lambda \leqq \mu$. *Then*

$$\frac{\lambda}{\mu+\lambda}\left(\frac{\sin x}{x}\right)^2 + \frac{\mu}{\mu+\lambda}\left(\frac{\tan x}{x}\right) > 1. \tag{30}$$

Corollary 5. *Let* $0 < x < \frac{\pi}{2}$ *and* $p \leqq q$. *Then, for* $q > 0$ *or* $q \leqq -2$, *the following inequality holds true*:

$$2\left(\frac{\sin x}{x}\right)^p + \left(\frac{\tan x}{x}\right)^q > 3. \tag{31}$$

Remark 5.

It is obvious that Wilker's inequality (22) would follow as a special case of the inequality (30) when $\lambda = \mu = 1$. Furthermore, in its special case when $p = q = 1$, the inequality (31) reduces to Huygens's inequality (23).

7 Applications and Consequences of Theorem 4

As an *Open Problem*, Sándor and Bencze [25] asked to prove that, for all

$$x \in \left(0, \frac{\pi}{2}\right) \quad \text{and} \quad \alpha \in (0, \infty),$$

the following trigonometric inequality holds true:

$$\left(\frac{\sin x}{x}\right)^{\alpha} > \frac{\cos^{\alpha} x}{1 + \cos^{\alpha} x}. \tag{32}$$

The Sándor-Bencze conjectured inequality (32) provides a good opportunity to illustrate the application of the foregoing results. Based upon the improved Wilker inequality (24) asserted by Theorem 4, we give here a sharp and generalized version of the Sándor-Bencze conjectured inequality (24).

Theorem 5. *Let*

$0 < x < \frac{\pi}{2}$. Then, for $\alpha > 0 \quad or \quad \alpha \leqq -1$, the following inequality holds true:

$$\left(\frac{\sin x}{x}\right)^{\alpha} > \frac{4 \cos^{\alpha} x}{1 + \sqrt{1 + 8 \cos^{2\alpha} x}}. \tag{33}$$

Proof. Putting $\lambda = \mu = 1, \ p = 2\alpha$ and $q = \alpha$ in Theorem 4, we get $\left(\frac{\sin x}{x}\right)^{2\alpha} + \cos^{-\alpha} x \left(\frac{\sin x}{x}\right)^{\alpha} - 2 > 0 \ \left(0 < x < \frac{\pi}{2}; \ \alpha > 0 \quad \text{or} \quad \alpha \leqq -1\right)$, which is equivalent to the following trigonometric inequality:

$$\left[\left(\frac{\sin x}{x}\right)^{\alpha} + \frac{\cos^{-\alpha} x + \sqrt{\cos^{-2\alpha} x + 8}}{2}\right] \cdot \left[\left(\frac{\sin x}{x}\right)^{\alpha} + \frac{\cos^{-\alpha} x - \sqrt{\cos^{-2\alpha} x + 8}}{2}\right] > 0.$$

We can now deduce from the above inequality that

$$\left(\frac{\sin x}{x}\right)^{\alpha} + \frac{\cos^{-\alpha} x - \sqrt{\cos^{-2\alpha} x + 8}}{2} > 0,$$

this is, that

$$\left(\frac{\sin x}{x}\right)^{\alpha} > \frac{-\cos^{-\alpha} x + \sqrt{\cos^{-2\alpha} x + 8}}{2} = \frac{4 \cos^{\alpha} x}{1 + \sqrt{1 + 8 \cos^{2\alpha} x}}. \tag{34}$$

The proof of Theorem 5 is thus completed.

As a consequence of Theorem 5, we immediately obtain the following refinement of the Sándor-Bencze conjectured inequality (32):

Corollary 6. *If*

$0 < x < \frac{\pi}{2} \quad and \quad \alpha > 0$, then

$$\left(\frac{\sin x}{x}\right)^{\alpha} > \frac{4\cos^{\alpha} x}{1+\sqrt{1+8\cos^{2\alpha} x}} > \frac{2\cos^{\alpha} x}{1+\cos^{\alpha} x} > \frac{\cos^{\alpha} x}{1+\cos^{\alpha} x}. \tag{35}$$

In addition, upon replacing α by $-\alpha$ in Theorem 5, a *reversed* version of the Sándor-Bencze conjectured inequality (32) is derived as follows.

Corollary 7. *If*

$0 < x < \frac{\pi}{2} \quad and \quad \alpha \geqq 1$, then

$$\left(\frac{\sin x}{x}\right)^{\alpha} < \frac{\cos^{\alpha} x+\sqrt{\cos^{2\alpha} x+8}}{4}. \tag{36}$$

Remark 6.

Corollary 6 and Corollary 7 show that the inequality (12) is sharper and more general than the Sándor-Bencze conjectured inequality (32).

We conclude this presentation by remarking further that, by making use of the definitions and concepts of *Time Scale Theory* (see, for details, [26] and the many references to relevant earlier works cited therein), one can investigate the interesting possibility of extending some of the inequalities in this work to hold true on time scales. For various further developments, some of which are based also upon the results surveyed in this article, the interested reader may refer to many recent investigations including (for example), [2], [15], [16], [17], [18] and [36].

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