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A New Class of Laguerre-Based Poly-Euler and Multi Poly-Euler Polynomials

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Abstract: In this paper, we introduce a new class of Laguerre Poly-Euler and Laguerre multi Poly-Euler polynomials. The concept of Poly-Euler number $E_n^{(k)}(a,b)$, generalized Poly-Euler polynomials $E_n^{(k)}(x;a,b,e)$ of Jolany et al., Hermite-Bernoulli polynomials ${}_HB_n(x,y)$ of Dattoli et al., ${}_HB_n^{(\alpha)}(x,y)$ of Pathan and Khan and Hermite based Poly-Euler polynomials ${}_HE_n^{(k)}(x,y;a,b,e)$ of Khan are generalized to the one ${}_LE_n^{(k)}(x,y,z;a,b,e)$. Some implicit summation formulae and general symmetry identities arising from different analytical means and applying generating functions.

Keywords: Laguerre polynomials, Hermite polynomials, Poly-Euler polynomials, Laguerre Poly-Euler polynomials, multi Poly-Euler polynomials, Laguerre multi Poly-Euler polynomials, summation formulae, symmetric identities.

1 Introduction

The two variable Laguerre polynomials $L_n(x,y)$ are defined by the generating function [5]

$$\sum_{n=0}^{\infty} L_n(x,y)\frac{t^n}{n!} = \exp yt C_0(xt), \tag{1.1}$$

where $C_0(x)$ is the 0-th order Tricomi function [2]

$$C_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{(r!)^2} \tag{1.2}$$

and are represented by the series

$$L_n(x,y) = \sum_{s=0}^{n} \frac{n!(-1)^s y^{n-s} x^s}{(n-s)!(s!)^2} \tag{1.3}$$

Recently, the generalized Poly-Euler polynomials are defined by Jolany et al. [6]-[9] as follows:

$$\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^t} e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x;a,b,e)\frac{t^n}{n!}, \tag{1.4}$$

$$|t| < \frac{2\pi}{|\ln a + \ln b|}$$

Note that the Poly-Euler polynomials of Sasaki and Bayad ([1],[14]) can be deduced from (1.4) by replacing t with $4t$ and taking $x = \frac{1}{2}$. when $x = 0$, (1.4) gives

$$E_n^{(k)}(0;a,b,e) = E_n^{(k)}(a,b)$$

$$\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^t} = \sum_{n=0}^{\infty} E_n^{(k)}(a,b)\frac{t^n}{n!}, \tag{1.5}$$

$$|t| < \frac{2\pi}{|\ln a + \ln b|}$$

and when $a = 1$ and $b = e$, we get

$$E_n^{(k)}(x;1,e,e) = E_n^{(k)}(x)$$

where

$$\frac{2Li_k(1-e^{-t})}{1+e^t} e^{xt} = \sum_{n=0}^{\infty} E_n^{(k)}(x)\frac{t^n}{n!}, \tag{1.6}$$

$$|t| < \frac{2\pi}{|\ln a + \ln b|}$$

On the other hand in the same paper by Jolany et al. [6]-[9], they defined certain multi Poly-Euler polynomials as

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follows

$$\frac{2Li_{k_1,\cdots,k_r}(1-(ab)^{-t})}{(a^{-t}+b^t)^r}e^{rxt}=\sum_{n=0}^{\infty}E_n^{(k_1,\cdots,k_r)}(x;a,b,e)\frac{t^n}{n!}, \tag{1.7}$$

$$|t|<\frac{2\pi}{|\ln a+\ln b|}$$

where

$$Li_{(k_1,\cdots,k_r)}(z)=\sum_{r,k=1}^{\infty}\frac{z^{mr}}{m_1^{k_1}\cdots m_r^{k_r}}$$

is the generalization of poly-logarithm.

In particular

$$E_n^{(k_1\cdots k_r)}(x;1,e,e)=E_n^{(k_1\cdots k_r)}(x)$$

$$E_n^{(k_1\cdots k_r)}(0;a,b,e)=E_n^{(k_1\cdots k_r)}(a,b)$$

Further by taking $r=1$, in (1.7) immediately yield (1.4).

Very recently, Pathan et al. [15]-[21] introduced the generalized Hermite-Bernoulli polynomials of two variables ${}_HB_n^{(\alpha)}(x,y)$ defined as

$$\left(\frac{t}{e^t-1}\right)^{\alpha}e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HB_n^{(\alpha)}(x,y)\frac{t^n}{n!} \tag{1.8}$$

which are essentially generalization of Bernoulli numbers, Bernoulli polynomials, Hermite polynomials and Hermite-Bernoulli polynomials ${}_HB_n(x,y)$ introduced by Dattoli et al. ([4], p.386 (1.6)) in the form

$$\left(\frac{t}{e^t-1}\right)e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HB_n(x,y)\frac{t^n}{n!} \tag{1.9}$$

The 2-variable Kampe de Feriet generalization of the Hermite polynomials [3] reads

$$H_n(x,y)=n!\sum_{r=0}^{[\frac{n}{2}]}\frac{y^rx^{n-2r}}{r!(n-2r)!} \tag{1.10}$$

These polynomials are usually defined by the generating function

$$e^{xt+yt^2}=\sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!} \tag{1.11}$$

and reduce to the ordinary Hermite polynomials $H_n(x)$ when $y=-1$ and x is replaced by $2x$.

Motivated by their importance and potential for applications in certain problems in number theory, combinatorics, classical and numerical analysis and other fields of applied mathematics, several kinds of some special numbers and polynomials were recently studied by many authors (see [1]-[24]).

In this note firstly, we will give the definition of the Laguerre Poly-Euler polynomials ${}_LE_n^{(k)}(x,y,z;a,b,e)$ and Laguerre multi Poly-Euler polynomials ${}_LE_n^{(k_1\cdots k_r)}(x,y,z;a,b,e)$ which generalize the concept stated above and then investigate their basic properties and relationships with Poly-Euler numbers $E_n^{(k)}(a,b)$, Poly-Euler polynomials $E_n^{(k)}(x)$, generalized Poly-Euler polynomials $E_n^{(k)}(x;a,b,e)$ of Jolany et al., Hermite-Bernoulli polynomials ${}_HB_n(x,y)$ of Dattoli et at., ${}_HB_n^{(\alpha)}(x,y)$ of Pathan and Khan and Hermite poly-Euler polynomials ${}_HE_n^{(k)}(x,y;a,b,e)$ of Khan. The reminder of this paper is organized as follows: We modify generating functions for the Poly-Euler polynomials and derive some identities related to Laguerre polynomials, Hermite polynomials, Poly-Euler polynomials and power sums. Some implicit summation formulae and general symmetry identities are derived by using different analytical means and applying generating functions. These result extended some known summations and identities of generalized Hermite-Bernoulli polynomials studied by Dattoli et al., Hermite Poly-Euler polynomials studied by Khan, Zhang et al., Yang, Pathan and Pathan and Khan.

2 Definition and Properties of the Laguerre Poly-Euler polynomials and Laguerre multi Poly-Euler polynomials

In this section, we will establish definitions and properties of Laguerre Poly-Euler polynomials ${}_LE_n^{(k)}(x,y,z;a,b,e)$ and Laguerre multi Poly-Euler polynomials ${}_LE_n^{(k_1,\cdots,k_r)}(x,y,z;a,b,e)$.

Definition 2.1. Let $a,b>0$ and $a\neq b$. The Laguerre Poly-Euler polynomials ${}_LE_n^{(k)}(x,y,z;a,b,e)$ for a nonnegative integer n is defined by

$$\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^t}e^{yt+zt^2}C_0(xt)$$

$$=\sum_{n=0}^{\infty}{}_LE_n^{(k)}(x,y,z;a,b,e)\frac{t^n}{n!},\quad |t|<\frac{2\pi}{|\ln a+\ln b|} \tag{2.1}$$

For $x=0$ in (2.1), the result reduces to the known result of Khan [11].

$$\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^t}e^{yt+zt^2}$$

$$=\sum_{n=0}^{\infty}{}_HE_n^{(k)}(y,z;a,b,e)\frac{t^n}{n!},\quad |t|<\frac{2\pi}{|\ln a+\ln b|} \tag{2.2}$$

As in the case $x=y=z=0$ and $e=1$ in (2.1), it leads to an extension of the generalized poly-Euler polynomials denoted by $E_n^{(k)}(a,b)$ for a nonnegative integer n, defined earlier by (1.5).

Definition 2.2. Let $a,b>0$ and $a\neq b$. The Laguerre multi Poly-Euler polynomials ${}_LE_n^{(k_1,\cdots,k_r)}(x,y,z;a,b,e)$ for a nonnegative integer n, is defined by

$$\frac{2Li_k(1-(ab)^{-t})}{(a^{-t}+b^t)^r}e^{r(yt+zt^2)}C_0(rxt)$$

$$=\sum_{n=0}^{\infty}{}_LE_n^{(k_1,\cdots,k_r)}(x,y,z;a,b,e)\frac{t^n}{n!},\ |t|<\frac{2\pi}{|\ln a+\ln b|} \tag{2.3}$$

For $x=0$ in (2.3), the result reduces to the known result of Khan [11].

$$\frac{2Li_k(1-(ab)^{-t})}{(a^{-t}+b^t)^r}e^{r(yt+zt^2)}$$

$$=\sum_{n=0}^{\infty}{}_HE_n^{(k_1,\cdots,k_r)}(y,z;a,b,e)\frac{t^n}{n!},\quad |t|<\frac{2\pi}{|\ln a+\ln b|} \tag{2.4}$$

As in the case $x=y=z=0$ and $e=1$ in (2.3), it leads to an extension of the generalized multi Poly-Euler polynomials denoted by $E_n^{(k,\cdots,k_r)}(a,b)$ for a nonnegative integer n, defined earlier by (1.7).

Theorem 2.1. Let $a,b>0$ and $a\neq b$. For $x,y,z\in R$ and $n\geq 0$. Then we have

$${}_LE_n^{(k)}(x,y,z;1,e,e)={}_LE_n^{(k)}(x,y,z),$$

$${}_LE_n^{(k)}(0,0,0;a,b,1)=E_n^{(k)}(a,b)$$

$${}_LE_n^{(k)}(0,0,0;1,e,1)=E_n^{(k)},$$

$${}_LE_n^{(k)}(x,y,z;a,b,e)={}_LE_n^{(k)}(x,y,z;a,b) \tag{2.5}$$

$${}_LE_n^{(k)}(x,y+z,v+u;a,b,e)$$

$$=\sum_{m=0}^{n}\binom{n}{m}{}_LE_{n-m}^{(k)}(x,z,v;a,b,e)H_m(y,u;a,b,e) \tag{2.6}$$

$${}_LE_n^{(k)}(x,y+v,z;a,b,e)=\sum_{m=0}^{n}\binom{n}{m}v^m{}_LE_{n-m}^{(k)}(x,y,z;a,b,e) \tag{2.7}$$

Proof. The formula in (2.6) are obvious. Applying definition (2.1), we have

$$\sum_{n=0}^{\infty}{}_LE_n^{(k)}(x,y+z,v+u;a,b,e)\frac{t^n}{n!}$$

$$=\left(\sum_{n=0}^{\infty}{}_LE_n^{(k)}(x,z,v;a,b,e)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}H_m(y,u)\frac{t^m}{m!}\right)$$

$$\sum_{n=0}^{\infty}{}_LE_n^{(k)}(x,y+z,v+u;a,b,e)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}{}_LE_{n-m}^{(k)}(x,z,v;a,b,e)H_m(x,y)\right)\frac{t^n}{n!}$$

Now equating the coefficient of $\frac{t^n}{n!}$ in the above equation, we get the result (2.6). Again by definition (2.1) of Laguerre Poly-Euler polynomials, we have

$$\sum_{n=0}^{\infty}{}_LE_n^{(k)}(x,y+v,z;a,b,e)\frac{t^n}{n!}$$

$$=\left(\frac{2Li_k(1-(ab)^{-1})}{a^{-t}+b^t}\right)e^{(y+v)t+zt^2}C_0(xt)$$

$$\sum_{n=0}^{\infty}{}_LE_n^{(k)}(x,y+v,z;a,b,e)\frac{t^n}{n!}$$

$$=\left(\frac{2Li_k(1-(ab)^{-1})}{a^{-t}+b^t}\ e^{yt+zt^2}C_0(xt)\right)e^{vt}$$

which can be written as $\sum_{n=0}^{\infty}{}_LE_n^{(k)}(x,y+v,z;a,b,e)\frac{t^n}{n!}=\sum_{n=0}^{\infty}{}_LE_n^{(k)}(x,y,z;a,b,e)\frac{t^n}{n!}\sum_{m=0}^{\infty}\frac{(vt)^m}{m!}$ $\sum_{n=0}^{\infty}{}_LE_n^{(k)}(x,y+v,z;a,b,e)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}v^m{}_LE_{n-m}^{(k)}(x,y,z;a,b,e)\right)\frac{t^n}{n!}$

On equating the coefficient of the like power of $\frac{t^n}{n!}$ in the above equation, we get the result (2.7). Hence we complete the proof of theorem.

Theorem 2.2. The Laguerre multi Poly-Euler polynomials satisfy the following relation:

$${}_LE_n^{(k_1,\cdots,k_r)}(x,y+z,u;a,b,e)$$

$$=\sum_{m=0}^{n}\binom{n}{m}(rz)^m{}_LE_{n-m}^{(k_1,\cdots,k_r)}(x,y,u;a,b,e) \tag{2.8}$$

Proof. Since

$$\sum_{n=0}^{\infty}{}_LE_n^{(k_1,\cdots,k_r)}(x,y+z,u;a,b,e)\frac{t^n}{n!}$$

$$=\frac{2Li_k(1-(ab)^{-t})}{(a^{-t}+b^t)^r}e^{r((y+z)t+ut^2)}C_0(rxt)$$

$$\sum_{n=0}^{\infty}{}_LE_n^{(k_1,\cdots,k_r)}(x,y+z,u;a,b,e)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}{}_LE_n^{(k_1,\cdots,k_r)}(x,y+z,u;a,b,e)\frac{t^n}{n!}\sum_{m=0}^{\infty}\frac{(rzt)^m}{m!}$$

Replacing n by $n-m$ in the above equation and equating the coefficients of t^n, we get the result (2.8).

Theorem 2.3. The Laguerre multi Poly-Euler polynomials satisfy the following relation:

$${}_LE_n^{(k_1,\cdots,k_r)}(x,y,z;a,b,e)$$

$$=\sum_{m=0}^{[\frac{n}{2}]}\sum_{k=0}^{n-2m}\frac{(-1)^k(r)^{k+m}x^kz^mE_{n-k-2m}^{(k_1,\cdots,k_r)}(y;a,b,e)}{(n-k-2m)!(k!)^2!m!} \tag{2.9}$$

$$\sum_{n=0}^{\infty}{}_LE_n^{(k_1,\cdots,k_r)}(x,y,z;a,b,e)\frac{t^n}{n!}$$

$$= \frac{2Li_k(1-(ab)^{-t})}{(a^{-t}+b^t)^r}\, e^{r(yt+zt^2)} C_0(rxt)$$

$$\sum_{n=0}^{\infty} {}_LE_n^{(k_1,\cdots,k_r)}(x,y,z;a,b,e)\frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} E_n^{(k_1,\cdots,k_r)}(y;a,b,e)\frac{t^n}{n!}\right)$$

$$\times\left(\sum_{m=0}^{\infty}\frac{(rzt^2)^m}{m!}\right)\left(\sum_{k=0}^{\infty}\frac{(-1)^k(rxt)^k}{(k!)^2}\right)$$

Replacing n by $n-k$, we get

$$\sum_{n=0}^{\infty} {}_LE_n^{(k_1,\cdots,k_r)}(x,y,z;a,b,e)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{(-1)^k(rx)^k E_{n-k}^{(k_1,\cdots,k_r)}(y;a,b,e)}{(n-k)!(k!)^2}\right) t^n \left(\sum_{m=0}^{\infty}\frac{(rzt^2)^m}{m!}\right)$$

Replacing n by $n-2m$ in the above equation and equating the coefficients of t^n, we get the result (2.9).

3 Implicit Summation Formulae Involving Laguerre Poly-Euler Polynomials

For the derivation of implicit summation formulae involving Laguerre Poly-Euler polynomials ${}_LE_n^{(k)}(x,y,z;a,b,e)$ the same consideration as developed for the ordinary Hermite and related polynomials in Khan et al. [10] and Hermite-Bernoulli polynomials in Pathan [15] and Pathan et al. [16]-[21] holds as well. First we prove the following results involving Laguerre Poly-Euler polynomials ${}_LE_n^{(k)}(x,y,z;a,b,e)$.

Theorem 3.1. Let $a,b>0$ and $a\neq b$. Then, for $x,y,z\in R$ and $m,n\geq 0$, the following implicit summation formula for Laguerre Poly-Euler polynomials ${}_LE_n^{(k)}(x,y,z;a,b,e)$ holds true:

$${}_LE_{m+n}^{(k)}(x,v,z;a,b,e) = \sum_{s,k=0}^{m,n}\binom{m}{s}\binom{n}{k}(v-y)^{s+k}$$

$$\times {}_LE_{m+n-s-k}^{(k)}(x,y,z;a,b,e) \tag{3.1}$$

Proof. We replace t by $t+u$ and rewrite the generating function (2.1) as

$$\left(\frac{2Li_k(1-(ab)^{-(t+u)})}{a^{-(t+u)}+b^{(t+u)}}\right) e^{z(t+u)^2} C_0(x(t+u))$$

$$= e^{-y(t+u)}\sum_{m,n=0}^{\infty} {}_LE_{m+n}^{(k)}(x,y,z;a,b,e)\frac{t^n}{n!}\frac{u^m}{m!} \tag{3.2}$$

Replacing y by v in the above equation and equating the resulting equation to the above equation, we get

$$e^{(v-y)(t+u)}\sum_{m,n=0}^{\infty} {}_LE_{m+n}^{(k)}(x,y,z;a,b,e)\frac{t^n}{n!}\frac{u^m}{m!}$$

$$= \sum_{m,n=0}^{\infty} {}_LE_{m+n}^{(k)}(x,v,z;a,b,e)\frac{t^n}{n!}\frac{u^m}{m!} \tag{3.3}$$

on expanding exponential function (3.3) gives

$$\sum_{N=0}^{\infty}\frac{[(v-y)(t+u)]^N}{N!}\sum_{m,n=0}^{\infty} {}_LE_{m+n}^{(k)}(x,y,z;a,b,e)\frac{t^n}{n!}\frac{u^m}{m!}$$

$$= \sum_{m,n=0}^{\infty} {}_LE_{m+n}^{(k)}(x,v,z;a,b,e)\frac{t^n}{n!}\frac{u^m}{m!} \tag{3.4}$$

which on using formula [[22], p. 52(2)]

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(m+n)\frac{x^n}{n!}\frac{y^m}{m!} \tag{3.5}$$

in the left hand side becomes

$$\sum_{k,s=0}^{\infty}\frac{(v-y)^{k+s}\,t^k u^s}{k!s!}\sum_{m,n=0}^{\infty} {}_LE_{m+n}^{(k)}(x,y,z;a,b,e)\frac{t^n}{n!}\frac{u^m}{m!}$$

$$= \sum_{m,n=0}^{\infty} {}_LE_{m+n}^{(k)}(x,v,z;a,b,e)\frac{t^n}{n!}\frac{u^m}{m!} \tag{3.6}$$

Now replacing n by $n-k$, s by $n-s$ and using the lemma [[22], p. 100(1)] in the left hand side of (3.6), we get $\sum_{m,n=0}^{\infty}\sum_{k,s=0}^{\infty}\frac{(v-y)^{k+s}}{k!s!}\,{}_LE_{m+n-k-s}^{(k)}(x,y,z;a,b,e)\frac{t^n}{(n-k)!}\frac{u^m}{(m-s)!}$

$$= \sum_{m,n=0}^{\infty} {}_LE_{m+n}^{(k)}(x,v,z;a,b,e)\frac{t^n}{n!}\frac{u^m}{m!} \tag{3.7}$$

Finally, on equating the coefficient of the like powers of t^n and u^m in the above equation, we get the required result.

Remark 1. By taking $m=0$ in equation (3.1), we immediately deduce the following result.

Corollary 3.1. The following implicit summation formula for Laguerre Poly-Euler polynomials ${}_LE_n^{(k)}(x,y,z;a,b,e)$ holds true:

$${}_LE_n^{(k)}(x,v,z;a,b,e) = \sum_{k=0}^{n}\binom{n}{k}(v-y)^k\,{}_LE_{n-k}^{(k)}(x,y,z;a,b,e) \tag{3.8}$$

Remark 2. On replacing v by $v+y$ and setting $x=z=0$ in Theorem (3.1), we get the following result involving Laguerre Poly-Euler polynomial of one variable

$${}_LE_{m+n}^{(k)}(v+y;a,b,e) = \sum_{s,k=0}^{m,n}\binom{m}{s}\binom{n}{k}(v)^{k+s}$$

$$\times {}_LE_{m+n-k-s}^{(k)}(y;a,b,e) \tag{3.9}$$

whereas by setting $v=0$ in Theorem (3.1), we get another result involving Laguerre Poly-Euler polynomial of one and two variable

$${}_LE_{m+n}^{(k)}(x,z;a,b,e) = \sum_{s,k=0}^{m,n}\binom{m}{s}\binom{n}{k}(-y)^{k+s}$$

$$\times {}_{L}E^{(k)}_{m+n-k-s}(x,y,z;a,b,e) \tag{3.10}$$

Remark 3. Along with the above result we will exploit extended forms of Laguerre Poly-Euler polynomial ${}_{L}E^{(k)}_{m+n}(x,v;a,b,e)$ by setting $z=0$ in the Theorem (3.1) to get

$${}_{L}E^{(k)}_{m+n}(x,v;a,b,e) = \sum_{k,s=0}^{m,n} \binom{m}{s}\binom{n}{k}(v-y)^{k+s}$$

$$\times {}_{H}E^{(k)}_{m+n-k-s}(x,y;a,b,e) \tag{3.11}$$

Remark 4. A straight forward expression of the ${}_{L}E_{m+n}(x,v,z;a,b,e)$ is suggested by a special case of the Theorem (3.1) for $k=1$ in the following form

$${}_{L}E_{m+n}(x,v,z;a,b,e) = \sum_{s,k=0}^{m,n} \binom{m}{s}\binom{n}{k}(v-y)^{s+k}$$

$$\times {}_{L}E_{m+n-s-k}(x,y,z;a,b,e) \tag{3.12}$$

Theorem 3.2. Let $a,b>0$ and $a\neq b$. Then, for $x,y,z\in R$ and $m,n\geq 0$, the following implicit summation formula for Laguerre Poly-Euler polynomials ${}_{L}E^{(k)}_{n}(x,y,z;a,b,e)$ holds true:

$${}_{L}E^{(k)}_{n}(x,y+u,z;a,b,e) = \sum_{j=0}^{n}\binom{n}{j} u^{j}\,{}_{L}E^{(k)}_{n-j}(x,y,z;a,b,e) \tag{3.13}$$

Proof. Since

$$\sum_{n=0}^{\infty} {}_{L}E^{(k)}_{n}(x,y+u,z;a,b,e)\frac{t^n}{n!}$$

$$= \frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^{t}}\, e^{(y+u)t+zt^2} C_0(xt)$$

$$\sum_{n=0}^{\infty} {}_{L}E^{(k)}_{n}(x,y+u,z;a,b,e)\frac{t^n}{n!}$$

$$= \left(\sum_{n=0}^{\infty} {}_{L}E^{(k)}_{n}(x,y,z;a,b,e)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty} u^{j}\frac{t^j}{j!}\right)$$

Now, replacing n by $n-j$ and comparing the coefficient of t^n, we get the result (3.13).

Theorem 3.3. Let $a,b>0$ and $a\neq b$. Then, for $x,y,z\in R$ and $m,n\geq 0$, the following implicit summation formula for Laguerre Poly-Euler polynomials ${}_{L}E^{(k)}_{n}(x,y,z;a,b,e)$ holds true:

$${}_{L}E^{(k)}_{n}(x,y+u,z+w;a,b,e)$$

$$= \sum_{m=0}^{n}\binom{n}{m}\,{}_{L}E^{(k)}_{n-m}(x,y,z;a,b,e)H_m(u,w) \tag{3.14}$$

Proof. By the definition of Laguerre Poly-Euler polynomials and the definition (1.11), we have

$$\left(\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^{t}}\right) e^{(y+u)t+(z+w)t^2} C_0(xt)$$

$$= \left(\sum_{n=0}^{\infty} {}_{L}E^{(k)}_{n}(x,y,z)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} H_m(u,w)\frac{t^m}{m!}\right)$$

Now, replacing n by $n-m$ and comparing the coefficient of t^n, we get the result (3.14).

Theorem 3.4. Let $a,b>0$ and $a\neq b$. Then, for $x,y,z\in R$ and $m,n\geq 0$, the following implicit summation formula for Laguerre Poly-Euler polynomials ${}_{L}E^{(k)}_{n}(x,y,z;a,b,e)$ holds true:

$${}_{L}E^{(k)}_{n}(x,y,z;a,b,e) = \sum_{m=0}^{n-2j}\sum_{j=0}^{[\frac{n}{2}]} \frac{E^{(k)}_{m}(a,b)L_{n-m-2j}(x,y)z^{j}n!}{m!\,j!\,(n-m-2j)!} \tag{3.15}$$

Proof. Applying the definition (2.1) to the term $\left(\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^{t}}\right)$ and expanding the exponential and tricomi function $e^{yt+zt^2}C_0(xt)$ at $t=0$ yields

$$\left(\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^{t}}\right) e^{yt+zt^2} C_0(xt)$$

$$= \left(\sum_{m=0}^{\infty} E^{(k)}_{m}(a,b)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty} L_n(x,y)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty} z^{j}\frac{(t)^{2j}}{j!}\right)$$

$$\sum_{n=0}^{\infty} {}_{L}E^{(k)}_{n}(x,y,z;a,b,e)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} E^{(k)}_{m}(a,b)L_{n-m}(x,y)\right)\frac{t^n}{n!}\left(\sum_{j=0}^{\infty} z^{j}\frac{t^{2j}}{j!}\right)$$

Now, replacing n by $n-2j$ and comparing the coefficient of t^n, we get the result (3.15).

Theorem 3.5. Let $a,b>0$ and $a\neq b$. Then, for $x,y,z\in R$ and $m,n\geq 0$, the following implicit summation formula for Laguerre Poly-Euler polynomials ${}_{L}E^{(k)}_{n}(x,y,z;a,b,e)$ holds true:

$${}_{L}E^{(k)}_{n}(x,y+1,z;a,b,e) = \sum_{m,j=0}^{n} \frac{n!(-1)^{j}(x)^{j}\,{}_{H}E^{(k)}_{n-m-j}(y,z;a,b,e)}{(n-m-j)!m!(j!)^2} \tag{3.16}$$

Proof. By the definition of Laguerre Poly-Euler polynomials, we have

$$\sum_{n=0}^{\infty} {}_{L}E^{(k)}_{n}(x,y+1,z;a,b,e)\frac{t^n}{n!}$$

$$= \left(\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^{t}}\right) e^{(y+1)t+zt^2} C_0(xt)$$

$$= \left(\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{{}_{H}E_{n-m}^{(k)}(y,z;a,b,e)}{(n-m)!n!}\right)t^{n}\right)\left(\sum_{j=0}^{\infty}\frac{(-1)^{j}(xt)^{j}}{(j!)^{2}}\right)$$

$$= \left(\sum_{n=0}^{\infty}\left(\sum_{j=0}^{\infty}\sum_{m=0}^{n}\frac{(-1)^{j}(x)^{j}{}_{H}E_{n-m}^{(k)}(y,z;a,b,e)}{(n-m)!n!(j!)^{2}}\right)t^{n+j}\right)$$

Replacing n by $n-j$, we have

$$\sum_{n=0}^{\infty}{}_{L}E_{n}^{(k)}(x,y+1,z;a,b,e)\frac{t^{n}}{n!}$$

$$= \left(\sum_{n=0}^{\infty}\left(\sum_{m,j=0}^{n}\frac{(-1)^{j}(x)^{j}{}_{H}E_{n-m}^{(k)}(y,z;a,b,e)}{(n-m)!n!(j!)^{2}}\right)t^{n+j}\right)$$

On comparing the coefficient of t^n, we get the result (3.16).

Theorem 3.6. Let $a,b>0$ and $a\neq b$. Then, for $x,y,z\in R$ and $m,n\geq 0$, the following implicit summation formula for Laguerre Poly-Euler polynomials ${}_{L}E_{n}^{(k)}(x,y,z;a,b,e)$ holds true:

$$ {}_{L}E_{n}^{(k)}(x,y+1,z;a,b,e)=\sum_{m=0}^{n}\binom{n}{m}{}_{L}E_{n-m}^{(k)}(x,y,z;a,b,e) \tag{3.17}$$

Proof. By the definition of Laguerre Poly-Euler polynomials, we have

$$\sum_{n=0}^{\infty}{}_{L}E_{n}^{(k)}(x,y+1,z;a,b,e)\frac{t^{n}}{n!}-\sum_{n=0}^{\infty}{}_{L}E_{n}^{(k)}(x,y,z;a,b,e)\frac{t^{n}}{n!}$$

$$=\left(\frac{2Li_{k}(1-(ab)^{-t})}{a^{-t}+b^{t}}\right)(e^{t}-1)e^{yt+zt^{2}}C_{0}(xt)$$

$$=\sum_{n=0}^{\infty}{}_{L}E_{n}^{(k)}(x,y,z;a,b,e)\frac{t^{n}}{n!}\left(\sum_{m=0}^{\infty}\frac{t^{m}}{m!}-1\right)$$

$$=\sum_{n=0}^{\infty}{}_{L}E_{n}^{(k)}(x,y,z;a,b,e)\frac{t^{n}}{n!}\sum_{m=0}^{\infty}\frac{t^{m}}{m!}-\sum_{n=0}^{\infty}{}_{L}E_{n}^{(k)}(x,y,z;a,b,e)\frac{t^{n}}{n!}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}{}_{L}E_{n-m}^{(k)}(x,y,z;a,b,e)\frac{t^{n}}{m!(n-m)!}$$

$$-\sum_{n=0}^{\infty}{}_{L}E_{n}^{(k)}(x,y,z;a,b,e)\frac{t^{n}}{n!}$$

Finally equating the coefficient of the like powers of t^n in the above equation, we get the result (3.17).

4 General Symmetry Identities for Laguerre Poly-Euler Polynomials

In this section, we will obtain general symmetry identities for the Laguerre Poly-Euler polynomials ${}_{L}E_{n}^{(k)}(x,y,z;a,b,e)$ by applying the generating function (2.1). It turns out that some known identities of Khan [11]-[13], Pathan et al. [15]-[21], Yang et al. [23], Zhang et al. [24].

Theorem 4.1. Let $a,b>0$ and $a\neq b$. Then, for $x,y,z\in R$ and $m,n\geq 0$. Then the following identity holds true:

$$\sum_{m=0}^{n}\binom{n}{m}b^{m}a^{n-m}{}_{E}G_{n-m}^{(k)}(x,by,b^{2}z;b,e){}_{L}E_{m}^{(k)}(x,ay,a^{2}z;a,e)$$

$$=\sum_{m=0}^{n}\binom{n}{m}a^{m}b^{n-m}{}_{L}E_{n-m}^{(k)}(x,ay,a^{2}z;a,e){}_{L}E_{m}^{(k)}(x,by,b^{2}z;b,e) \tag{4.1}$$

Proof. Start with

$$g(t)=\left(\frac{(2Li_{k}(1-(ab)^{-t})C_{0}(xt))^{2}}{(a^{-at}+b^{at})(a^{-bt}+b^{bt})}\right)e^{abyt+a^{2}b^{2}zt^{2}} \tag{4.2}$$

Then the expression for $g(t)$ is symmetric in a and b and we can expand $g(t)$ into series in two ways to obtain $g(t)=\sum_{n=0}^{\infty}{}_{L}E_{n}^{(k)}(x,by,b^{2}z;b,e)\frac{(at)^{n}}{n!}\sum_{m=0}^{\infty}{}_{L}E_{m}^{(k)}(x,ay,a^{2}z;a,e)\frac{(bt)^{m}}{m!}$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^{m}{}_{L}E_{m}^{(k)}(x,by,b^{2}z;b,e;)$$

$$\times{}_{L}E_{n-m}^{(k)}(x,ay,a^{2}z;a,e;)\frac{t^{n}}{n!}$$

On the similar lines we can show that $g(t)=\sum_{n=0}^{\infty}{}_{L}E_{n}^{(k)}(x,ay,a^{2}z;a,e)\frac{(bt)^{n}}{n!}\sum_{m=0}^{\infty}{}_{L}E_{m}^{(k)}(x,by,b^{2}z;b,e)\frac{(at)^{m}}{m!}$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}a^{m}b^{n-m}{}_{L}E_{n-m}^{(k)}(x,ay,a^{2}z;a,e)$$

$$\times{}_{L}E_{m}^{(k)}(x,by,b^{2}z;b,e)\frac{t^{n}}{n!}$$

Comparing the coefficient of t^n on the right hand sides of the last two equations, we arrive at the desired result.

Remark 1. By setting $b=1$ in Theorem (4.1), we immediately following result

$$\sum_{m=0}^{n}\binom{n}{m}a^{n-m}{}_{L}E_{n-m}^{(k)}(x,y,z;1,e){}_{L}E_{m}^{(k)}(x,ay,a^{2}z;a,e)$$

$$=\sum_{m=0}^{n}\binom{n}{m}a^{m}{}_{L}E_{n-m}^{(k)}(x,ay,a^{2}z;a,e){}_{L}E_{m}^{(k)}(x,y,z;1,e) \tag{4.3}$$

Theorem 4.2. Let $a,b>0$ and $a\neq b$. Then, for $x,y,z\in R$ and $m,n\geq 0$. Then the following identity holds true:

$$\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}b^m a^{n-m}{}_LE_{n-m}^{(k)}\left(x,by+\frac{b}{a}i+j,b^2u;A,B,e\right)$$

$$\times{}_LE_m^{(k)}(x,az,a^2v;A,B,e)$$

$$=\textstyle\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}a^m b^{n-m}{}_LE_{n-m}^{(k)}\left(x,ay+\frac{a}{b}i+j,a^2u;A,B,e\right)$$

$$\times{}_LE_m^{(k)}(x,bz,b^2v;A,B,e) \tag{4.4}$$

Proof. Let

$$g(t)=\left(\frac{(2Li_k(1-(ab)^{-t})C_0(xt))^2}{(A^{-at}+B^{at})(A^{-bt}+B^{bt})}\right)$$

$$\times\left(\frac{(e^{abt}-1)^2e^{ab(y+z)t+a^2b^2(u+v)t^2}}{(e^{at}-1)(e^{bt}-1)}\right)$$

$$=\left(\frac{2Li_k(1-(ab)^{-t})C_0(xt)}{(A^{-at}+B^{at})}\right)e^{abyt+a^2b^2ut^2}\left(\frac{e^{abt}-1}{e^{bt}-1}\right)$$

$$\times\left(\frac{2Li_k(1-(ab)^{-t})C_0(xt)}{(A^{-bt}+B^{bt})}\right)e^{abzt+a^2b^2vt^2}\left(\frac{e^{abt}-1}{e^{at}-1}\right)$$

$$=\left(\frac{2Li_k(1-(ab)^{-t})C_0(xt)}{(A^{-at}+B^{at})}\right)e^{abyt+a^2b^2ut^2}$$

$$\times\sum_{i=0}^{a-1}e^{bti}\left(\frac{2Li_k(1-(ab)^{-t})C_0(xt)}{(A^{bt}+B^{bt})}\right)e^{abzt+a^2b^2vt^2}\sum_{j=0}^{b-1}e^{atj} \tag{4.5}$$

$$=\left(\frac{2Li_k(1-(ab)^{-t})C_0(xt)}{(A^{-at}+B^{at})}\right)e^{a^2b^2ut^2}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}e^{(by+\frac{b}{a}i+j)at}$$

$$\times\sum_{m=0}^{\infty}{}_LE_m^{(k)}(x,az,a^2v;A,B,e)\frac{(bt)^m}{m!}$$

$$=\sum_{n=0}^{\infty}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}{}_LE_{n-m}^{(k)}\left(x,by+\frac{b}{a}i+j,b^2u:A,B,e\right)\frac{(at)^n}{n!}$$

$$\times\sum_{m=0}^{\infty}{}_LE_m^{(k)}(x,az,a^2v;A,B,e)\frac{(bt)^m}{m!}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}{}_LE_{n-m}^{(k)}\left(x,by+\frac{b}{a}i+j,b^2u;A,B,e\right)$$

$$\times\sum_{m=0}^{\infty}{}_LE_m^{(k)}(x,az,a^2v;A,B,e)b^m a^{n-m}t^n \tag{4.6}$$

On the other hand

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}{}_LE_{n-m}^{(k)}\left(x,ay+\frac{a}{b}i+j,a^2u;A,B,e\right)$$

$$\times\sum_{m=0}^{\infty}{}_LE_m^{(k)}(x,bz,b^2v;A,B,e)b^{n-m}a^m t^n \tag{4.7}$$

By comparing the coefficients of t^n on the right hand sides of the last two equations, we arrive at the desired result.

5 Conclusion

In this paper, we modify generating functions for the Poly-Euler and multi Poly-Euler polynomials and derive some identities related to Laguerre polynomials, Hermite polynomials, Poly-Euler polynomials and power sums. These generating functions have wide applications in certain problems in number theory, combinatorics, classical and numerical analysis and other fields of applied mathematics, several kinds of special numbers and polynomials.

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