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Transmuted Exponentiated Inverse Rayleigh Distribution

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Abstract: A generalized form of Inverse Rayleigh distribution was derived by using quadratic rank transmutation map (QTRM), known as Transmuted Exponentiated Inverse Rayleigh (TEIR) distribution. A comprehensive account of the mathematical properties of TEIR distribution was discussed. We have derived its moments, incomplete moments, renyi entropy, random number generator and quantile function. Furthermore, expressions of hazard and survival function were derived. The plots of probability density function and hazard function were also present. Maximum likelihood equations and the order statistics densities were derived. The utility of the distribution is optimized through a real data set application.

Keywords: Transmuted, Inverse Rayleigh, Moments, Entropy, ML Estimation

1 Introduction

Some compounding distributions have been proposed in the literature to model lifetime data sets. Voda[1], pioneered one parameter Inverse Rayleigh (IR) distribution. Inverse Rayleigh distribution is one of the comprehensive and relevant lifetime model, and its applications are in reliability and survival data sets. A lot of work has been done in the literature on Inverse Rayleigh distribution. IR distribution was championed by [1], discussed its properties and ML estimation of the scale parameter. Further, [2] provide closed-form expressions for the mean, harmonic mean, geometric mean, mode and the median of this distribution. Moreover, [2][3] estimated the parameters using different classical and Bayesian estimation methods. In recent years generalization of distribution theory has received considerable attention. Many extensions for the Rayleigh distribution, using the quadratic rank transmutation map (QRTM), have been introduced in the literature. For example, using the QRTM, Merovci [4] [5] developed transmuted Rayleigh and transmuted generalized Rayleigh distribution; Ahmed [6] generalized the Inverse Rayleigh distribution to develop a transmuted inverse Rayleigh distribution. Beta Inverse Rayleigh [7]. Rehman and Dar [8] studied Exponentiated inverse Rayleigh distribution, examined its mathematical properties and estimated the parameter using Bayesian estimation. The probability density and cumulative distribution function of Exponentiated Inverse Rayleigh distribution are given below,

$$f(x) = \frac{2\alpha\theta}{x^3}\left(e^{-\frac{\theta\alpha}{x^2}}\right) \,\&\, F(x) = \left(e^{-\frac{\theta\alpha}{x^2}}\right)$$

where α, θ are scale and shape parameters respectively. We define and studied a new model Transmuted Exponentiated Inverse Rayleigh distribution. A random variable x is said to have transmuted distribution if its cumulative distribution function satisfies the following relationship

$$F(x) = (1+\lambda)\,G(x) - \lambda G^2(x) \tag{1}$$

$$f(x) = g(x)\left[1+\lambda-2\lambda G(x)\right], \quad |\lambda| \leqslant 1 \tag{2}$$

where G(x) is cumulative distribution function of the base line distribution and f(x) and g(x) are the corresponding the probability density functions of F(x) and G(x) respectively. The range of λ is from negative one to positive one, for further information see [9]. The generalized distribution reduces to parent distribution for $\lambda = 0$. Using this approach various generalized distributions were generated; Transmuted Power Function, Kumaraswamy Exponentiated Inverse Rayleigh, Transmuted Lindley and Transmuted Pareto [10][11][12][13].

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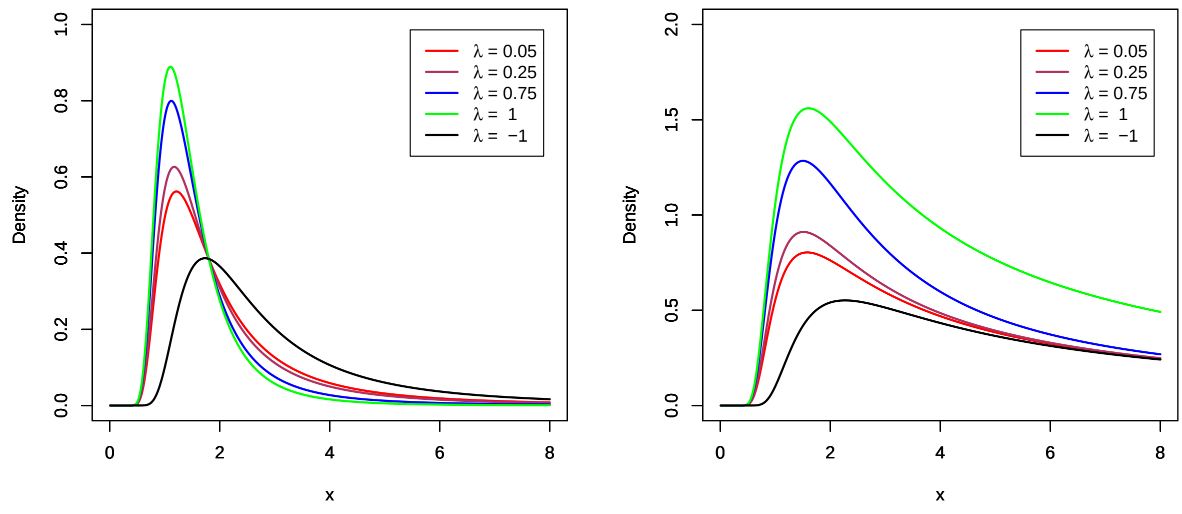

(a) (b)

Fig. 1: Part (a) is density plot and (b) is hazard function plot for selected values ($\alpha = \theta = 1.5$)

2 The TEIR Distribution

The probability density and cumulative distribution function of Transmuted Exponentiated Inverse Rayleigh distribution are obtained incorporating the EIR distribution pdf and cdf in equation 1 and 2

$$f(x) = \frac{2\alpha\theta}{x^3}\left(e^{-\frac{\theta\alpha}{x^2}}\right)\left[1+\lambda-2\lambda\left(e^{-\frac{\theta\alpha}{x^2}}\right)\right]; \qquad x \geqslant 0; \alpha,\theta > 0; \ |\lambda| \leqslant 1$$

$$F(x) = \left(e^{-\frac{\theta\alpha}{x^2}}\right)\left[1+\lambda-\lambda\left(e^{-\frac{\theta\alpha}{x^2}}\right)\right]$$

The TEIR distribution is a protracted distribution that support to analyse more multifaceted data sets. The Exponentiated Inverse Rayleigh distribution is special case when $\lambda = 0$. TEIR distribution reduce to Transmuted Inverse Rayleigh distribution for $\alpha = 1$.

2.1 Survival function

The survival function is obtained by using the relation S(x)=1-F(x), so the expression of survival function of TEIR distribution is given below,

$$S(x) = 1-\left[1+\lambda-\lambda\left(e^{-\frac{\theta\alpha}{x^2}}\right)\right]\left(e^{-\frac{\theta\alpha}{x^2}}\right)$$

2.2 Hazard function

The hazard rate function of random variable X with the pdf f(x) and a survival function S(x) is given by

$$H(x) = \frac{f(x)}{S(x)}$$

$$H(x) = \frac{2\alpha\theta\left(e^{-\frac{\theta\alpha}{x^2}}\right)\left[1+\lambda-2\lambda\left(e^{-\frac{\theta\alpha}{x^2}}\right)\right]}{x^3\left[1-\left(1+\lambda-\lambda\left(e^{-\frac{\theta\alpha}{x^2}}\right)\right)\left(e^{-\frac{\theta\alpha}{x^2}}\right)\right]}$$

3 Moments

3.1 rth moments

The rth moment of the Transmuted Exponentiated Inverse Rayleigh distribution, says μ_r, is given the following form

$$\mu_r = 2(\alpha\theta)^{\frac{r}{2}+1}\left[1+\lambda-\lambda(2)^{\frac{r}{2}}\right]\Gamma\left(1-\frac{r}{2}\right); \qquad r<2$$

Proof: By definition the moments obtained by using the expression given below,

$$\mu_r = \int_0^\infty x^r f(x;\alpha,\theta,\lambda)\,dx$$

$$\mu_r = \int_0^\infty x^r \frac{2\alpha\theta}{x^3}\left(e^{-\frac{\theta\alpha}{x^2}}\right)\left[1+\lambda-2\lambda\left(e^{-\frac{\theta\alpha}{x^2}}\right)\right]dx$$

Making substitution

$$y=\frac{\theta\alpha}{x^2}, \qquad x=\sqrt{\frac{\theta\alpha}{y}}, \qquad \& \quad dy=-\frac{2\theta\alpha}{x^3}\,dx$$

Putting these values the expression will be

$$= -2\theta\alpha\int_\infty^0 \left(\sqrt{\frac{\theta\alpha}{y}}\right)^{r-3} e^{-y}\left[1+\lambda-2\lambda e^{-y}\right]\left(\frac{\theta\alpha}{y}\right)^{\frac{3}{2}} dy$$

$$= 2(\theta\alpha)^{\frac{r}{2}+1}\left[\int_0^\infty y^{-\frac{r}{2}}e^{-y}\,dy+\lambda\int_0^\infty y^{-\frac{r}{2}}e^{-y}\,dy-2\lambda\int_0^\infty y^{-\frac{r}{2}}e^{-2y}\,dy\right] \tag{3}$$

The integral of the expression is

$$\int_0^\infty y^{-\frac{r}{2}}e^{-y}\,dy=\Gamma\left(1-\frac{r}{2}\right)$$

$$\int_0^\infty y^{-\frac{r}{2}}e^{-2y}\,dy=(2)^{\frac{r}{2}-1}\,\Gamma\left(1-\frac{r}{2}\right); \qquad r<2$$

Eq. 3 becomes,

$$\mu_r = 2(\theta\alpha)^{\frac{r}{2}+1}\left[\Gamma\left(1-\frac{r}{2}\right)+\lambda\Gamma\left(1-\frac{r}{2}\right)-\lambda\,(2)^{\frac{r}{2}}\,\Gamma\left(1-\frac{r}{2}\right)\right]; \quad r<2$$

$$\mu_r = 2(\theta\alpha)^{\frac{r}{2}+1}\left[1+\lambda-\lambda\,(2)^{\frac{r}{2}}\right]\Gamma\left(1-\frac{r}{2}\right); \quad r<2$$

3.2 Moment generating function

The expression of moment generating function is given below,

$$M_x(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}2(\theta\alpha)^{\frac{r}{2}+1}\left[1+\lambda-\lambda\,(2)^{\frac{r}{2}}\right]\Gamma\left(1-\frac{r}{2}\right);\qquad r<2$$

Proof: The moment generating function derive using the relation

$$M_x(t)=E\left(e^{tx}\right)=\int_0^{\infty}e^{tx}f(x;\alpha,\theta,\lambda)\,dx$$
$$=\int_0^{\infty}\left(1+tx+\frac{(tx)^2}{2!}-----\right)f(x;\alpha,\theta,\lambda)\,dx$$
$$=\int_0^{\infty}\sum_{r=0}^{\infty}\frac{t^r}{r!}x^r f(x;\alpha,\theta,\lambda)\,dx$$
$$=\sum_{r=0}^{\infty}\frac{t^r}{r!}E(x^r)$$
$$M_x(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}2(\theta\alpha)^{\frac{r}{2}+1}\left[1+\lambda-\lambda\,(2)^{\frac{r}{2}}\right]\Gamma\left(1-\frac{r}{2}\right);\qquad r<2$$

4 Random number generator

The random number the TEIR distribution is obtained by using the subsequent relation F(x)=R where R U(0,1). The Final expression of Random number generator is given below

$$\left(e^{-\frac{\theta\alpha}{x^2}}\right)\left[1+\lambda-\lambda\left(e^{-\frac{\theta\alpha}{x^2}}\right)\right]=R$$
$$x=\sqrt{-\frac{\theta\alpha}{\sqrt{\ln\left[\frac{(1+\lambda)-\sqrt{(1+\lambda)^2-4R\lambda}}{2\lambda}\right]}}}$$

5 Order statistics

Let $X_1,\ X_2,\ X_3,\ \ldots\ldots\ldots.,X_n$ be random sample of TEIR distribution and its ordered values is denoted as $X_{(1)},\ X_{(2)},\ X_{(3)},\ \ldots\ldots\ldots.,X_{(n)}$. The probability density function (pdf) of order statistics is obtained using the below function

$$f_{s:n}(x)=\frac{n!}{(s-1)!\,(n-s)!}f(x)\,[F(x)]^{s-1}[1-F(x)]^{n-s}$$

The density of the nth ordered statistics follows the Transmuted Exponentiated Inverse Rayleigh distribution is derived as follow

$$f_{s:n}(x)=\frac{n!\,2\alpha\theta}{(s-1)!\,(n-s)!\,x^3}\left(e^{-\frac{\theta\alpha}{x^2}}\right)^s\left[1+\lambda-2\lambda\left(e^{-\frac{\theta\alpha}{x^2}}\right)\right]\left[1+\lambda-\lambda\left(e^{-\frac{\theta\alpha}{x^2}}\right)\right]^{s-1}\left[1-\left(e^{-\frac{\theta\alpha}{x^2}}\right)\left[1+\lambda-\lambda\left(e^{-\frac{\theta\alpha}{x^2}}\right)\right]\right]^{n-s}$$

The density of smallest order statistics is given below,

$$f_{n:n}(x)=\frac{2n\alpha\theta}{x^3}\left(e^{-\frac{\theta\alpha}{x^2}}\right)\left[1+\lambda-2\lambda\left(e^{-\frac{\theta\alpha}{x^2}}\right)\right]\left[\left(e^{-\frac{\theta\alpha}{x^2}}\right)\left[1+\lambda-\lambda\left(e^{-\frac{\theta\alpha}{x^2}}\right)\right]\right]^{n-1}$$

The largest density of order statistics is

$$f_{n:n}(x)=\frac{2n\alpha\theta n}{x^3}\left(e^{-\frac{\theta\alpha}{x^2}}\right)\left[1+\lambda-2\lambda\left(e^{-\frac{\theta\alpha}{x^2}}\right)\right]\left[1-\left(e^{-\frac{\theta\alpha}{x^2}}\right)\left[1+\lambda-\lambda\left(e^{-\frac{\theta\alpha}{x^2}}\right)\right]\right]^{n-1}$$

6 Estimation and data analysis

We consider the maximum likelihood estimation procedure to estimate the unknown parameters of TEIR distribution. The sample values $x_1, x_2, x_3, \ldots\ldots., x_n$ consisting of n observations. The Log-likelihood function of probability density function is given by

$$l=n\log(2)+n\log(\alpha)+n\log(\theta)-\sum_{i=1}^{n}\ln x_i{}^3-\sum_{i=1}^{n}\log\left[1+\lambda-2\lambda\left(e^{-\frac{\theta\alpha}{x_i{}^2}}\right)\right]$$

Hence, the parameters are obtained by differentiating the above equation on parameters and equating them equal to zero. The maximum likelihood differential equations are,

$$\frac{\partial L}{\partial\theta}=\frac{n}{\theta}+\frac{2\alpha\lambda\left(e^{-\frac{\theta\alpha}{x_i{}^2}}\right)}{x^2\left(1+\lambda-2\lambda\left(e^{-\frac{\theta\alpha}{x_i{}^2}}\right)\right)}$$

$$\frac{\partial L}{\partial\alpha}=\frac{n}{\alpha}+\frac{2\theta\lambda\left(e^{-\frac{\theta\alpha}{x_i{}^2}}\right)}{x^2\left(1+\lambda-2\lambda\left(e^{-\frac{\theta\alpha}{x_i{}^2}}\right)\right)}$$

$$\frac{\partial L}{\partial\lambda}=\frac{1-2\left(e^{-\frac{\theta\alpha}{x_i{}^2}}\right)}{1+\lambda-2\lambda\left(e^{-\frac{\theta\alpha}{x_i{}^2}}\right)}$$

It is clear that equations are not in explicit form, so estimates are obtained by the solution of nonlinear equations simultaneously. The estimated values can be obtained numerically by using the iteratively based procedures such as the Newton-Raphson algorithm. AdequacyModel library uses in R for estimation of parameters.

7 Application

In this section usefulness of the under discussion distribution is illustrates using real-life data example. The data consist of thirty successive March precipitation (in inches) observations given by [14]. The data set is provided below 0.77, 1.74, 0.81, 1.2, 1.95, 1.2, 0.47, 1.43, 3.37, 2.2, 3, 3.09, 1.51, 2.1, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.9, and 2.05. Three distributions are fitted to this data set; Transmuted Exponentiated Inverse Rayleigh

Table 1: Descriptive statistic of for precipitation data

Min	Median	Mean	Mode	Variance	Skewness	Kurtosis	Max
0.32	1.47	1.675	1.5	1.00123	1.08668	1.20688	4.75

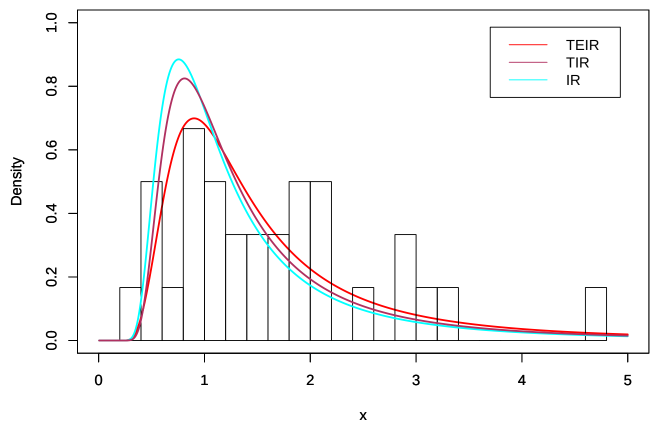

Fig. 2: Fitted densities of three distributions

(TEIR), Transmuted Inverse Rayleigh (TIR) and Inverse Rayleigh (IR) distributions. Table 1 presents basic descriptive statistics for the samples data set.

The estimated parameters of this data are given in Table 2. The diagnostic measures are present in Table 3. It is observed that the TEIR distribution is competitive distribution compared with other distributions. In fact, based on the values of AIC, AICC, BIC, HQBIC and Kolmogorov-Smirnov (KS) test statistic. Among the all considered fitted distributions, TEIR distribution is the best distribution. The probability density functions are presented in Fig 2.

Table 2: Estimated parameter values for precipitation data

Model	α	θ	λ
TEIR	1.1878(5.739)	0.6362(4.778)	-0.6701(0.2661)
TIR		0.8588(0.2136)	0.0001 (0.4017)
IR		0.8588 (0.1568)	

Table 3: The measures -2$\hat{l}$, AIC, CAIC, BIC and HQIC

Model	$-2\hat{l}$	AIC	CAIC	BIC	HQIC	KS
TEIR	84.202	90.202	91.125	94.406	90.547	0.1818
TIR	85.073	91.073	91.717	94.759	92.169	0.2397
IR	96.292	92.292	92.416	92.674	91.721	0.2696

8 Conclusion

We studied a new distribution, entitled the TEIR distribution, which is a modification of EIR distribution. In this distribution, a new parameter has inducted that increase the flexibility of the distribution. We plot the density and hazard rate curves for selected values of parameters. The expressions of moment generating function incomplete moments, quantile function, and entropy are drive. The largest and smallest densities of ordered statistics are also calculated. The maximum likelihood equations are derived. The utility of the model is exemplified in an application of sample data set by using maximum likelihood method. The derived model is the best model among other fitted models. We hope that in the field of modeling of reliability analysis, the proposed model may attract wider application.

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