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Limit Distributions of Maximal Random Chord Length

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Abstract: Taking in consideration several methods of choosing the chord at random, we have described the asymptotic distribution of the maximum length of the chord on the unit circle. The appropriate normalizing constants are presented and the rate of convergence is established, for each case.

Keywords: Extreme value distribution, Bertrand's paradox, random chord length, rate of convergence.
MSC: 60DO5, 60G70

1 Introduction

The famous Bertrand paradox, besides alerting us to be very careful when choosing geometrical objects "at random", also gave rise to many different methods for the choice of the random chord. Bertrand himself considered three ways of choosing the chord "at random", and some papers that followed proposed more. Some of them can be found in [4, 5, 1, 6]. Similar problems with other geometrical objects were considered in e.g. [7, 8]. With this paradox as a guide, for every way of choosing a chord in a circle, we have different distribution functions for random chord length. The goal of this paper is to analyze the limit distribution of the maximal length of the chord depending of the way it is chosen "at random".

Let the random variable X be the length of the random chord in a unit circle, and its distribution function $F(x) = P\{X \leq x\}, x \in \mathbf{R}$.

We shall consider (X_n) sequence of i.i.d. random variables, $n \in \mathbf{N}$, with the distribution function F. Let $M_n = \max\{X_1, X_2, \ldots, X_n\}$ for $n \in \mathbf{N}$. The distribution function for M_n is $F_{M_n}(x) = P\{M_n \leq x\} = F^n(x)$, for $x \in \mathbf{R}$ and $n \in \mathbf{N}$.

This paper is organized as follows. In section Preliminaries we present different methods for generating a chord "at random" on unit circle and distribution functions for the length of a random chord which differs respectively for each case. The section 3 is devoted to obtain the normalizing constants and the rate of convergence for the i.i.d. sequence of random variables with distribution functions obtained in section 2.

2 Preliminaries

Here we shall present several cases of methods of choosing a chord "at random" on unit circle and five distribution functions to which they correspond. See [1] for more details.

Case 1: This appears when choosing two random points uniformly and independently on the circumference of the circle, from which a random chord is formed.

$$F(x) = \begin{cases} 0 & x < 0, \\ \frac{2}{\pi}\arcsin\frac{x}{2} & 0 \leq x < 2, \\ 1 & x \geq 2. \end{cases} \tag{1}$$

Case 2: This case is fulfilled when the chord center is randomly selected inside the circle.

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$$F(x)=\begin{cases}0 & x<0,\\ \frac{x^2}{4} & 0\le x<2,\\ 1 & x\ge 2.\end{cases} \tag{2}$$

Case 3: This case appears when the chord center is uniformly distributed on a reference radius.

$$F(x)=\begin{cases}0 & x<0,\\ 1-\frac{\sqrt{4-x^2}}{2} & 0\le x<2,\\ 1 & x\ge 2.\end{cases} \tag{3}$$

Case 4: This case appears within three different methods of choosing a chord "at random". First method is described by defining a random chord as one that separates the circle into two parts, with the smaller area, being equally likely in $(0,0.5\pi)$. The second method appears by selecting a random point R inside the circle, with spinning a random angle at R relative to reference line OR. And the last method appears by choosing one random point on the circumference and one inside the circle which determines the random chord.

$$F(x)=\begin{cases}0 & x<0,\\ \frac{2}{\pi}\left(\arcsin\frac{x}{2}-\frac{x\sqrt{4-x^2}}{4}\right) & 0\le x<2,\\ 1 & x\ge 2.\end{cases} \tag{4}$$

Case 5: Select two random points inside the circle. Connect and extend those random points in order to form a random chord.

$$F(x)=\begin{cases}0 & x<0,\\ \frac{2}{\pi}\arccos\frac{\sqrt{4-x^2}}{2}-\frac{x(6+x^2)\sqrt{4-x^2}}{12\pi} & 0\le x<2,\\ 1 & x\ge 2.\end{cases} \tag{5}$$

In order to find the limit distribution we shall use the Theorem 1.6.1 from [9]. Distribution function F from all five cases satisfies necessary conditions of this theorem, with the appropriate finite limit

$$\alpha=\lim_{x\to x_F}\frac{(x_F-x)F'(x)}{1-F(x)}, \tag{6}$$

where $x_F=\sup\{x:F(x)<1\}=2<\infty$ and F' denotes nonzero derivation of F. We easily infer that the appropriate extreme value distribution belongs to the Weibull-type distributions. Therefore, by fundamental theorem of extreme value theory (see e.q. [2, 3]), there exist some constants $a_n>0$ and $b_n\in\mathbf{R}$, $n\in\mathbf{N}$, such that

$$P\left\{\frac{M_n-b_n}{a_n}\le x\right\}\xrightarrow{\omega}\begin{cases}e^{-(-x)^{\alpha}} & x<0,\\ 1 & x\ge 0,\end{cases}\quad n\to\infty,$$

where ω denotes convergence in distribution. The normalizing constants a_n and b_n, $n\in\mathbf{N}$, meets the limit $a_nx+b_n\to x_F=2$ as $n\to\infty$, for $x\in\mathbf{R}$.

The next two well-known theorems in extreme value theory will have a significant part in this paper (sequence (X_n) is proposed to be arbitrary):

Theorem 1. ([9], Th. 1.5.1) Let (X_n) be a sequence of independent and identically distributed random variables with joint distribution function F. Let $0\le\tau\le+\infty$ and (u_n) be a sequence of real numbers. Then

$$\lim_{n\to\infty}n(1-F(u_n))=\tau,$$

if and only if $\lim_{n\to\infty}P\{M_n\le u_n\}=e^{-\tau}$.

Theorem 2. ([9], Th. 2.4.2) Let $\{X_n\}$ be i.i.d. sequence, put $\tau_n=n(1-F(u_n))$, and write $\Delta_n=\left(1-\frac{\tau_n}{n}\right)^n-e^{-\tau_n}$, $\Delta_n'=e^{-\tau_n}-e^{-\tau}$, so that

$$P\{M_n\le u_n\}-e^{-\tau}=\Delta_n+\Delta_n'.$$

Then

$$0 \le -\Delta_n \le \frac{\tau_n^2 e^{-\tau_n}}{2} \frac{1}{n-1} \le 0.3 \frac{1}{n-1},$$

where the first bound is asymptotically sharp, in the sense that if $\tau_n \to \tau$ then $\Delta_n \sim -(\tau^2 e^{-\tau}/2)/2$. Furthermore, for $\tau - \tau_n \le \log 2$,

$$\Delta_n^{'} = e^{-\tau}\{(\tau - \tau_n) + \theta(\tau - \tau_n)^2\},$$

with $0 < \theta < 1$.
So, it is easy to conclude that as $\tau_n \to \tau$ we have

$$\Delta_n \sim -\frac{e^{-\tau}\tau^2}{2n}, \quad \Delta_n^{'} \sim e^{-\tau}(\tau - \tau_n),$$

as $n \to \infty$.

3 Main result

This paper contains one dominant theorem which provides the asymptotic values of normalizing constants of the extreme value theory, provided for i.i.d. sequence $\{X_n{}^i\}$, $n \in \mathbf{N}$, with distribution function F_i, $i = \overline{1,5}$, with the appropriate rate of convergence for each case.

Theorem 3. Let $X_1^i, X_2^i, \ldots$ be independent and with the same distribution function F_i, which coincides with the distribution function for the length of a random chord on unit circle for cases i, $i = \overline{1,5}$. Let $M_n^i = \max\{X_1^i, X_2^i, \ldots, X_n^i\}$, $n \in \mathbf{N}$, be the maximal length of n random chords in unit circle, chosen by different methods. The constants $a_n^i > 0$ and $b_n^i \in \mathbf{R}$, $n \in \mathbf{N}$, such that $P\left\{\frac{M_n^i - b_n^i}{a_n^i} \le x\right\}$ converges in distribution to Weibull distribution with some shape parameter, are asymptotically equal to $a_n^1 = \frac{\pi^2}{4n^2}$, $a_n^2 = 1/n$, $a_n^3 = 1/n^2$, $a_n^4 = \frac{\pi^2}{16n^2}$ and $a_n^5 = \frac{9\pi^2}{256n^2}$, with $b_n^i = 2$ for each $i = \overline{1,5}$ and $n \in \mathbf{N}$. The rate of convergence is $\mathscr{O}(\frac{1}{n})$, for each case.

Proof of Theorem 3. To proof this theorem well known limit approximations will be helpful:

$$\ln x \sim x - 1, x \to 1, \tag{7}$$

$$\ln(1-x) \sim -x, x \to 0, \tag{8}$$

and

$$\arccos x \sim \frac{\pi}{2} - x, x \to 0. \tag{9}$$

Let us denote $u_n = a_n x + b_n$, for $x \in \mathbf{R}$ and $n \in \mathbf{N}$.

Case 1: Distribution function F_{M_n} with the argument u_n can be easily described in the following manner:

$$F_{M_n}(u_n) = \left(\frac{2}{\pi}\arcsin\frac{u_n}{2}\right)^n = \exp\left\{n \ln\left(\frac{2}{\pi}\arcsin\frac{u_n}{2}\right)\right\}. \tag{10}$$

With the limit (7), we can deduce

$$n \ln\left(\frac{2}{\pi}\arcsin\frac{u_n}{2}\right) \sim \frac{2n}{\pi}\arcsin\frac{u_n}{2} - n, n \to \infty. \tag{11}$$

The parameter α of Weibull distribution is equal to $1/2$, considered by (6).
Hence, for each $x < 0$,

$$\frac{2n}{\pi}\arcsin\frac{u_n}{2} - n \sim -(-x)^{1/2}, n \to \infty. \tag{12}$$

Clearly next approximation holds

$$\arcsin\frac{u_n}{2} = \arcsin\frac{a_n x + b_n}{2} \sim -\frac{\pi(-x)^{1/2}}{2n} + \frac{\pi}{2}, n \to \infty, \tag{13}$$

and additionally, using (12),

$$\frac{a_n x + b_n}{2} \sim \cos\frac{\pi(-x)^{1/2}}{2n} \sim 1 + \frac{x\pi^2}{8n^2}, n \to \infty, \tag{14}$$

instructs us that the asymptotic normalizing constants are the form of $a_n = \frac{\pi^2}{4n^2}$ and $b_n = 2$, $n \in \mathbf{N}$.

With Theorem 2 as a guide, the limit , $n(1-F(u_n)) \sim (-x)^{1/2}, n \to \infty$, is established and accordingly the next limit follows

$$\arcsin\left(1 - \frac{\pi^2(-x)}{8n^2}\right) \sim \frac{\pi}{2}\left(1 - \frac{(-x)^{1/2}}{n}\right), n \to \infty. \tag{15}$$

We note by (15), using identical notation as in Theorem 2, that

$$\tau_n = n\left(1 - \frac{2}{\pi}\arcsin\frac{u_n}{2}\right) = n\left(1 - \frac{2}{\pi}\arcsin\left(\frac{x\pi^2}{8n^2} + 1\right)\right), n \to \infty, \tag{16}$$

and $\tau = (-x)^{1/2}$. Therefore, the following limit relation holds

$$\Delta_n + \Delta_n' \sim -\frac{e^{-(-x)^{1/2}}(-x)}{2n}, n \to \infty.$$

Case 2: The beginning is similar like in case 1, using distribution function F_{M_n} with argument u_n, and with further reasoning that

$$F_{M_n}(a_n x + b_n) = \exp\{n(2\ln(a_n x + b_n) - \ln 4)\}. \tag{17}$$

The adequate parameter of Weibull distribution is obtained by (6), and found to be 1. Then, for each $x < 0$, the next limit relation holds

$$n(2\ln(a_n x + b_n) - \ln 4) \sim x, n \to \infty. \tag{18}$$

and consequently

$$2\ln(a_n x + b_n) \sim \frac{x}{n} + \ln 4, \tag{19}$$

which is equivalent to

$$\ln\frac{a_n x + b_n}{2} \sim \frac{x}{2n}, n \to \infty. \tag{20}$$

After considering limit (7), approximation

$$\ln\frac{a_n x + b_n}{2} \sim -1 + \frac{a_n x + b_n}{2}, n \to \infty, \tag{21}$$

holds, and the normalizing constants follow the approximation

$$a_n x + b_n \sim 2 + \frac{x}{n}, n \to \infty, \tag{22}$$

and, therefore, they are asymptotically equal to $a_n = 1/n$, $b_n = 2$, $n \in \mathbf{N}$. Introducing τ_n as

$$\tau_n = n(1 - F(u_n)) = n\left(1 - \frac{(x/n+2)^2}{4}\right) = -\frac{x^2}{4n} - x, \tag{23}$$

with

$$\tau - \tau_n = -x - \left(-\frac{x^2}{4n} - x\right) = \frac{x^2}{4n}, \tag{24}$$

so the following estimation holds

$$\Delta_n+\Delta_n^{'}\sim -\frac{e^x x^2}{2n}+\frac{e^x x^2}{4n}=-\frac{e^x x^2}{4n}, n\to\infty.$$

Case 3: With similar form as before,

$$F_{M_n}(u_n)=\exp\left\{n\ln\left(1-\frac{\sqrt{4-u_n^2}}{2}\right)\right\}, \tag{25}$$

and with the appropriate Weibull distribuiton parameter α, which is equal to 1/2 by (6), for each $x<0$, the sequence (u_n) satisfies the limit relation

$$\ln\left(1-\frac{\sqrt{4-u_n^2}}{2}\right)\sim\frac{-(-x)^{1/2}}{n}, n\to\infty. \tag{26}$$

Reformulating (26) with some help of (8),

$$\ln\left(1-\frac{\sqrt{4-u_n^2}}{2}\right)\sim -\frac{\sqrt{4-u_n^2}}{2}, n\to\infty, \tag{27}$$

enables us to reason that

$$\sqrt{1-\frac{u_n^2}{4}}\sim\frac{(-x)^{1/2}}{n}, n\to\infty. \tag{28}$$

Furthermore, from results (27) and (28),

$$\frac{u_n}{2}\sim\sqrt{1+\frac{x}{n^2}}\sim 1+\frac{x}{2n^2}, n\to\infty, \tag{29}$$

we can conclude that the asymptotic normalizing constants are the form of $a_n=1/n^2$ and $b_n=2$, $n\in\mathbf{N}$. Considering the asymptotic estimation of τ_n as

$$\tau_n=\frac{n}{2}\sqrt{4-u_n^2}=\sqrt{-x}\sqrt{1+\frac{x}{4n^2}}\sim\sqrt{-x}\left(1+\frac{x}{8n^2}\right), n\to\infty, \tag{30}$$

with $\tau=(-x)^{1/2}$, the following estimation holds

$$\Delta_n+\Delta_n^{'}\sim -\frac{e^{-(-x)^{1/2}}(-x)}{2n}+\frac{e^{-(-x)^{1/2}}(-x)^{3/2}}{8n^2}, n\to\infty.$$

Case 4: Like in previous cases, with one step ahead and with Weibull-distribution parameter $\alpha=1/2$ (6), for each $x<0$ and (u_n) defined previously, following limit is

$$\ln\left[\frac{2}{\pi}\left(\arcsin\frac{u_n}{2}-\frac{u_n\sqrt{4-u_n^2}}{2}\right)\right]\sim -\frac{(-x)^{1/2}}{n}, n\to\infty. \tag{31}$$

With (7) as a limit guide, estimating the left side of (31) as

$$\ln\left[\frac{2}{\pi}\left(\arcsin\frac{u_n}{2}-\frac{u_n\sqrt{4-u_n^2}}{2}\right)\right]\sim -1+\frac{2}{\pi}\left(\arcsin\frac{u_n}{2}-\frac{u_n\sqrt{4-u_n^2}}{2}\right), n\to\infty, \tag{32}$$

and after some easy notable limits

$$\frac{u_n\sqrt{4-u_n^2}}{2}=\frac{u_n}{2}\sqrt{(2-u_n)(2+u_{n)}}\sim u_n\sqrt{2-u_n}\sim 2\sqrt{2-u_n}, n\to\infty, \tag{33}$$

which are supported by the assumption that $u_n \to 2, n \to \infty$, the following expression holds

$$-1+\frac{2}{\pi}\left(\arcsin\frac{u_n}{2}-\frac{u_n\sqrt{4-u_n^2}}{2}\right)\sim-\frac{4}{\pi}\sqrt{2-u_n}, n\to\infty. \tag{34}$$

Therefore, for each $x<0$,

$$-\frac{4}{\pi}\sqrt{2-u_n}\sim-\frac{(-x)^{1/2}}{n}, n\to\infty, \tag{35}$$

and after simple algebra, we get the limit

$$2-a_nx+b_n\sim\frac{\pi^2}{16}\frac{(-x)}{n^2}, n\to\infty, \tag{36}$$

which is helpful in realizing that the asymptotic values of the normalizing constants are $a_n=\frac{\pi^2}{16n^2}$ and $b_n=2$, $n\in\mathbf{N}$. Asymptotically

$$\tau_n=n\left(1-\frac{2}{\pi}\left(\arcsin\frac{u_n}{2}-\frac{u_n\sqrt{4-u_n^2}}{2}\right)\right)\sim\frac{2}{\pi}\frac{nu_n\sqrt{4-u_n}}{2}=$$
$$\frac{2}{\pi}\sqrt{-x}\frac{\pi}{4n}n\left(\frac{\pi^2}{16n^2}x+2\right)\sqrt{1+\frac{\pi^2x}{64n^2}}\sim\sqrt{-x}\left(1+\frac{\pi^2x}{128n^2}\right), n\to\infty,$$

and with $\tau=(-x)^{1/2}$, enable us to deduce the limit relation

$$\Delta_n+\Delta_n^{'}\sim-\frac{e^{-(-x)^{1/2}}(-x)}{2n}+\frac{e^{-(-x)^{1/2}}(-x)^{3/2}\pi^2}{128n^2}, n\to\infty.$$

Case 5: By a well-known method of finding the Weibull-distribution parameter, we conclude that $\alpha=1/2$, and in the same way of reasoning and going several steps ahead, next limit relation is, for each $x<0$,

$$n\ln\left(\frac{2}{\pi}\arccos\frac{\sqrt{4-u_n^2}}{2}-\frac{u_n(6+u_n^2)\sqrt{4-u_n^2}}{12\pi}\right)\sim-(-x)^{1/2}, n\to\infty, \tag{37}$$

which is equivalent to

$$\ln\left(\frac{2}{\pi}\arccos\frac{\sqrt{4-u_n^2}}{2}-\frac{u_n(6+u_n^2)\sqrt{4-u_n^2}}{12\pi}\right)\sim-\frac{(-x)^{1/2}}{n}, n\to\infty. \tag{38}$$

Similarly like before, by (7), (37) and (38), and with identical reasoning, the following estimation holds

$$\ln\left(\frac{2}{\pi}\arccos\frac{\sqrt{4-u_n^2}}{2}-\frac{u_n(6+u_n^2)\sqrt{4-u_n^2}}{12\pi}\right)\sim$$
$$-1+\frac{2}{\pi}\arccos\frac{\sqrt{4-u_n^2}}{2}-\frac{u_n(6+u_n^2)\sqrt{4-u_n^2}}{12\pi}\sim-\frac{(-x)^{1/2}}{n}, n\to\infty,$$

and with the help of (9) for the last medium term estimation, we obtain

$$\frac{2}{\pi}\arccos\frac{\sqrt{4-u_n^2}}{2}-\frac{u_n(6+u_n^2)\sqrt{4-u_n^2}}{12\pi}\sim$$
$$1-\frac{\sqrt{4-u_n^2}}{\pi}-\frac{u_n(6+u_n^2)\sqrt{4-u_n^2}}{12\pi}\sim1-\frac{16}{3\pi}\sqrt{2-u_n}, n\to\infty.$$

Thereby, for each $x<0$,

$$\frac{16}{3\pi}\sqrt{2-u_n}\sim\frac{(-x)^{1/2}}{n}, n\to\infty, \tag{39}$$

and with some help of simple algebra we conclude that

$$2-(a_n x+b_n)\sim\frac{9\pi^2}{256n^2}(-x),n\to\infty, \tag{40}$$

so the normalizing constants are easily noted to be $a_n=\frac{9\pi^2}{256n^2}$ and $b_n=2$, $n\in\mathbf{N}$. Discussing the asymptotic behavior of τ_n we see that

$$\tau_n=n\left(1-\frac{2}{\pi}\arccos\frac{\sqrt{4-u_n^2}}{2}+\frac{u_n(6+u_n^2)\sqrt{4-u_n^2}}{12\pi}\right)\sim$$
$$n\frac{\sqrt{4-u_n^2}}{\pi}\left(1+\frac{u_n(6+u_n^2)}{12}\right)\sim$$
$$(-x)^{1/2}\sqrt{1+\frac{9x\pi^2}{1024n^2}}\sim(-x)^{1/2}\left(1+\frac{9x\pi^2}{2048n^2}\right),n\to\infty,$$

and with $\tau=(-x)^{1/2}$, the following asymptotic estimation holds

$$\Delta_n+\Delta_n^{'}\sim-\frac{e^{-(-x)^{1/2}}(-x)}{2n}+\frac{e^{-(-x)^{1/2}}9\pi^2(-x)^{3/2}}{2048n^2},n\to\infty.$$

This completes the proof of this theorem. □

4 Conclusion

The main theorem in this paper gives the asymptotic normalizing constants and the rate of convergence for the appropriate extreme value theory problem. Since there are more different ways, than presented in this paper, of choosing a random chord on unit circle, it will be interesting to find distributions functions of random chord length based on new methods of choosing a chord ”at random”. This is a open problem.

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